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# SOME MISCELLANEOUS PROPERTIES AND APPLICATIONS OF CERTAIN OPERATORS OF FRACTIONAL CALCULUS

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**Abstract.** In recent years, various operators of fractional calculus (that is, calculus of integrals and derivatives of arbitrary real or complex orders) have been investigated and applied in many remarkably diverse fields. The main object of this paper is to consider some miscellaneous properties and applications which are associated with several fractional differintegral operators. We first investigate, in a systematic and unified manner, various families of series identities which emerged in connection with some of these fractional differintegral formulas. By using such operators of fractional calculus, a number of integral formulas as well as fractional differintegral formulas involving inverse hyperbolic functions are also evaluated.

## 1. Introduction, Definitions and Preliminaries

During the past three decates or so, the extensively-investigated The subject of fractional calculus (that is, calculus of integrals and derivatives of any *arbitrary* real or complex orders) has been investigated rather extensively due mainly to its demonstrated applications in numerous seemingly diverse fields of science and engineering (see, for details, [1, 11-13, 20, 45-47, 53] and [56]). One of the operators of fractional calculus, which has presumably been used in the literature most commonly and most widely, is the Riemann-Liouville fractional differintegral

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operator $\mathcal{D}_z^\mu$ of order $\mu$ $(\mu \in \mathbb{C})$ given by Definition 1 below (see, for details, [4, Chapter 13], [12, 20, 45, 47] and [53]).

**Definition 1.** The (Riemann-Liouville) *fractional differintegral operator* $\mathcal{D}_z^\mu$ is defined by

$$\mathcal{D}_z^\mu \{f(z)\} = \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-\zeta)^{-\mu-1} f(\zeta)\, d\zeta & [\Re(\mu) < 0] \\[2mm] \dfrac{d^n}{dz^n} \left\{ \mathcal{D}_z^{\mu-n} \{f(z)\} \right\} & [n-1 \leqq \Re(\mu) < n;\ n \in \mathbb{N}], \end{cases} \tag{1}$$

provided that the defining integral in (1) exists.

Throughout our present investigation, we denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. Furthermore, we have

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{N} := \{1, 2, 3, \cdots\}.$$

It readily follows from Definition 1 that

$$\mathcal{D}_z^\mu \left\{ z^\lambda \right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, z^{\lambda-\mu} \qquad [\Re(\lambda) > -1]. \tag{2}$$

In terms of the Srivastava-Daoust multivariable hypergeometric function defined by (*cf.* [57] and [58]; see also [9], [60, p. 37 *et seq.*] and [62, p. 64 *et seq.*])

$$F^{p:\, q_1; \cdots;\, q_r}_{\ell: m_1; \cdots; m_r} \left[ \begin{matrix} \left(a_j; \alpha'_j, \cdots, \alpha_j^{(r)}\right)_{1,p} : \left(c'_j, \gamma'_j\right)_{1,q_1}; \cdots; \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,q_r}; \\ \left(b_j; \beta'_j, \cdots, \beta_j^{(r)}\right)_{1,\ell} : \left(d'_j, \delta'_j\right)_{1,m_1}; \cdots; \left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r}; \end{matrix} \; z_1, \cdots, z_r \right]$$
$$:= \sum_{n_1, \cdots, n_r = 0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n_1\alpha'_j + \cdots + n_r\alpha_j^{(r)}} \prod_{j=1}^{q_1} (c'_j)_{n_1\gamma'_j} \cdots \prod_{j=1}^{q_r} \left(c_j^{(r)}\right)_{n_r\gamma_j^{(r)}}}{\prod_{j=1}^{\ell} (b_j)_{n_1\beta'_j + \cdots + n_r\beta_j^{(r)}} \prod_{j=1}^{m_1} (d'_j)_{n_1\delta'_j} \cdots \prod_{j=1}^{m_r} \left(d_j^{(r)}\right)_{n_r\delta_j^{(r)}}} \frac{z_1^{n_1}}{n_1!} \cdots \frac{z_r^{n_r}}{n_r!}, \tag{3}$$

in which the multiple hypergeometric series converges absolutely under the parametric and variable constraints detailed in the aforecited works and $(\lambda)_\nu$ denotes the Pochhammer symbol (or the *shifted factorial*, since $(1)_n = n!$ for $n \in \mathbb{N}_0$) given (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{4}$$

we recall the following potentially useful analogue of a well-known result [62, p. 303, Problem 1] for the familiar Riemann-Liouville operator $\mathcal{D}_z^{\mu}$ (*cf.* [19, p. 54, Equation (3.13)]; see also [14, p. 1178, Equation (2.3)]):

$$\mathcal{D}_z^{\lambda-\mu}\left\{z^{\lambda-1}\prod_{j=1}^{r}\left\{\left(1-a_j\,z^{\mu_j}\right)^{-\alpha_j}\right\}\right\}$$

(5)

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\,z^{\mu-1}\,F_{1:0;\,\cdots\,;0}^{1:1;\,\cdots\,;1}\left[\begin{array}{l}(\lambda;\mu_1,\cdots,\mu_r)\;:(\alpha_1,1)\,;\cdots;(\alpha_r,1)\,;\\(\mu;\mu_1,\cdots,\mu_r)\;:\underline{\qquad}\,;\cdots;\underline{\qquad}\,;\end{array}a_1\,z^{\mu_1},\cdots,a_r\,z^{\mu_r}\right]$$

$$\left[\Re(\lambda)>0;\;\mu_j>0\;\;(j=1,\cdots,r);\;\max\left\{\left|a_1\,z^{\mu_1}\right|,\cdots,\left|a_r\,z^{\mu_r}\right|\right\}<1\right].$$

By applying the reflection formula for the Gamma function:

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}\qquad\left(z\in\mathbb{C}\setminus\mathbb{Z};\;\mathbb{Z}:=\{0,\pm1,\pm2,\cdots\}\right), \tag{6}$$

we can easily rewrite (5) as follows:

$$\mathcal{D}_z^{\lambda-\mu}\left\{z^{\lambda-1}\prod_{j=1}^{r}\left\{\left(1-a_j\,z^{\mu_j}\right)^{-\alpha_j}\right\}\right\}=\frac{\Gamma(1-\mu)}{\Gamma(1-\lambda)}\,\frac{\sin(\pi\mu)}{\sin(\pi\lambda)}\,z^{\mu-1}$$

(7)

$$\cdot\,F_{1:0;\,\cdots\,;0}^{1:1;\,\cdots\,;1}\left[\begin{array}{l}(\lambda;\mu_1,\cdots,\mu_r):(\alpha_1,1)\;;\cdots;\;(\alpha_r,1)\,;\\(\mu;\mu_1,\cdots,\mu_r):\underline{\qquad}\;;\cdots;\underline{\qquad}\;;\end{array}a_1\,z^{\mu_1},\cdots,a_r\,z^{\mu_r}\right]$$

$$\left[\Re(\lambda)>0;\;\mu_j>0\;\;(j=1,\cdots,r);\;\max\left\{\left|a_1\,z^{\mu_1}\right|,\cdots,\left|a_r\,z^{\mu_r}\right|\right\}<1\right].$$

We next recall that, in a series of recent investigations dealing with power, composite, rational, exponential and logarithm functions as well as many other special functions, Nishimoto *et al.* and other authors (*cf., e.g.*, [27-43]; see also [7, 17-19, 21-24, 54, 55, 61]) made use of a certain *fractional differintegral operator* $\mathcal{N}_z^{\nu}$ [that is, *fractional derivative operator* $\mathcal{N}_z^{\nu}$ *of order* $\nu\in\mathbb{C}$ *when* $\Re(\nu)>0$ and *fractional integral operator* $\mathcal{N}_z^{\nu}$ of order $\nu\in\mathbb{C}$ when $\Re(\nu)<0$] (see Definition 2 below), which is based essentially upon the familiar *Cauchy-Goursat Integral Formula* (see, for details, [65, Chapter 5]).

**Definition 2** (*cf.* [25, 26] and [63]). If the function $f(z)$ is analytic (regular) and has no branch points inside and on $\mathcal{C}$, where

$$\mathcal{C}:=\{\mathcal{C}^-,\mathcal{C}^+\}, \tag{8}$$

$\mathcal{C}^-$ is a contour along the cut joining the points $z$ and $-\infty+i\Im(z)$, which starts from the point at $-\infty$, encircles the point $z$ once counter-clockwise, and returns to the point at $-\infty$, $\mathcal{C}^+$ is a contour along the cut joining the points $z$ and $\infty+i\Im(z)$, which starts from the point at $\infty$, encircles the point $z$ once counter-clockwise, and returns to the point at $\infty$,

$$\mathcal{N}_z^{\nu}\{f(z)\} := \frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,\mathrm{d}\zeta \tag{9}$$

$$(\nu \in \mathbb{C} \setminus \mathbb{Z}^-;\ \mathbb{Z}^- := \{-1,-2,-3,\ \cdots\})$$

and

$$\mathcal{N}_z^{-n}\{f(z)\} := \lim_{\nu \to -n}\left\{\mathcal{N}_z^{\nu}\{f(z)\}\right\} \qquad (n \in \mathbb{N} := \{1,2,3,\ \cdots\}), \tag{10}$$

where $\zeta \neq z$,

$$-\pi \leqq \arg(\zeta - z) \leqq \pi \quad \text{for} \quad \mathcal{C}^-, \tag{11}$$

and

$$0 \leqq \arg(\zeta - z) \leqq 2\pi \quad \text{for} \quad \mathcal{C}^+, \tag{12}$$

then

$$\mathcal{N}_z^{\nu}\{f(z)\} \qquad [\Re(\nu) > 0]$$

is said to be the *fractional derivative of* $f(z)$ *of order* $\nu$ and

$$\mathcal{N}_z^{\nu}\{f(z)\} \qquad [\Re(\nu) < 0]$$

is said to be the *fractional integral of* $f(z)$ *of order* $-\nu$, provided that

$$|\mathcal{N}_z^{\nu}\{f(z)\}| < \infty \qquad (\nu \in \mathbb{C}). \tag{13}$$

**Remark 1.** Throughout our present investigation, in case the differintegrated function $f(z)$ is a many-valued function, we shall tacitly consider the *principal value* of $f(z)$.

We choose to recall here the following potentially useful lemmas and properties associated with the fractional differintegration which is given by Definition 2 above (*cf.*, *e.g.*, [25] and [26]).

**Lemma 1**. (Linearity Property). *If the functions* $f(z)$ *and* $g(z)$ *are single-valued and analytic in some domain* $\Omega \subseteq \mathbb{C}$, *then*

$$\mathcal{N}_z^{\nu}\left\{k_1\, f(z) + k_2\, g(z)\right\} = k_1\, \mathcal{N}_z^{\nu}\{f(z)\} + k_2\, \mathcal{N}_z^{\nu}\{g(z)\} \quad (\nu \in \mathbb{C};\ z \in \Omega) \tag{14}$$

*for any constants* $k_1$ *and* $k_2$.

**Lemma 2**. (Index Law). *If the function $f(z)$ is single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$\mathcal{N}_z^{\nu}\big\{\mathcal{N}_z^{\mu}\{f(z)\}\big\} = \mathcal{N}_z^{\mu+\nu}\{f(z)\} = \mathcal{N}_z^{\mu}\big\{\mathcal{N}_z^{\nu}\{f(z)\}\big\} \tag{15}$$
$$\big(\mathcal{N}_z^{\mu}\{f(z)\} \neq 0;\ \mathcal{N}_z^{\nu}\{f(z)\} \neq 0;\ \mu, \nu \in \mathbb{C}\,;\ z \in \Omega\big).$$

**Lemma 3**. (Generalized Leibniz Rule). *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$\mathcal{N}_z^{\nu}\{f(z) \cdot g(z)\} = \sum_{n=0}^{\infty} \binom{\nu}{n} \mathcal{N}_z^{\nu-n}\{f(z)\} \cdot \mathcal{N}_z^{n}\{g(z)\} \qquad (\nu \in \mathbb{C}\,;\ z \in \Omega), \tag{16}$$

*where $\mathcal{N}_z^{n}\{g(z)\}$ is the ordinary derivative of $g(z)$ of order $n$* $\ (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$, *it being tacitly assumed* (*for simplicity*) *that $g(z)$ is the polynomial part* (*if any*) *of the product $f(z) \cdot g(z)$.*

**Property 1**. For a constant $\lambda$,

$$\mathcal{N}_z^{\nu}\big\{e^{\lambda z}\big\} = \lambda^{\nu}\, e^{\lambda z} \qquad (\lambda \neq 0;\ \nu \in \mathbb{C}\,;\ z \in \mathbb{C}) \tag{17}$$

and

$$\mathcal{N}_z^{\nu}\big\{e^{-\lambda z}\big\} = e^{-i\pi\nu}\, \lambda^{\nu}\, e^{-\lambda z} \qquad (\lambda \neq 0;\ \nu \in \mathbb{C}\,;\ z \in \mathbb{C}). \tag{18}$$

**Property 2**. For a constant $\lambda$,

$$\mathcal{N}_z^{\nu}\big\{z^{\lambda}\big\} = e^{-i\pi\nu}\, \frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\, z^{\lambda-\nu} \tag{19}$$
$$\left(\nu \in \mathbb{C}\,;\ z \in \mathbb{C} \setminus \{0\};\ \left|\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right).$$

**Property 3**. For a constant $\lambda$,

$$\mathcal{N}_z^{\nu}\big\{(z-c)^{\lambda}\big\} = e^{-i\pi\nu}\, \frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\, (z-c)^{\lambda-\nu} \tag{20}$$
$$\left(\nu \in \mathbb{C}\,;\ z \in \mathbb{C} \setminus \{c\};\ c \in \mathbb{C}\,;\ \left|\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right).$$

By suitably applying Property 3 in conjunction with the definition (3), it is not difficult to derive the following analogue of the fractional differintegral formula (5):

$$\begin{aligned}
&\mathcal{N}_z^{\lambda-\mu}\left\{z^{\lambda-1}\ \prod_{j=1}^{r}\left\{\left(1-a_j\ z^{\mu_j}\right)^{-\alpha_j}\right\}\right\}=\frac{\Gamma(1-\mu)}{\Gamma(1-\lambda)}\ e^{-i\pi(\lambda-\mu)}\ z^{\mu-1}\\
&\cdot F_{1:0;\ \cdots\ ;0}^{1:1;\ \cdots\ ;1}\left[\begin{array}{l}(\lambda;\mu_1,\cdots,\mu_r):(\alpha_1,1)\ ;\cdots;\ (\alpha_r,1)\,;\\ (\mu;\mu_1,\cdots,\mu_r):\text{———}\ ;\cdots;\text{———}\ ;\end{array}\ a_1\ z^{\mu_1},\cdots,a_r\ z^{\mu_r}\right]
\end{aligned} \tag{21}$$

$$\left(\left|\frac{\Gamma(1-\mu)}{\Gamma(1-\lambda)}\right|<\infty;\ \mu_j>0\ \ (j=1,\cdots,r);\ \max\left\{|a_1\ z^{\mu_1}|\,,\cdots,|a_r\ z^{\mu_r}|\right\}<1\right),$$

which may be compared with the fractional differintegral formula (7).

In a considerably large number of recent investigations (some of which have already been referred to above), several fractional differintegral formulas involving the fractional differintegral operators $\mathcal{D}_z^{\mu}$ $(\mu\in\mathbb{C})$ and $\mathcal{N}_z^{\nu}$ $(\nu\in\mathbb{C})$ were derived by applying such fractional differintegral formulas as those depicted above in (for example) Equations (2), (5), (7) and (21), Lemma 3, Property 2 and Property 3. Some of these fractional differintegral formulas were also shown to lead to several closely-related series identities (see, for details, [14, 15, 36] and [44]). Each of the fractional differintegral operators $\mathcal{D}_z^{\mu}$ $(\mu\in\mathbb{C})$ and $\mathcal{N}_z^{\nu}$ $(\nu\in\mathbb{C})$ has indeed been investigated and applied in many remarkably diverse fields. In the present sequel to some of these earlier works, we consider several miscellaneous properties and applications which are associated with each of the fractional differintegral operators $\mathcal{D}_z^{\mu}$ $(\mu\in\mathbb{C})$ and $\mathcal{N}_z^{\nu}$ $(\nu\in\mathbb{C})$. We first investigate, in a systematic and unified manner, various families of series identities which emerged in connection with some of the fractional differintegral formulas referred to above. We then use such operators of fractional calculus with a view to deriving a number of integral formulas as well as fractional differintegral formulas involving inverse hyperbolic functions.

## 2. Series Identities Derived by Means of the Fractional Differintegral Operator $\mathcal{N}_z^{\nu}$

The series identities (22) and (23) asserted by Theorems 1 and 2, respectively, were derived recently by Nishimoto [36] and Nishimoto *et al.* [44] by comparing different expressions which they obtained for the following fractional differintegrals:

$$\mathcal{N}_z^{\gamma}\left\{\log\left(\left[(z-b)^2-c\right]^2-d\right)\right\}\quad\text{and}\quad\mathcal{N}_z^{\gamma}\left\{\log\left(\left[\sqrt{z-b}-c\right]^2-d\right)\right\}.$$

**Theorem 1**. (*cf.* [36, p. 35, Theorem 4]). *The following series identity holds true:*

$$\begin{aligned}(22)\qquad &\sum_{\ell=0}^{\infty}\sum_{m=1}^{\infty}\frac{(2m)_\ell\,\Gamma(2\ell+4m+\gamma)}{m\cdot\ell!\,\Gamma(2\ell+4m)}\left(\frac{c}{(z-b)^2}\right)^\ell\left(\frac{d}{(z-b)^4}\right)^m\\ &=2\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+\gamma)}{(2\ell)!}\left[\left(\frac{c+\sqrt{d}}{(z-b)^2}\right)^\ell+\left(\frac{c-\sqrt{d}}{(z-b)^2}\right)^\ell-2\left(\frac{c}{(z-b)^2}\right)^\ell\right]\\ &\qquad\left(|z-b|^2>\max\left\{|c|,\sqrt{d}\right\};\ z\in\mathbb{C}\setminus\{b\}\right),\end{aligned}$$

*provided that each of the series involved is absolutely convergent.*

**Theorem 2**. (*cf.* [44, p. 22, Theorem 3]). *The following series identity holds true*:

$$\begin{aligned}(23)\qquad &\sum_{\ell=0}^{\infty}\sum_{m=1}^{\infty}\frac{(2m)_\ell\,\Gamma\left(\frac{1}{2}\,\ell+m+\gamma\right)}{m\cdot\ell!\,\Gamma\left(\frac{1}{2}\,\ell+m\right)}\left(\frac{c}{\sqrt{z-b}}\right)^\ell\left(\frac{d}{z-b}\right)^m\\ &=\frac{1}{2}\sum_{\ell=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\,\ell+\gamma\right)}{\Gamma\left(\frac{1}{2}\,\ell+1\right)}\left[\left(\frac{c+\sqrt{d}}{\sqrt{z-b}}\right)^\ell+\left(\frac{c-\sqrt{d}}{\sqrt{z-b}}\right)^\ell-2\left(\frac{c}{\sqrt{z-b}}\right)^\ell\right]\\ &\qquad\left(\sqrt{|z-b|}>\max\left\{|c|,\sqrt{d}\right\};\ z\in\mathbb{C}\setminus\{b\}\right),\end{aligned}$$

*provided that each of the series involved is absolutely convergent.*

**Remark 2.** In their statements of Theorems 1 and 2, respectively, Nishimoto [36, p. 35, Theorem 4] and Nishimoto *et al.* [44, p. 22, Theorem 3] also included the trivially obvious special cases of the assertions (22) and (23) when $\gamma=n\ \ (n\in\mathbb{N})$.

In the existing literature, there are remarkably many families of interesting (and potentially useful) series identities. We choose to present an important member of one of these families of series identities, which is due to Chen and Srivastava [2, p. 586, Equation (2.10)], as Theorem 3 below.

**Theorem 3.** *Let* $\{\Omega(n)\}_{n\in\mathbb{N}_0}$ *be a bounded sequence of complex numbers. Then*

$$(24)\qquad \sum_{\ell,m=0}^{\infty}\frac{\Omega(\ell+2m)}{(\mu)_m}\frac{x^\ell}{\ell!}\,\frac{y^{2m}}{m!}=\sum_{\ell,m=0}^{\infty}\Omega(\ell+m)\,\frac{\left(\mu-\frac{1}{2}\right)_m}{(2\mu-1)_m}\,\frac{(x+2y)^\ell}{\ell!}\,\frac{(-4y)^m}{m!},$$

*provided that each of the double series involved is absolutely convergent.*

By letting $\mu\to\frac{1}{2}$ and by setting $\mu=\frac{3}{2}$ in the assertion (24) of Theorem 3, and then making use of the following hypergeometric reduction formulas [50, p. 461, Entry 7.3.1.106]:

$$(25)\qquad {}_2F_1\left[\begin{array}{c}\lambda,\lambda+\frac{1}{2};\\ \frac{1}{2};\end{array}\; z\right]=\frac{1}{2}\left[\left(1+\sqrt{z}\,\right)^{-2\lambda}+\left(1-\sqrt{z}\,\right)^{-2\lambda}\right]$$

and [50, p. 461, Entry 7.3.1.107]:

$$(26)\qquad {}_2F_1\left[\begin{array}{c}\lambda,\lambda+\frac{1}{2};\\ \frac{3}{2};\end{array}\; z\right]=\frac{1}{2(2\lambda-1)\sqrt{z}}\left[\left(1-\sqrt{z}\,\right)^{1-2\lambda}-\left(1+\sqrt{z}\,\right)^{1-2\lambda}\right],$$

respectively, Lin *et al.* [15] derived two general series identities which are asserted by Theorem 4 below (see also a recent paper [64] dealing extensively and systematically with the theory and applications of such hypergeometric reduction formulas as (25) and (26) above).

**Theorem 4**. (*cf.* [15, p. 744, Theorem 3]). *Let* $\{\Omega(n)\}_{n\in\mathbb{N}_0}$ *be a bounded sequence of complex numbers. Then*

$$(27)\qquad \sum_{\ell,m=0}^{\infty}\Omega(\ell+2m)\,\frac{x^{\ell}}{\ell!}\,\frac{y^{2m}}{(2m)!}=\frac{1}{2}\sum_{\ell=0}^{\infty}\Omega(\ell)\,\left(\frac{(x+y)^{\ell}+(x-y)^{\ell}}{\ell!}\right)$$

*and*

$$(28)\qquad \sum_{\ell,m=0}^{\infty}\Omega(\ell+2m)\,\frac{x^{\ell}}{\ell!}\,\frac{y^{2m+1}}{(2m+1)!}=\frac{1}{2}\sum_{\ell=0}^{\infty}\Omega(\ell)\,\left(\frac{(x+y)^{\ell+1}-(x-y)^{\ell+1}}{(\ell+1)!}\right),$$

*provided that each of the double series involved in the assertions* (27) *and* (28) *is absolutely convergent.*

**Remark 3.** As already observed by (for example) Lin *et al.* [15, p. 745], the hypergeometric reduction formula (26) is an essentially *differentiated* version of the hypergeometric reduction formula (25), since it is easily verified that [51, p. 69, Exercise 1]

$$(29)\qquad \frac{d}{dz}\left\{{}_2F_1\left[\begin{array}{c}\lambda,\mu;\\ \nu;\end{array}\; z\right]\right\}=\frac{\lambda\mu}{\nu}\,{}_2F_1\left[\begin{array}{c}\lambda+1,\mu+1;\\ \nu+1;\end{array}\; z\right].$$

The details involved may be left as an exercise for the interested reader.

Upon suitable specializations of the complex sequence $\{\Omega(n)\}_{n\in\mathbb{N}_0}$, each of the assertions (27) and (28) of Theorem 4 would easily yield a remarkably large number of series identities including (for example) those in the class of series identities represented in Theorems 1 and 2. Here, in our present investigation, we derive the following unification and generalization of Theorems and 2 by applying only the first assertion (27) of Theorem 4. Analogous series identities can similarly be derived by appying the second assertion (28) of Theorem 4.

**Theorem 5.** *The following series identity holds true*:

$$\sum_{\ell=0}^{\infty}\sum_{m=1}^{\infty}\frac{\left(\frac{1}{2}\,\ell+m\right)\cdot(2m)_{\ell}\,\prod_{j=1}^{p}\Gamma\big(r_j(\ell+2m)+\gamma_j\big)}{m\cdot\ell!\,\prod_{j=1}^{q}\Gamma\big(s_j(\ell+2m)+\delta_j\big)}\,x^{\ell}\,y^{2m}$$

$$= \frac{1}{2}\sum_{\ell=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}\,\left[(x+y)^{\ell}+(x-y)^{\ell}-2x^{\ell}\right] \tag{30}$$

$$\big[r_j, s_k\in\mathbb{R}^+ \ \ (j=1,\cdots,p;\ k=1,\cdots,q);$$
$$\gamma_j\in\mathbb{C}\ \ (j=1,\cdots,p);\ \delta_j\in\mathbb{C}\setminus\mathbb{Z}_0^+\ \ (j=1,\cdots,q)\big],$$

*provided that each of the series involved is absolutely convergent.*

*Proof.* First of all, upon setting

$$\Omega(n)=\frac{n!\,\prod_{j=1}^{p}\Gamma\big(r_jn+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_jn+\delta_j\right)} \tag{31}$$

$$\big[n\in\mathbb{N}_0;\ r_j, s_k\in\mathbb{R}^+\ \ (j=1,\cdots,p;\ k=1,\cdots,q);$$
$$\gamma_j\in\mathbb{C}\ \ (j=1,\cdots,p);\ \ \delta_j\in\mathbb{C}\setminus\mathbb{Z}_0^+\ \ (j=1,\cdots,q)\big]$$

in the first assertion (27) of Theorem 4, we find that

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\ell+2m)!\,\prod_{j=1}^{p}\Gamma\big(r_j(\ell+2m)+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\big(s_j(\ell+2m)+\delta_j\big)}\,\frac{x^{\ell}}{\ell!}\,\frac{y^{2m}}{(2m)!}$$

$$=\frac{1}{2}\sum_{\ell=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}\,\left[(x+y)^{\ell}+(x-y)^{\ell}\right]. \tag{32}$$

We denote, for convenience, the first member of (32) by $\mathcal{S}(x,y)$ and observe from the definition (4) that

$$\frac{(\ell+2m)!}{\ell!\,(2m)!}=\frac{\Gamma(\ell+2m+1)}{\ell!\,\Gamma(2m+1)}=\frac{\left(\frac{1}{2}\,\ell+m\right)\cdot(2m)_{\ell}}{m\cdot\ell!}\qquad(\ell\in\mathbb{N}_0;\ m\in\mathbb{N}),$$

which readily yields

$$\begin{aligned}\mathcal{S}(x,y) &:= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\ell+2m)!\ \prod_{j=1}^{p}\Gamma\big(r_j(\ell+2m)+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\big(s_j(\ell+2m)+\delta_j\big)}\ \frac{x^\ell}{\ell!}\ \frac{y^{2m}}{(2m)!} \\
&= \sum_{\ell=0}^{\infty}\sum_{m=1}^{\infty} \frac{\left(\frac{1}{2}\ \ell+m\right)\cdot(2m)_\ell\ \prod_{j=1}^{p}\Gamma\big(r_j(\ell+2m)+\gamma_j\big)}{m\cdot \ell!\ \prod_{j=1}^{q}\Gamma\big(s_j(\ell+2m)+\delta_j\big)}\ x^\ell\ y^{2m} \\
&\quad + \sum_{\ell=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}\ x^\ell.\end{aligned} \tag{33}$$

If we substitute this last expression for $\mathcal{S}(x,y)$ from (33) for the first member of (32) and transpose the resulting single $\ell$-series to the right-hand side of (32), we shall arrive at the assertion (30) of Theorem 5 under the constraints stated already.

Alternatively, we can prove the assertion (30) of Theorem 5 *directly* by appealing to the hypergeometric reduction formula (25), that is, *without* using the assertion (27) of Theorem 4. Indeed, if in the left-hand side of (30), we set

$$\ell \mapsto \ell - 2m \qquad \left(0 \leqq m \leqq \left[\frac{\ell}{2}\right];\ \ell, m \in \mathbb{N}_0\right)$$

and make use of the following well-known series identity for a suitably bounded *double* sequence $\{\Lambda(\ell,m)\}_{\ell,m\in\mathbb{N}_0}$ (see, for example, [51, p. 57, Lemma 11 (7)] and [62, p. 100, Lemma 2 (3)]):

$$\sum_{\ell,m=0}^{\infty}\Lambda(\ell,m) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{[\ell/2]}\Lambda(\ell-2m,m), \tag{34}$$

we find for the first member of (27) that

$$\begin{aligned}&\Theta(x,y) \\
&:= \sum_{\ell=0}^{\infty}\sum_{m=1}^{\infty} \frac{\left(\frac{1}{2}\ \ell+m\right)\cdot(2m)_\ell\ \prod_{j=1}^{p}\Gamma\big(r_j(\ell+2m)+\gamma_j\big)}{m\cdot \ell!\ \prod_{j=1}^{q}\Gamma\big(s_j(\ell+2m)+\delta_j\big)}\ x^\ell\ y^{2m} \\
&= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\ell+2m)!\ \prod_{j=1}^{p}\Gamma\big(r_j(\ell+2m)+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\big(s_j(\ell+2m)+\delta_j\big)}\ \frac{x^\ell}{\ell!}\frac{y^{2m}}{(2m)!} - \sum_{\ell=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}\ x^\ell \\
&= \sum_{\ell=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}\ x^\ell\ {}_2F_1\left[\begin{matrix}-\frac{\ell}{2}, -\frac{\ell}{2}+\frac{1}{2}; \\ \frac{1}{2};\end{matrix}\ \frac{y^2}{x^2}\right] - \sum_{\ell=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}\ x^\ell,\end{aligned} \tag{35}$$

which, by applying the hypergeometric reduction formula (25), yields

$$
\begin{aligned}
&\Theta(x,y)\\
&= \frac{1}{2}\sum_{\ell=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}\left[(x+y)^{\ell}+(x-y)^{\ell}\right]-\sum_{\ell=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}x^{\ell}\\
&= \frac{1}{2}\sum_{\ell=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\big(r_j\ell+\gamma_j\big)}{\prod_{j=1}^{q}\Gamma\left(s_j\ell+\delta_j\right)}\left[(x+y)^{\ell}+(x-y)^{\ell}-2x^{\ell}\right],
\end{aligned}
\tag{36}
$$

which evidently completes our *alternative* demontration of (30) under the hypotheses stated already with Theorem 5. ∎

**Remark 4.** The assertion (22) of Theorem 1 would follow readily upon specializing Theorem 5 by setting

$$
p=q=1,\ r_1=s_1=2\quad(\gamma_1=\gamma;\ \delta_1=1),\quad x=\frac{c}{(z-b)^2}\quad\text{and}\quad y=\frac{\sqrt{d}}{(z-b)^2}
$$
$$
\left(|z-b|^2>\max\left\{|c|,\sqrt{d}\right\};\ z\in\mathbb{C}\setminus\{b\}\right).
$$

Theorem 2, on the other hand, is a special case of Theorem 5 when

$$
p=q=1,\ r_1=s_1=\frac{1}{2}\quad(\gamma_1=\gamma;\ \delta_1=1),\quad x=\frac{c}{\sqrt{z-b}}\quad\text{and}\quad y=\frac{\sqrt{d}}{\sqrt{z-b}}
$$
$$
\left(\sqrt{|z-b|}>\max\left\{|c|,\sqrt{d}\right\};\ z\in\mathbb{C}\setminus\{b\}\right).
$$

**Remark 5.** Just as we have demonstrated above in our alternative proof of Theorem 5, each of Theorems 1 and 2 can be proven *directly* by applying the hypergeometric reduction formula (25). The details involved are being left as an exercise for the interested reader.

## 3. Evaluation of Integrals by Using Fractional Calculus

In this section, we propose to illustrate how several elementary integrals can be evaluated by using fractional calculus.

**Example 1.** Consider the following elementary integral:

$$
\mathcal{I}_1 := \int^{z} z^{\kappa}\left(az^{\rho}+b\right)^{\sigma}\, dz \tag{37}
$$

where (and in what follows) the *lower* terminal of the integral can be specified appropriately. By suitably applying the fractional differintegral formula (5) with

$$r = 1 \qquad \text{and} \qquad \lambda = \mu - 1 = \kappa + 1 \qquad \left(a_1 = -\frac{a}{b};\ \alpha_1 = -\delta;\ \mu_1 = \rho\right),$$

it is not difficult to find from (37) that

$$\mathcal{I}_1 = \frac{b^\sigma\ z^{\kappa+1}}{\kappa+1}\ {}_2\Psi_1^*\left[(\kappa+1,\rho),(-\sigma,1);(\kappa+2,\rho);-\frac{a}{b}\ z^\rho\right] \tag{38}$$
$$[\Re(\kappa) > -1;\ \rho > 0;\ b \neq 0],$$

in terms of the Fox-Wright function ${}_p\Psi_q^*$ defined by the following special case of the Srivastava-Daoust multivariable hypergeometric function (3) when $r = 1$ (*cf.* [5, 66] and [67]; see also [3, p. 183] and [60, p. 21])

$$\begin{aligned} &{}_p\Psi_q^*\left[(a_1,A_1),\ \cdots,(a_p,A_p);(b_1,B_1),\ \cdots,(b_q,B_q);z\right] \\ &:= \sum_{k=0}^\infty \frac{(a_1)_{A_1k}\ \cdots (a_p)_{A_pk}}{(b_1)_{B_1k}\ \cdots (b_q)_{B_qk}}\ \frac{z^k}{k!} \\ &= \frac{\Gamma(b_1)\ \cdots \Gamma(b_q)}{\Gamma(a_1)\ \cdots \Gamma(a_p)}\ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ \cdots,(a_p,A_p);\\ \\ (b_1,B_1),\ \cdots,(b_q,B_q);\end{array} z\right] \end{aligned} \tag{39}$$

$$\left(A_j > 0\quad (j=1,\ \cdots,p);\ B_j > 0\quad (j=1,\ \cdots,q);\ 1+\sum_{j=1}^q B_j - \sum_{j=1}^p A_j \geqq 0\right),$$

where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \left(\prod_{j=1}^p A_j^{-A_j}\right)\cdot\left(\prod_{j=1}^q B_j^{B_j}\right).$$

Clearly, we have (see, for details, [59, p. 19])

$$\begin{aligned} &{}_p\Psi_q^*\left[(a_1,1),\ \cdots,(a_p,1);(b_1,1),\ \cdots,(b_q,1);z\right] \\ &= {}_pF_q\left[\begin{array}{c}a_1,\ \cdots,a_p;\\ b_1,\ \cdots,b_q;\end{array} z\right] = \frac{\Gamma(b_1)\ \cdots \Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\ {}_p\Psi_q\left[\begin{array}{c}(a_1,1),\cdots,(a_p,1);\\ (b_1,1),\cdots,(b_q,1);\end{array} z\right] \end{aligned} \tag{40}$$

and (see, for details, [6])

$$\begin{aligned} &{}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ \cdots,(a_p,A_p);\\ (b_1,B_1),\ \cdots,(b_q,B_q);\end{array} z\right] \\ &= H_{p,q+1}^{1,p}\left[-z\ \left|\ \begin{array}{c}(1-a_1,A_1),\ \cdots,(1-a_p,A_p)\\ (0,1),(1-b_1,B_1),\ \cdots,(1-b_q,B_q)\end{array}\right.\right] \end{aligned} \tag{41}$$

in terms of the familiar and widely-investigated $F$ and $H$ functions, respectively (see also [61]).

**Remark 6.** By using the definition (39) and such relationships as (40), we can derive a number of simpler cases of the integral formula (38). In particular, when $\sigma = n \;\; (n \in \mathbb{N}_0)$, we find from (38) that [49, p. 27, Entry 1.2.2.7]

$$\int^z z^\kappa \, (az^\rho + b)^n \;\, dz = \sum_{k=0}^{n} \binom{n}{k} \frac{a^k \; b^{n-k} z^{\kappa+\rho k+1}}{\kappa + \rho k + 1} \qquad [\, n \in \mathbb{N}_0; \; \Re(\kappa) > -1; \; \rho > 0] \,. \tag{42}$$

**Example 2.** By simple changes of the variable $w$ of integration, the problem of evaluation of such integrals as

$$\int^w \log(\log w) \; dw, \quad \int^w \left(\frac{1}{\log w}\right) \; dw \quad \text{and} \quad \int^w e^{e^w} \; dw$$

can be reduced to that of the evaluation of integrals of the following class:

$$\mathcal{I}_2 := \int^z z^\kappa \; e^{az^\rho} \; dz = \lim_{n\to\infty} \int^z z^\kappa \left(1 + \frac{az^\rho}{n}\right)^n \; dz, \tag{43}$$

which, upon evaluation by means of the fractional differintegral formula (5) with

$$r = 1 \quad \text{and} \quad \lambda = \mu - 1 = \kappa + 1 \qquad \left(a_1 = -\frac{a}{b}; \; \alpha_1 = -\delta; \; \mu_1 = \rho\right),$$

yields

$$\mathcal{I}_2 = \frac{z^{\kappa+1}}{\kappa+1} \lim_{n\to\infty} \; {}_2\Psi_1^* \left[(\kappa+1, \rho), (-n, 1); (\kappa+2, \rho); -\frac{a}{n} \; z^\rho\right] \qquad [\Re(\kappa) > -1; \; \rho > 0] \,. \tag{44}$$

Since

$$\frac{(-n)_k}{(-n)^k} = \frac{n(n-1)\cdots(n-k+1)}{n^k} \to 1 \quad \text{as} \quad n \to \infty \qquad (k = 0, 1, \cdots, n),$$

we find from (44) that

$$\begin{aligned} \int^z z^\kappa \; e^{az^\rho} \; dz &= \frac{z^{\kappa+1}}{\kappa+1} \;\; {}_1\Psi_1^* \left[(\kappa+1, \rho); (\kappa+2, \rho); az^\rho\right] \\ &= \sum_{k=0}^{\infty} \frac{a^k \;\, z^{\kappa+\rho k+1}}{(\kappa + \rho k + 1) \cdot k!} \qquad [\Re(\kappa) > -1; \; \rho > 0] \,. \end{aligned} \tag{45}$$

**Remark 7.** In terms of the *incomplete Gamma function* $\gamma(z,\alpha)$ defined by

$$\gamma(z,\alpha) := \int_0^\alpha t^{z-1}\ e^{-t}\ dt = \frac{z^\alpha}{\alpha}\ {}_1F_1(\alpha;\alpha+1;-z) \tag{46}$$
$$\left[\Re(z)>0;\ |\arg(\alpha)| \leqq \pi-\epsilon\ \ (0<\epsilon<\pi)\right],$$

a *special* case of the integral formula (45) when $\rho=1$ and $a=-\alpha$ can be put in the following form [49, p. 137, Entry 1.3.2.3]:

$$\int_0^z z^\kappa\ e^{-\alpha z}\ dz = \frac{1}{\alpha^{\kappa+1}}\ \gamma(\kappa+1,\alpha z) \qquad [\Re(\kappa)>-1]. \tag{47}$$

**Remark 8.** In an obviously *exceptional* case of the integral $\mathcal{I}_2$ in (43) when $\kappa=-n\ \ (n\in\mathbb{N})$, if we *further* set $\rho=1$, we can apply Lemma 3 with

$$\nu=-1,\ \ f(z)=z^{-n}\ \ (n\in\mathbb{N}) \quad \text{and} \quad g(z)=e^{az},$$

so that

$$\int^z \frac{e^{az}}{z^n}\ dz = \sum_{k=0}^\infty \binom{-1}{k} \mathcal{N}_z^{-1-k}\left\{z^{-n}\right\}\cdot \frac{d^k}{dz^k}\{e^{az}\} \qquad (n\in\mathbb{N}). \tag{48}$$

Since, by definition,

$$\binom{-1}{k} = \frac{(-1)(-2)(-3)\cdots(-1-k+1)}{k!} = (-1)^k \qquad (k\in\mathbb{N}_0),$$

by separating the infinite sum into three parts as follows, we find that

$$\begin{aligned}\int^z \frac{e^{az}}{z^n}\ dz &= \frac{e^{az}}{az^n} + e^{az}\left(\sum_{k=0}^{n-1}(-a)^{k-1}\ \mathcal{N}_z^{-k}\left\{z^{-n}\right\}\right.\\ &\qquad \left. + \sum_{k=n}^{\infty}(-a)^{k-1}\ \mathcal{N}_z^{-k}\left\{z^{-n}\right\}\right)\\ &= \frac{e^{az}}{az^n} + e^{az}\left(\sum_{k=0}^{n-1}(-a)^{k-1}\ \mathcal{N}_z^{-k}\left\{z^{-n}\right\}\right.\\ &\qquad \left. + \sum_{k=0}^{\infty}(-a)^{n+k-1}\ \mathcal{N}_z^{-n-k}\left\{z^{-n}\right\}\right) \qquad (n\in\mathbb{N}).\end{aligned} \tag{49}$$

If we set

$$\lambda=-n \quad \text{and} \quad \nu=-k$$

in Property 2, we get

$$\begin{aligned}\mathcal{N}_z^{-k}\left\{z^{-n}\right\} &= e^{i\pi k}\ \frac{\Gamma(n-k)}{\Gamma(n)}\ z^{k-n}\\ &= (-1)^k\ \frac{(n-k-1)!}{(n-1)!}\ z^{k-n}\\ &\qquad (k\in\{0,1,2,\cdots,n-1\};\ n\in\mathbb{N}).\end{aligned} \tag{50}$$

We now recall the following easy consequence of Lemma 3 and Property 2 [26, p. 69, Equation (27)]:

$$\mathcal{N}_z^{-n}\{\log z\} = \frac{z^n}{n!}\left[\log z - \sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}}{n-k}\binom{n}{k}\right] \qquad (n \in \mathbb{N}), \tag{51}$$

which, by means of the familiar combinatorial identity (see, for example, [8, p. 6, Entry (1.45)]):

$$\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}}{n-k}\binom{n}{k} = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k} = \sum_{k=1}^{n}\frac{1}{k} \qquad (n \in \mathbb{N}), \tag{52}$$

assumes a remarkably simpler form given by

$$\mathcal{N}_z^{-n}\{\log z\} = \frac{z^n}{n!}\left(\log z - H_n\right) \qquad (n \in \mathbb{N}_0), \tag{53}$$

where $H_n$ denotes the harmonic numbers defined by

$$H_n := \sum_{k=1}^{n}\frac{1}{k} \qquad \left(n \in \mathbb{N};\ H_0 := 0\right). \tag{54}$$

Next, by operating upon both sides of the fractional differintegral formula [26, p. 51, Equation (12)]:

$$\mathcal{N}_z^{-\nu}\left\{z^{-\nu}\right\} = -\frac{e^{i\pi\nu}}{\Gamma(\nu)}\ \log z \qquad \Big(\nu \in \mathbb{C};\ |\Gamma(\nu)| < \infty\Big) \tag{55}$$

by $\mathcal{N}_z^{-n}$ and applying the result (53), we find that

$$\begin{aligned}\mathcal{N}_z^{-\nu-n}\left\{z^{-\nu}\right\} &= -\frac{e^{i\pi\nu}}{\Gamma(\nu)}\,\mathcal{N}_z^{-n}\{\log z\}\\ &= -\frac{e^{i\pi\nu}}{\Gamma(\nu)}\,\frac{z^n}{n!}\left(\log z - H_n\right) \qquad \Big(\nu\in\mathbb{C};\ |\Gamma(\nu)| < \infty;\ n\in\mathbb{N}\Big).\end{aligned} \tag{56}$$

Finally, upon setting

$$\nu = m \quad \text{and} \quad n \mapsto n - m \qquad (m \leqq n;\ m, n \in \mathbb{N})$$

in this last result (56), we obtain

$$\mathcal{N}_z^{-n}\left\{z^{-m}\right\} = -\frac{e^{i\pi m}}{(m-1)!}\,\frac{z^{n-m}}{(n-m)!}\left(\log z - H_{n-m}\right) \qquad (m \leqq n;\ m, n \in \mathbb{N}) \tag{57}$$

in terms of the harmonic numbers $H_n$ defined by (54). An obviously erroneous version of the fractional differintegral formula (57) was presented and applied in a recent paper by Salinas de Romero *et al.* [52, p. 57, Equation (8)].

By appropriately applying the fractional differintegral formulas (50) and (57) to the last member of (49), we are led eventually to the following integral formula:

$$\begin{aligned}\int^z \frac{e^{az}}{z^n}\,dz &= \frac{e^{az}}{az^n} - \frac{e^{az}}{(n-1)!}\left(\sum_{k=0}^{n-1} \frac{(n-k-1)!\cdot a^{k-1}}{z^{n-k}}\right.\\ &\qquad \left. - \sum_{k=0}^{\infty} (\log z - H_k)\,\frac{a^{n+k-1}\,(-z)^k}{k!}\right) \qquad (n \in \mathbb{N}),\end{aligned} \tag{58}$$

which, for $n = 1$, yields

$$\int^z \frac{e^{az}}{z}\,dz = e^{az}\left(\log z + \sum_{k=1}^{\infty} (\log z - H_k)\,\frac{(-az)^k}{k!}\right) \tag{59}$$

in terms of the harmonic numbers $H_n$ defined by (54).

By means of a known relationship [10, p. 363, Entry (55.7.2)] of the infinite sums in (58) and (59) with the exponential integral $\mathrm{Ei}(z)$ (see also [16]), the integral formulas (58) and (59) can be shown to be essentially the same as the known results [49, p. 138, Entries 1.3.2.11 and 1.3.2.12].

**Example 3.** Consider the following integral:

$$\mathcal{I}_3 := \int^z z^{\kappa}\,\sin z\,dz, \tag{60}$$

which can be evaluated, for different constraints upon the parameter $\kappa$, by applying the method (using Lemma 3 and Property 2) *mutatis mutandis*. We thus find that

$$\int^z z^{\kappa}\,\sin z\,dz = z^{\kappa+1}\sum_{k=0}^{\infty} \frac{(-z)^k}{(\kappa+1)_{k+1}}\,\sin\left(z + \frac{k\pi}{2}\right) \qquad [\Re(\kappa) > -1], \tag{61}$$

$$\int^z z^n\,\sin z\,dz = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k!\cdot z^{n-k}\,\sin\left(z - \frac{(k+1)\pi}{2}\right) \qquad (n \in \mathbb{N}_0) \tag{62}$$

and

$$\begin{aligned}\int^z \frac{\sin z}{z^n}\,dz &= -\frac{\cos z}{z^n} - \frac{1}{(n-1)!}\left[\sum_{k=0}^{n-1} \frac{(n-k-1)!}{z^{n-k}}\,\cos\left(z + \frac{k\pi}{2}\right)\right.\\ &\qquad \left. - \sum_{k=0}^{\infty} \frac{(-z)^k}{k!}\,(\log z - H_k)\cos\left(z + \frac{(n+k)\pi}{2}\right)\right] \qquad (n \in \mathbb{N}),\end{aligned} \tag{63}$$

each of which may be compared with the corresponding known integral formulas recorded by (for example) Prudnikov *et al.* [49], $H_n$ being the harmonic numbers defined already by (54).

**Remark 9.** In its special case when $n = 1$, this last integral formula (63) reduces to the following interesting form:

$$\int^z \frac{\sin z}{z}\, dz = -\sum_{k=0}^{\infty} \frac{(-z)^k}{k!}\ (\log z - H_k)\cos\left(z + \frac{(k+1)\pi}{2}\right). \tag{64}$$

## 4. Applications of Fractional Calculus Involving Inverse Hyperbolic Functions

In a recent paper, Prieto *et al.* [48] considered several applications of the fractional differintegral operator $\mathcal{N}_z^\nu$ involving the inverse trigonometric functions $\sin^{-1} z$ and $\cos^{-1} z$. Salinas de Romero *et al.* [52], on the other hand, considered analogous applications involving the inverse hyperbolic functions $\sinh^{-1} z$ and $\cosh^{-1} z$. Here, in the present section, we show how such fractional differintegral formulas as (for example) (2) or (5) can be applied to derive, in a relatively simpler manner, several $n$-fold integral formulas for various inverse hyperbolic functions.

**Theorem 6.** *Each of the following $n$-fold integral formulas holds true for the inverse hyperbolic functions involved*:

$$\begin{aligned}&\mathcal{D}_z^{-n}\left\{\sinh^{-1} z\right\}\\ &= \frac{z^{n+1}}{(n+1)!}\ {}_3F_2\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{2}\,n+1,\frac{1}{2}\,n+\frac{3}{2};-z^2\right) \qquad (n\in\mathbb{N};\ |z|<1),\end{aligned} \tag{65}$$

$$\begin{aligned}&\mathcal{D}_z^{-n}\left\{\operatorname{sech}^{-1} z\right\}\\ &= \frac{z^n}{n!}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\ {}_3F_2\left(-\frac{k}{2},\frac{1}{2},1;\frac{1}{2}\,n+\frac{1}{2},\frac{1}{2}\,n+1;z^2\right)-\log z+H_n\right],\\ &(n\in\mathbb{N};\ |z|<1)\end{aligned} \tag{66}$$

$$\begin{aligned}&\mathcal{D}_z^{-n}\left\{\operatorname{csch}^{-1} z\right\}\\ &= \frac{z^n}{n!}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\ {}_3F_2\left(-\frac{k}{2},\frac{1}{2},1;\frac{1}{2}\,n+\frac{1}{2},\frac{1}{2}\,n+1;-z^2\right)-\log z+H_n\right]\\ &(n\in\mathbb{N};\ |z|<1),\end{aligned} \tag{67}$$

*and*

$$\begin{aligned}&\mathcal{D}_z^{-n}\left\{\tanh^{-1} z\right\} \\ &= \frac{z^{n+1}}{(n+1)!}\ {}_3F_2\left(\frac{1}{2},1,1;\frac{1}{2}\,n+1,\frac{1}{2}\,n+\frac{3}{2};z^2\right) \qquad (n\in\mathbb{N};\ |z|<1),\end{aligned} \tag{68}$$

*where $H_n$ denotes the harmonic numbers defined by* $(54)$.

*Proof.* For the inverse hyperbolic function $\sinh^{-1} z$, it is fairly well known that

$$\begin{aligned}\sinh^{-1} z &= \log\left(z+\sqrt{1+z^2}\right) \\ &= \mathcal{D}^{-1}\left\{\left(1+z^2\right)^{-\frac{1}{2}}\right\} \\ &= z\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-z^2\right) \qquad (|z|<1).\end{aligned} \tag{69}$$

In light of the fractional differintegral formula (2) or (5) (with $\lambda=\mu-n=1$) and the Legendre duplication formula for the Gamma function:

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right), \tag{70}$$

the first assertion (55) of Theorem 6 would follow easily when we apply the operator $\mathcal{D}_z^{-n}$ upon the second (or, alternatively, the third) member of (69).

Next, for the inverse hyperbolic function $\operatorname{sech}^{-1}z$, we have

$$\begin{aligned}\operatorname{sech}^{-1}z &= \log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = \log\left(1+\sqrt{1-z^2}\right)-\log z \\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\ \left(1-z^2\right)^{\frac{k}{2}}-\log z,\end{aligned} \tag{71}$$

which, in view of (5) in conjunction with the following easily derivable fractional differintegral formula by the principle of mathematical induction on $n\in\mathbb{N}$ [see also Equation (53)]:

$$\mathcal{D}_z^{-n}\{\log z\}=\frac{z^n}{n!}\ (\log z-H_n) \qquad (n\in\mathbb{N}), \tag{72}$$

readily yields the assertion (65) of Theorem 6. Formula (72) is, in fact, an obvious special case of the following known result (*cf.*, *e.g.*, [4, p. 188, Entry 13.1(24)]):

$$\begin{aligned}\mathcal{D}_z^{\mu}\left\{z^{\lambda}\ \log z\right\} &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\ z^{\lambda-\mu}\left[\log z+\psi(\lambda+1)-\psi(\lambda-\mu+1)\right] \\ &\qquad [\Re(\lambda)>-1;\ \mu\in\mathbb{C}]\end{aligned} \tag{73}$$

when

$$\lambda = 0 \quad \text{and} \quad \mu = -n \qquad (n \in \mathbb{N}),$$

since

$$\psi(n+1) - \psi(1) = H_n \qquad (n \in \mathbb{N}_0), \tag{74}$$

in terms of the Harmonic numbers $H_n$ given by (54), $\psi(z)$ being the Psi (or Digamma) function defined by

$$\psi(z) := \frac{d}{dz}\{\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(\tau)\, d\tau. \tag{75}$$

In order to prove the assertion (67) of Theorem 6, we observe from (69) that

$$\begin{aligned} \operatorname{csch}^{-1} z &= \sinh^{-1}\left(\frac{1}{z}\right) = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right) = \log\left(1+\sqrt{1+z^2}\right) - \log z \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(1+z^2\right)^{\frac{k}{2}} - \log z, \end{aligned} \tag{76}$$

which, just as in our demonstration of (65), leads us easily to (67) by means of the fractional differintegral formulas (5) and (72).

Finally, we can similary prove the assertion (68) of Theorem 6 by noting that

$$\tanh^{-1} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) = z \; {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right). \tag{77}$$

The details involved may be left as an exercise for the interested reader. ■

## 4. Further Remarks and Observations

In this concluding section, we present the following general fractional differintegral formulas involving the operator $\mathcal{N}_z^{\nu}$, which are proven by appealing appropriately to such fractional differintegral formulas as those depicted in (for example) Lemma 3, Property 2 and Property 3 of Section 1 (*cf.* [19, p. 53, Equations (3.6) and (3.7)] *and* [17, p. 92, Equation (2.4)]; see also [15, p. 740, Theorem 1]):

$$\begin{aligned} &\mathcal{N}_z^{\nu}\left\{\left[(z-\alpha)^{\mu} - \zeta\right]^{\kappa}\right\} \\ &= \frac{e^{-i\pi\nu}\,(z-\alpha)^{\kappa\mu-\nu}}{\Gamma(-\kappa)}\; {}_2\Psi_1\left[(\nu-\kappa\mu, \mu), (-\kappa, 1);\; (-\kappa\mu, \mu); \frac{\zeta}{(z-\alpha)^{\mu}}\right], \end{aligned} \tag{78}$$

$$\mathcal{N}_z^{\nu}\left\{\left(z^{\mu} - \zeta\right)^{\kappa}\right\} = \frac{e^{-i\pi\nu}\, z^{\kappa\mu-\nu}}{\Gamma(-\kappa)}\; {}_2\Psi_1\left[(\nu-\kappa\mu, \mu), (-\kappa, 1);\; (-\kappa\mu, \mu); \frac{\zeta}{z^{\mu}}\right] \tag{79}$$

*and*

$$
\begin{aligned}
&\mathcal{N}_z^{\nu}\left\{\frac{(\alpha z+\beta)^{\rho}}{[(\lambda z+\mu)^{\sigma}-\zeta]^{\kappa}}\right\} \\
&= e^{-i\pi\nu}\,\frac{\lambda^{\nu}}{\Gamma(\kappa)}\,\frac{(\alpha z+\beta)^{\rho}}{(\lambda z+\mu)^{\nu+\kappa\sigma}}\,\sum_{\ell=0}^{\infty}\binom{\nu}{\ell}\binom{\rho}{\ell}\cdot \ell!\left(-\frac{\alpha(\lambda z+\mu)}{\lambda(\alpha z+\beta)}\right)^{\ell} \\
&\quad \cdot {}_2\Psi_1\left[(\nu-\ell+\kappa\sigma,\sigma),(\kappa,1);\ (\kappa\sigma,\sigma);\frac{\zeta}{(\lambda z+\mu)^{\sigma}}\right] \\
&\qquad \left(\alpha\neq 0;\ \lambda\neq 0;\ \sigma\in\mathbb{R}^{+};\ \left|\frac{\zeta}{(\lambda z+\mu)^{\sigma}}\right|<1\right),
\end{aligned}
\tag{80}
$$

it being provided that both sides of each of the assertions (78), (79) and (80) exist.

The fractional differintegral formula (86) is equivalent to a result proven recently by Nishimoto [28, p. 37, Theorem 1 (i)]. The fractional differintegral formula (79), on the other hand, can easily be rewritten in the following form by appealing to the relationship (41) with the Fox $H$-function:

$$
\mathcal{N}_z^{\nu}\left\{\left(z^{\mu}-\zeta\right)^{\kappa}\right\}=\frac{e^{-i\pi\nu}\ z^{\kappa\mu-\nu}}{\Gamma(-\kappa)}\ H_{2,2}^{1,2}\left[-\frac{\zeta}{z^{\mu}}\ \middle|\ \begin{matrix}(1+\kappa\mu-\nu,\mu),(1+\kappa,1)\\(0,1),(1+\kappa\mu,\mu)\end{matrix}\right], \tag{81}
$$

provided that each member of (81) exists.

In light of the following rather elementary property of the Fox $H$-function involved in (41) and (81) (see [59, p. 15, Equation (2.3.6)]):

$$
\begin{aligned}
&z^{\sigma}\ H_{p,q}^{m,n}\left[z\ \middle|\ \begin{matrix}(a_1,A_1),\ \cdots,(a_p,A_p)\\(b_1,B_1),\ \cdots,(b_q,B_q)\end{matrix}\right] \\
&= H_{p,q}^{m,n}\left[z\ \middle|\ \begin{matrix}(a_1+\sigma A_1,A_1),\ \cdots,(a_p+\sigma A_p,A_p)\\(b_1+\sigma B_1,B_1),\ \cdots,(b_q+\sigma B_q,B_q)\end{matrix}\right] \qquad (\sigma\in\mathbb{C}),
\end{aligned}
\tag{82}
$$

we can at once rewrite the fractional differintegral formula (81) in its *equivalent* form:

$$
\begin{aligned}
&\mathcal{N}_z^{\nu}\left\{\left(z^{\mu}-\zeta\right)^{\kappa}\right\} \\
&= \frac{e^{-i\pi\nu}\ z^{\kappa-\nu}\ (-\zeta)^{\kappa-(\kappa/\mu)}}{\Gamma(-\kappa)}\ H_{2,2}^{1,2}\left[-\frac{\zeta}{z^{\mu}}\ \middle|\ \begin{matrix}(1+\kappa-\nu,\mu),\left(1+\dfrac{\kappa}{\mu},1\right)\\\left(\dfrac{\kappa}{\mu}-\kappa,1\right),(1+\kappa,\mu)\end{matrix}\right],
\end{aligned}
\tag{83}
$$

which, in fact, corresponds *precisely* to one of the *main* results of Saxena and Nishimoto [54, p. 59, Theorem 1] (see also [19] and [55]).

Upon setting

$$\rho=1,\quad \lambda=\alpha \qquad \text{and} \qquad \mu=\beta,$$

the fractional differintegral formula (80) can be reduced fairly easily to the following remarkably simple form:

$$\mathcal{N}_z^{\nu}\left\{\frac{\alpha z+\beta}{[(\alpha z+\beta)^{\sigma}-\zeta]^{\kappa}}\right\}=e^{-i\pi\nu}\,\frac{\alpha^{\nu}}{(\alpha z+\beta)^{\nu+\kappa\sigma-1}}$$

$$\cdot\ {}_2\Psi_1\left[(\nu+\kappa\sigma-1,\sigma),(\kappa,1);\ (\kappa\sigma-1,\sigma);\frac{\zeta}{(\alpha z+\beta)^{\sigma}}\right] \tag{84}$$

$$\left(\alpha\neq 0;\ \sigma\in\mathbb{R}^+;\ \left|\frac{\zeta}{(\alpha z+\beta)^{\sigma}}\right|<1\right).$$

Moreover, in its *further* special case when

$$\kappa=\sigma-1=1,\quad \alpha=a,\quad \beta=b\qquad\text{and}\qquad \zeta=b^2-ac,$$

the fractional differintegral formula (84) would readily yield

$$\mathcal{N}_z^{\nu}\left\{\frac{az+b}{az^2+2bz+c}\right\}$$

$$=e^{-i\pi\nu}\left(\frac{a}{az+b}\right)^{\nu+1}\ {}_2\Psi_1\left[(\nu+1,2),(1,1);\ (1,2);\frac{b^2-ac}{(az+b)^2}\right] \tag{85}$$

$$\left(az^2+2bz+c\neq 0;\ a\neq 0;\ |\Gamma(\nu+1)|<\infty;\ \left|\frac{b^2-ac}{(az+b)^2}\right|<1\right),$$

which can easily be rewritten in the following *equivalent* forms:

$$\mathcal{N}_z^{\nu}\left\{\frac{az+b}{az^2+2bz+c}\right\}$$

$$=e^{-i\pi\nu}\,\Gamma(\nu+1)\left(\frac{a}{az+b}\right)^{\nu+1}\sum_{m=0}^{\infty}\binom{\nu+2m}{2m}\left(\frac{b^2-ac}{(az+b)^2}\right)^{m} \tag{86}$$

$$\left(az^2+2bz+c\neq 0;\ a\neq 0;\ |\Gamma(\nu+1)|<\infty;\ \left|\frac{b^2-ac}{(az+b)^2}\right|<1\right)$$

and

$$\mathcal{N}_z^{\nu}\left\{\frac{az+b}{az^2+2bz+c}\right\}$$

$$=e^{-i\pi\nu}\left(\frac{a}{az+b}\right)^{\nu+1}\ {}_2F_1\left[\begin{array}{r}\frac{1}{2}\,\nu+\frac{1}{2},\frac{1}{2}\,\nu+1;\\ \frac{1}{2};\end{array}\ \frac{b^2-ac}{(az+b)^2}\right] \tag{87}$$

$$\left(az^2+2bz+c\neq 0;\ a\neq 0;\ |\Gamma(\nu+1)|<\infty;\ \left|\frac{b^2-ac}{(az+b)^2}\right|<1\right).$$

The fractional differintegral formula (73) happens to be one of the *main* results proven recently by Nishimoto and Miyakoda [41, p. 56, Theorem 2(i)]. More interestingly, by appealing appropriately to hypergeometric reduction formula (25), we can deduce a remarkably simpler (closed-form) version of the fractional differintegral formula (87) given by

$$\begin{aligned}\mathcal{N}_z^{\nu}\left\{\frac{az+b}{az^2+2bz+c}\right\} &= \frac{1}{2}\, e^{-i\pi\nu}\left(\frac{a}{az+b}\right)^{\nu+1} \\ &\cdot \left[\left(1+\frac{\sqrt{b^2-ac}}{az+b}\right)^{-\nu-1} + \left(1-\frac{\sqrt{b^2-ac}}{az+b}\right)^{-\nu-1}\right] \\ &\left(az^2+2bz+c \neq 0;\ a \neq 0;\ |\Gamma(\nu+1)|<\infty;\ \left|\frac{b^2-ac}{(az+b)^2}\right|<1\right).\end{aligned} \tag{88}$$

Numerous obvious *further* special cases and applications of each of the fractional differintegral formulas (78), (79) and (80), as well as of their aforementioned and other corollaries and consequences (see also [15]), can be found to be derived in a considerably large number of recent investigations which appeared and continue to appear in the *Journal of Fractional Calculus* (see, for details, [7, 22-24, 32-35, 39] and [42]; see also many of the closely-related earlier as well as more recent references cited in our bibliography). For instance, Miyakoda [22] considered an obvious special case of the fractional differintegral formula (78) when

$$\mu = m \qquad \text{and} \qquad \kappa = \pm\,\frac{1}{n} \qquad (m, n \in \mathbb{N}).$$

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