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$\mathbb{T}$-EPIDERIVATIVES OF SET-VALUED MAPS AND ITS APPLICATION TO SET OPTIMIZATION AND GENERALIZED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we first define a $\mathbb{T}$-cone which is a unified version of several cones, namely, contingent cone, radial cone, C-tangent cone, Clarke tangent cone, S-cone, adjacent cone, etc. Then, we define the $\mathbb{T}$-epiderivative of a set-valued map which includes the contingent epiderivative, radial epiderivative, S-epiderivative, adjacent epiderivative etc. as special cases. We present several properties of such an epiderivative. The generalized vector $\mathbb{T}$-variational inequality problem is also considered. We provide necessary and sufficient conditions for a solution of a set optimization problem. Several existence results for solutions of set optimization problems and a generalized vector $\mathbb{T}$-variational inequality problem are given.

1. INTRODUCTION

The contingent derivative of a set-valued map at a given point is a set-valued map whose graph equals the contingent cone to the graph of the set-valued map at that given point. This concept of derivative of a set-valued map was introduced by Aubin [1] in 1981. Several authors used this notion of contingent derivative to derive the optimality conditions for set-valued vector optimization problems; see, for example, [9, 22] and references therein. It is clear from the definition of the contingent derivative that two concepts are used. One is the contingent cone and the other is the graph of the set-valued map. Several authors generalized this kind of derivative by replacing different kinds of cones. Namely, Shi [26] introduced

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the concept of the TP-cone (we shall call it S-cone) to the graph of a set-valued map at a given point. By using this cone, he introduced the so-called TP-derivative (we shall call it S-derivative) of a set-valued map. He gave a relationship between the TP-derivative and the contingent derivative and some applications to perturbed set-valued vector optimization problems. This kind of derivative is further used by Taa [27] to investigate the necessary and sufficient conditions for an optimal solution of a set-valued vector optimization problem. Flores-Bazán [10] and Taa [28] introduced the radial derivative by replacing the contingent cone in the definition of the contingent derivative with the radial cone [6]. They also examined several properties of such derivative along with some optimality conditions for set-valued vector optimization problems. In 1997, Jahn and Rauh [16] replaced the graph of the set-valued map in the definition of the contingent derivative by the epigraph and the resulting derivative is called contingent epiderivative. More precisely, the contingent epiderivative of a set-valued map at a given point is a single-valued map whose epigraph equals the contingent cone to the epigraph of the set-valued map at that given point. They have shown that the contingent epiderivative has important properties and is one possible generalization of directional derivatives in the single-valued convex case. They also presented necessary and sufficient conditions for a solution of a set-valued vector optimization problem. In [10, 11], Flores-Bazán generalized the contingent epiderivative by replacing the contingent cone with the radial cone. Such kind of derivative is called radial epiderivative. He discussed several properties of such a derivative, relationships with the contingent epiderivative, and necessary and sufficient optimality conditions for a solution of a set-valued vector optimization problem. Lalitha et al. [18] introduced the concept of Clarke epiderivative of a set-valued map by considering the Clarke tangent cone in place of the contingent cone in the definition of contingent epiderivative. Later, Chen [8] introduced the concept of generalized Clarke epiderivative. Instead of considered the Clarke tangent cone to the epigraph of a set-valued map $F : X \to 2^Y$ at $(\bar{x}, \bar{y})$ in the epigraph of F, its projection on the image space is taken at that point and minimizers of this projection set is the value of the generalized Clarke epiderivative at that point. Lalitha et al. [18] and Chen [8] investigated several properties of their epiderivatives and gave a Fritz John type necessary optimality condition and Karush Kuhn Tucker type necessary and sufficient optimality conditions for a solution of a set-valued vector optimization problem. Bigi and Castellani [5] also extended the contingent epiderivative to a so-called K-epiderivative, and they presented optimality conditions with this differentiability concept.

In the last decade, the existence of an efficient or a weak efficient solution of a single-valued vector optimization problem is studied by using vector variational inequalities; see, for example, [3, 7, 12, 17, 19, 20, 30] and references therein. In almost of all the papers mentioned above, the necessary and sufficient conditions for a weak minimizer of a set-valued optimization problem are given in terms of some kind of generalized vector variational inequalities, without naming generalized vector variational inequalities. That is, a weak minimizer of a set-valued optimization problem is a solution of some kind of generalized vector variational inequality problem and vice-versa under certain conditions. But no author has discussed the existence of a weak minimizer of a set-valued optimization problem by proving the existence of a solution of the corresponding generalized vector variational inequality problem. This paper is an effort in this direction. One of the main motivations of this paper is to introduce a unified closed cone (called $\mathbb{T}$-cone) which includes almost all the cones mentioned above. We derive almost all the results mentioned in the above references in a more general and unified frame work.

In this paper, we introduce the concept of a $\mathbb{T}$-cone which includes, in general, all the closed cones, in particular, contingent cone, radial cone, S-cone, Clarke tangent cone, adjacent cone, etc. By using this cone, we introduce the $\mathbb{T}$-epiderivative of a set-valued map. More precisely, we define the $\mathbb{T}$-epiderivative of a set-valued map at a given point being a single-valued map whose epigraph is equal to the $\mathbb{T}$-cone to the epigraph of the set-valued map at that given point. This kind of derivative is a unified version of all the derivatives mentioned above. Several properties of the $\mathbb{T}$-epiderivative and its relations with known epiderivatives are given. We consider the generalized vector $\mathbb{T}$-variational inequality problem and give some optimality conditions for a solution of a set-valued vector optimization problem. At the end, we establish some existence results for solutions of set-valued vector optimization problems and generalized vector $\mathbb{T}$-variational inequality problems.

2. $\mathbb{T}$-Cones and $\mathbb{T}$-Epiderivatives

Let X and Y be two real normed spaces and C be a convex cone in Y inducing a partial order in Y. Let K be a nonempty subset of X and $F : X \to 2^Y$ be a set-valued map with nonempty values, where 2^Y denotes the family of all subsets of Y. The set

$$\text{graph}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$$

is called the *graph* of the map F. Throughout the paper, unless otherwise specified, we denote by int K and $\overline{K}$, the interior and closure of K, respectively.

Let $\bar{x} \in K$ be given. The *contingent cone* (also called, *Bouligand's cone*) *to* K *at* $\bar{x}$ [6] (see also [2, 4, 13]) is defined as

$$C(K; \bar{x}) = \{d \in X : \exists \{x_n\} \subset K \text{ and } \{t_n\} \subset (0, \infty) \text{ such that } x_n \to \bar{x} \text{ and } t_n(x_n - \bar{x}) \to d\}$$

(Severi [25] remarked that he has independently introduced this cone concept). It is easy to see that the above definition of the contingent cone can be written as

$$C(K;\bar{x}) = \{d \in X : \exists\{x_n\} \subset K \text{ and } \{t_n\} \text{ such that } x_n \to \bar{x},\ t_n \to 0^+ \text{ and } \frac{x_n - \bar{x}}{t_n} \to d\Big\}.$$

If $\bar{x} \in \operatorname{int} K$, then $C(K;\bar{x})$ is clearly the whole space. Since we require $x_n \to \bar{x}$ in the definition of $C(K;\bar{x})$, it is obvious that several authors considered $\bar{x} \in \overline{K}$.

If $d_n = \dfrac{x_n - \bar{x}}{t_n}$ $[\to d]$, that is, $x_n = \bar{x} + t_n d_n$ $[\in K]$, then we have $d \in C(K;\bar{x})$ if and only if there exist sequences $\{d_n\}$ and $\{t_n\}$ with $d_n \to d$ and $t_n \to 0^+$ such that $\bar{x} + t_n d_n \in K$ for all $n \in \mathbb{N}$. It is equivalent to saying that $d \in C(K;\bar{x})$ if and only if there exist sequences $\{d_n\}$ with $d_n \to d$ and $\{t_n\} \subset \mathbb{R}_+$ such that

$$\bar{x} + t_n d_n \in K, \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad t_n x_n \to 0$$

(see for example [28]).

Let $(\bar{x}, \bar{y}) \in \text{graph}(F)$ be given. Then

$$C\,(\text{graph}(F);(\bar{x},\bar{y})) = \{(x,y) \in X \times Y :\ \text{there exist } \{t_n\} \subset (0,\infty),\ \{x_n\} \subset K \text{ and } y_n \in F(x_n) \text{ with } (x_n,y_n) \to (\bar{x},\bar{y}) \text{ such that } t_n\,((x_n,y_n)-(\bar{x},\bar{y})) \to (x,y)\}$$

is the contingent cone to the graph of F, graph(F), at $(\bar{x},\bar{y})$. The set $S(\text{graph}(F); (\bar{x},\bar{y})) \subset X \times Y$, defined by

$$S(\text{graph}(F);(\bar{x},\bar{y})) = \{(x,y) \in X \times Y : \exists\{t_n\} \subset (0,\infty),\ \{x_n\} \subset K \text{ with } x_n \to \bar{x}, \text{ and } y_n \in F(x_n) \text{ such that } t_n((x_n,y_n) - (\bar{x},\bar{y})) \to (x,y)\}$$

is called the *S-cone to* $\text{graph}(F)$ *at* $(\bar{x},\bar{y})$. It is introduced and studied by Shi [26]. He also pointed out that

$$C(\text{graph}(F);(\bar{x},\bar{y})) \subset S(\text{graph}(F);(\bar{x},\bar{y})) \tag{2.1}$$

and that

$$C(\text{graph}(F);(\bar{x},\bar{y})) = S(\text{graph}(F);(\bar{x},\bar{y}))$$

if $\text{graph}(F)$ is convex.

We observe the following relationship.

Lemma 2.1. *If the first component of every element of* $S(\text{graph}(F);(\bar{x},\bar{y}))$ *is nonzero, then*

$$S(\text{graph}(F);(\bar{x},\bar{y})) = C(\text{graph}(F);(\bar{x},\bar{y})).$$

Proof. Let $(x,y) \in S(\text{graph}(F);(\bar{x},\bar{y}))$ with $x \neq 0$ be arbitrarily chosen. Then by the definition of the S-cone, there exist $\{t_n\} \subset (0,\infty)$, $\{x_n\} \subset K$ with $x_n \to \bar{x}$ and $y_n \in F(x_n)$ such that

$$t_n((x_n,y_n) - (\bar{x},\bar{y})) \to (x,y) \neq (0,y).$$

Since $x \neq 0$ we conclude $t_n \to \infty$ and therefore $t_n(y_n - \bar{y}) \to y$ implies $y_n \to \bar{y}$. Hence $(x, y) \in C(\mathrm{graph}(F); (\bar{x}, \bar{y}))$. The opposite inclusion is formulated in (2.1). ■

The *radial cone* $R(K; \bar{x})$ *to* K *at* $\bar{x}$ [6] (see also [10, 11, 28]) is defined by

$$R(K;\bar{x}) = \{d \in X : \exists \{d_n\} \to d,\ \{t_n\} \subset (0,\infty) \text{ such that } \bar{x} + t_n d_n \in K \text{ for all } n \in \mathbb{N}\}.$$

It is easy to see that (i) $C(K; \bar{x}) \subset R(K; \bar{x})$ and (ii) $C(K; \bar{x}) = R(K; \bar{x})$ whenever K is a convex set.

The radial cone is related to the *adjacent cone* $A(K; \bar{x})$ to K at $\bar{x}$ [29] being defined by

$$A(K; \bar{x}) = \{d \in X : \forall \{t_n\} \subset (0, \infty) \text{ with } t_n \to 0_+ \ \exists \{d_n\} \text{ with } d_n \to d \text{ such that}$$
$$\bar{x} + t_n d_n \in K \text{ for all } n \in \mathbb{N}\}.$$

It is obvious that $A(K; \bar{x}) \subset R(K; \bar{x})$.

The C*-tangent cone* $T(K; \bar{x})$ *to* K *at* $\bar{x}$ [8] is defined by

$$T(K; \bar{x}) = \{d \in X : \forall \{\bar{x}_n\} \subset K \text{ with } \bar{x}_n \to \bar{x} \text{ and } \{t_n\} \subset (0, +\infty) \text{ with } t_n \to +\infty,$$
$$\exists \{x_n\} \subset K \text{ such that } x_n \to \bar{x} \text{ and } t_n(x_n - \bar{x}) \to d\}.$$

Notice that the sequence $\{\bar{x}_n\}$ plays no role in the original definition of $T(K; \bar{x})$. It is easy to see that the C-tangent cone equals the adjacent cone.

Lemma 2.2. (Thanks to the referees to provide this result.) *For an arbitrary nonempty set* K *and every* $\bar{x} \in K$ *we have*

$$T(K; \bar{x}) = A(K; \bar{x}).$$

Proof. The C-tangent cone can equivalently be written as

$$T(K; \bar{x}) = \{d \in X : \forall \{t_n\} \subset (0, +\infty) \text{ with } t_n \to +\infty,$$
$$\exists \{x_n\} \subset K \text{ such that } x_n \to \bar{x} \text{ and } t_n(x_n - \bar{x}) \to d\}.$$

By setting $d_n := t_n(x_n - \bar{x})$ and $\theta_n := \frac{1}{t_n}$ for all $n \in \mathbb{N}$, we obtain $d_n \to d$, $\theta_n \to 0_+$,

$$\bar{x} + \theta_n d_n = x_n \in K \quad \text{for all } n \in \mathbb{N}$$

and $x_n \to \bar{x}$. Hence, we conclude $T(K; \bar{x}) = A(K; \bar{x})$. ■

The concept of C-tangent cone was first used by Chen [8]. He also pointed out that $T(K; \bar{x})$ is a closed convex cone while $C(K; \bar{x})$ is a closed cone, $T(K; \bar{x}) \subset C(K; \bar{x})$, and $T(K; \bar{x}) = C(K; \bar{x})$ whenever K is a convex set.

From the above remark, it is clear that $C(K;\bar{x}) = R(K;\bar{x}) = A(K;\bar{x})$ whenever K is a convex set.

Lemma 2.3. *The C-tangent cone (and so the adjacent cone) is a superset of the well-known Clarke tangent cone (e.g., see [14, p. 82]) defined by*

$$T_{Clarke}(K;\bar{x}) := \{d \in X : \forall \{\bar{x}_n\} \subset K \text{ with } \bar{x}_n \to \bar{x} \text{ and } \{\lambda_n\} \subset (0,\infty) \text{ with } \lambda_n \to 0$$
$$\exists \{d_n\} \text{ with } d_n \to d \text{ and } \bar{x}_n + \lambda_n d_n \in K \text{ for all } n \in \mathbb{N}\}.$$

Proof. Let $d \in T_{Clarke}(K;\bar{x})$ be arbitrarily given. Then for every sequence $\{\bar{x}_n\} \subset K$ with $\bar{x}_n \to \bar{x}$ and $\{\lambda_n\} \subset (0,\infty)$ with $\lambda_n \to 0$ there exists a sequence $\{d_n\}$ with $d_n \to d$ and $\bar{x}_n + \lambda_n d_n \in K$ for all $n \in \mathbb{N}$. If we set $x_n := \bar{x}_n + \lambda_n d_n$ for all $n \in \mathbb{N}$ and $t_n := \frac{1}{\lambda_n}$ for all $n \in \mathbb{N}$, we obtain

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \bar{x}_n + \lambda_n d_n = \bar{x}$$

and

$$\lim_{n\to\infty} t_n(x_n - \bar{x}) = \lim_{n\to\infty} \frac{1}{\lambda_n}(\bar{x} + \lambda_n d_n - \bar{x}) = \lim_{n\to\infty} d_n = d.$$

Consequently, we have $d \in T(K;\bar{x})$. ■

We now define a unified version of tangent cones.

Definition 2.1. Let Z be a real normed space. The set-valued map $\mathbb{T} : 2^Z \times Z \to 2^Z$ defined as

$$\mathbb{T}(P;\bar{z}) = \begin{cases} \text{closed cone}, & \text{if } \bar{z} \in P \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $P \subseteq Z$, is called a *tangent map* and $\mathbb{T}(P;\bar{z})$ is called $\mathbb{T}$*-cone to P at $\bar{z}$.*

The contingent cone $C(K;\bar{x})$, the radial cone $R(K;\bar{x})$, the Clarke tangent cone $T_{Clarke}(K;\bar{x})$, the S-cone and the adjacent cone $A(K;\bar{x})$ $(= T(K;\bar{x}))$ are examples of a $\mathbb{T}$-cone.

Let $F : K \to 2^Y$ be a set-valued map. The *epigraph of F* (with respect to K), denoted by epi(F), is defined as

$$\text{epi}(F) = \{(x,y) \in X \times Y : x \in K,\ y \in F(x) + C\}.$$

Let $(\bar{x},\bar{y}) \in X \times Y$ with $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ be given. A single-valued map $D^e_{\mathbb{T}}F(\bar{x},\bar{y}) : X \to Z$ is called $\mathbb{T}$*-epiderivative of F at* $(\bar{x},\bar{y})$ if its epigraph is equal to the $\mathbb{T}$-cone to the epigraph of F at $(\bar{x},\bar{y})$, that is,

$$\text{epi}(D^e_{\mathbb{T}}F(\bar{x},\bar{y})) = \mathbb{T}(\text{epi}(F);(\bar{x},\bar{y})).$$

Remark 2.1.

(a) If we replace the $\mathbb{T}$-cone by the contingent cone, the S-cone, the radial cone or the adjacent cone, then the $\mathbb{T}$-epiderivative is called *contingent epiderivative* denoted by $D^e_C F(\bar{x}, \bar{y})$ [16], *S-epiderivative* denoted by $D^e_S F(\bar{x}, \bar{y})$, *radial epiderivative* denoted by $D^e_R F(\bar{x}, \bar{y})$ [10, 11] or *adjacent epiderivative* denoted by $D^e_A F(\bar{x}, \bar{y})$ [8], respectively. Similar notions could be introduced with the Clarke tangent cone and the adjacent cone.

(b) If we replace the epigraph by the graph in the definition of the $\mathbb{T}$-epiderivative, then the set-valued map $D_{\mathbb{T}} F(\bar{x}, \bar{y}) : X \to 2^Y$ is called *$\mathbb{T}$-derivative of F at* $(\bar{x}, \bar{y})$. In other words, $D_{\mathbb{T}} F(\bar{x}, \bar{y}) : X \to !2^Y$ is the $\mathbb{T}$-derivative! F! at $(\bar{x}, \bar{y})$ if

$$\text{graph}\left(D_{\mathbb{T}} F(\bar{x}, \bar{y})\right) = \mathbb{T}\left(\text{graph}(F); (\bar{x}, \bar{y})\right).$$

(c) If we replace the $\mathbb{T}$-cone by the contingent cone, the S-cone, the radial cone or the C-tangent cone, then the $\mathbb{T}$-derivative is called *contingent derivative* denoted by $D_C F(\bar{x}, \bar{y})$ [1, 2, 4], *S-derivative* denoted by $D_S F(\bar{x}, \bar{y})$ [26, 27], *radial derivative* denoted by $D_R F(\bar{x}, \bar{y})$ [10, 28] or *adjacent derivative* denoted by $D_A F(\bar{x}, \bar{y})$ [8], respectively.

The adjacent epiderivative introduced and studied by Chen [8] is a set-valued map. He discussed several properties of the adjacent epiderivative and gave some optimality conditions of set-valued optimization problems. He proved that if $D^e_A F(\bar{x}, \bar{y})$ and $D^e_C F(\bar{x}, \bar{y})$ exist, then

$$D^e_A F(\bar{x}, \bar{y})(x) \subset \{D^e_C F(\bar{x}, \bar{y})(x)\} + C \quad \text{for all } x \in X,$$

where C is a closed pointed convex cone in Y; and

$$D^e_A F(\bar{x}, \bar{y})(x) = D^e_C F(\bar{x}, \bar{y})(x) \quad \text{for all } x \in X,$$

whenever F is C-convex on a convex set K.

The radial epiderivative was introduced and studied by Flores-Bazán [10, 11]. He derived a necessary and sufficient condition for a point to be a weak minimal solution of a non-convex set-valued vector optimization problem.

Throughout the paper, $D^e_G F(\bar{x}, \bar{y})$ stands for all $D^e_C F(\bar{x}, \bar{y})$, $D^e_S F(\bar{x}, \bar{y})$, $D^e_R F(\bar{x}, \bar{y})$ and $D^e_A F(\bar{x}, \bar{y})$; and $D_G F(\bar{x}, \bar{y})$ stands for all $D_C F(\bar{x}, \bar{y})$, $D_S F(\bar{x}, \bar{y})$, $D_R F(\bar{x}, \bar{y})$ and $D_A F(\bar{x}, \bar{y})$.

Remark 2.2. It is clear from the above definitions that

$$\text{epi}\left(D^e_R F(\bar{x}, \bar{y})\right) \subseteq \text{epi}\left(D^e_S F(\bar{x}, \bar{y})\right) \subseteq \text{epi}\left(D^e_C F(\bar{x}, \bar{y})\right).$$

One of the main motivations of this paper is to establish some results in connection with the $\mathbb{T}$-epiderivative which will be unified results for the contingent epiderivative, the S-epiderivative, the radial epiderivative, the adjacent epiderivative, etc.

3. Some Properties of $\mathbb{T}$-Epiderivatives

Before proving the existence of a $\mathbb{T}$-epiderivative in a special case we discuss some properties of these derivatives. By using the same argument as in the proof of Theorem 2 in [16], one can easily establish the following result.

Theorem 3.1. *Let $F : K \to 2^Y$ be a set-valued map and $(\bar{x}, \bar{y}) \in X \times Y$ be given such that $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$. If the $\mathbb{T}$-epiderivative $D^e_{\mathbb{T}} F(\bar{x}, \bar{y})$ exists, then it is unique.*

Definition 3.1. Let K be a nonempty convex subset of X. A set-valued map $F : K \to 2^Y$ is called *C-convex* if for all $x_1, x_2 \in K$ and $\lambda \in [0, 1]$

$$\lambda F(x_1) + (1 - \lambda) F(x_2) \subset F(\lambda x_1 + (1 - \lambda) x_2) + C.$$

Lemma 3.1. [16]. *If K is nonempty convex and $F : K \to 2^Y$ is C-convex, then epi(F) is convex.*

Theorem 3.2. *Let $K = X$, C be a closed convex cone in Y and $F : X \to 2^Y$ be C-convex. If the $\mathbb{T}$-derivative $D_{\mathbb{T}} F(\bar{x}, \bar{y})$ and the $\mathbb{T}$-epiderivative $D^e_{\mathbb{T}} F(\bar{x}, \bar{y})$ exist, then*

$$\textit{epi}\,(D_{\mathbb{T}} F(\bar{x}, \bar{y})) \subset \textit{epi}\,(D^e_{\mathbb{T}} F(\bar{x}, \bar{y}))\,.$$

Proof. It is similar to the proof of Theorem 3 in [16]. ■

Definition 3.2. A single-valued map $f : X \to Y$ is called

(i) *positively homogeneous* if for all $x \in X$ and $\alpha \geq 0$, $f(\alpha x) = \alpha f(x)$;

(ii) *subadditive* if for all $x_1, x_2 \in X$,

$$f(x_1 + x_2) \in \{f(x_1) + f(x_2)\} - C;$$

(iii) *sublinear* if it is both positively homogeneous and subadditive.

Condition A. If P is a nonempty convex set, then $\mathbb{T}(P; \bar{x})$ is a closed convex cone.

Theorem 3.3. *Let C be a pointed convex cone in Y, K be a convex set in X, $F : K \to 2^Y$ be a C-convex set-valued map with nonempty values, and $\bar{x} \in K$ and*

$\bar{y} \in F(\bar{x})$ be given. If Condition A is satisfied and the $\mathbb{T}$-epiderivative $D^{\,e}_{\mathbb{T}}F(\bar{x},\bar{y})$ exists, then it is sublinear.

Proof. Since F is C-convex, epi(F) is a convex set. Then by Condition A, the $\mathbb{T}$-cone $\mathbb{T}(\text{epi}(F);(\bar{x},\bar{y}))$ is a closed convex cone and, therefore, the epigraph of $D^e_{\mathbb{T}}F(\bar{x},\bar{y})$ is a closed convex cone. The rest of the proof lies on the lines of the proof of Theorem 4 in [16]. ■

Remark 3.1. We note that the adjacent cone $T(\text{epi}(F);(\bar{x},\bar{y}))$ is a closed convex cone and hence epi $(D^e_A F(\bar{x},\bar{y}))$ is a closed convex cone. Therefore, Theorem 3.3 holds without convexity of K, C-convexity of F and Condition A. For further detail, we refer to [8].

Theorem 3.4. *Let $Y=\mathbb{R}$ and assume that there are functions $f,g : X \to \mathbb{R}$ with $\text{epi}(f) \supset \mathbb{T}(\text{epi}(F);(\bar{x},\bar{y})) \supset \text{epi}(g)$. Then, the $\mathbb{T}$-epiderivative $D^e_{\mathbb{T}}F(\bar{x},\bar{y})$ is given by*

$$D^e_{\mathbb{T}}F(\bar{x},\bar{y})(x)=\min\{y\in\mathbb{R} : (x,y)\in\mathbb{T}(\text{epi}(F);(\bar{x},\bar{y}))\} \quad \text{for all } x\in X. \tag{3.1}$$

Proof. Although it is similar to the proof of Theorem 1 in [16], we include it for the sake of reader's convenience. We define the functional $D^e_{\mathbb{T}}F(\bar{x},\bar{y}) : X \to \mathbb{R}\bigcup\{-\infty\}$ by

$$D^e_{\mathbb{T}}F(\bar{x},\bar{y})(x) = \inf\{y \in \mathbb{R} : (x,y) \in \mathbb{T}(\text{epi}(F);(\bar{x},\bar{y}))\} \quad \text{for all } x \in X. \tag{3.2}$$

Since $\text{epi}(g) \subset \mathbb{T}(\text{epi}(F);(\bar{x},\bar{y}))$ for every $x \in X$, there is at least one $y \in \mathbb{R}$ with $(x,y) \in \mathbb{T}(\text{epi}(F);(\bar{x},\bar{y}))$. So, $D^e_{\mathbb{T}}F(\bar{x},\bar{y})$ is well defined on X. Now we will show that it is a $\mathbb{T}$-epiderivative.

Let $x \in X$ be an arbitrary element. Then by (3.1), there is an infimal sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ converging to $D^e_{\mathbb{T}}F(\bar{x},\bar{y})$ with $(x,y_n) \in \mathbb{T}(\text{epi}(F);(\bar{x},\bar{y}))$. Since the $\mathbb{T}$-cone is closed, we have

$$(x, D^e_{\mathbb{T}}F(\bar{x},\bar{y})(x)) \in \mathbb{T}\,(\text{epi}(F);(\bar{x},\bar{y}))\,. \tag{3.3}$$

Since $\text{epi}(f) \supset \mathbb{T}(\text{epi}(F);(\bar{x},\bar{y})) \supset \text{epi}(g)$, we have $-\infty < f(x) \leq D^e_{\mathbb{T}}F(\bar{x},\bar{y})(x)$, and hence (3.1) is satisfied. It follows that

$$\text{epi}\,(D^e_{\mathbb{T}}F(\bar{x},\bar{y})) = \mathbb{T}\,(\text{epi}(F);(\bar{x},\bar{y}))\,.$$

Hence, $D^e_{\mathbb{T}}F(\bar{x},\bar{y})$ is a $\mathbb{T}$-epiderivative of F at $(\bar{x},\bar{y})$. ■

By using the same argument as in the proof of Corollary 1 in [16], we can easily derive the following result.

Corollary 3.1. *Let $Y = \mathbb{R}$ and $K = X$. Let $F : X \to \mathbb{R}$ be a single-valued convex functional which is continuous at $\bar{x}$. Assume that the Condition A is satisfied. Then the $\mathbb{T}$-epiderivative $D^e_{\mathbb{T}}F(\bar{x}, \bar{y})$ is given by (3.1).*

Theorem 3.5. *Let $(\bar{x}, \bar{y}) \in$ graph(F) be given and $F : X \to \mathbb{R}$ be a single-valued function. If the $\mathbb{T}$-epiderivative $D^e_{\mathbb{T}}F(\bar{x}, \bar{y})$ exists, then it is lower semicontinuous.*

Proof. Since the $\mathbb{T}$-cone is closed, the $\mathbb{T}$-epigraph of $D^e_{\mathbb{T}}F(\bar{x}, \bar{y})$ is also closed. The conclusion follows from the fact that a function is lower semicontinuous if and only if its epigraph is closed. ■

Theorem 3.6. *Let $(\bar{x}, \bar{y}) \in$ graph(F) be given. If $D^e_{\mathbb{T}}F(\bar{x}, \bar{y})$ and $D^e_G F(\bar{x}, \bar{y})$ exist, and $G\,(\text{epi}(F); (\bar{x}, \bar{y})) \subset \mathbb{T}\,(\text{epi}(F); (\bar{x}, \bar{y}))$, then $D^e_G F(\bar{x}, \bar{y})(x) \in \{D^e_{\mathbb{T}}F(\bar{x}, \bar{y})(x)\} + C$ for all $x \in K$, where C is a closed pointed convex cone in Y.*

Proof. Let $x \in K$ be any arbitrary element. Then

$$\begin{aligned}(x, D^e_G F(\bar{x}, \bar{y})(x)) &\in \text{epi}\,(D^e_G F(\bar{x}, \bar{y})) = G\,(\text{epi}(F); (\bar{x}, \bar{y})) \\ &\subset \mathbb{T}\,(\text{epi}(F); (\bar{x}, \bar{y})) = \text{epi}\,(D^e_{\mathbb{T}}F(\bar{x}, \bar{y}))\,,\end{aligned}$$

and so $D^e_G F(\bar{x}, \bar{y})(x) \in \{D^e_{\mathbb{T}}F(\bar{x}, \bar{y})(x)\} + C$. ■

Following the lines in [15] one can also prove certain calculus rules for epiderivatives. We restrict ourselves to a simple rule for scalar multiplication (cf. [15, Thm. 2.1]).

Condition B. For every $\lambda > 0$

$$\mathbb{T}(m(\text{epi}(F)); m(\bar{x}, \bar{y})) = m\mathbb{T}(\text{epi}(F), (\bar{x}; \bar{y}))$$

for $m : X \times Y \to X \times Y$ with $m(x, y) = (x, \lambda y)$.

Theorem 3.7. *Let $(\bar{x}, \bar{y}) \in$ graph(F) be given, let $D^e_{\mathbb{T}}F(\bar{x}, \bar{y})$ exist, and let Condition B be satisfied. Then for every $\lambda > 0$, $D^e_{\mathbb{T}}(\lambda F)(\bar{x}, \lambda\bar{y})$ exists and*

$$D^e_{\mathbb{T}}(\lambda F)(\bar{x}, \lambda\bar{y}) = \lambda D^e_{\mathbb{T}}F(\bar{x}, \bar{y}).$$

Proof. The assertion follows from the equations

$$\begin{aligned}
\text{epi}(D^e_{\mathbb{T}}(\lambda F)(\bar{x},\lambda\bar{y}) &= \mathbb{T}(\text{epi}(\lambda F);(\bar{x},\lambda\bar{y}))\\
&= \mathbb{T}(\{(x,\tilde{y}) : x\in X,\ \tilde{y}\in\lambda F(x)+C\};(\bar{x},\lambda\bar{y}))\\
&= \mathbb{T}(\{(x,\lambda y) : x\in X,\ y\in F(x)+C\};(\bar{x},\lambda\bar{y}))\\
&= \mathbb{T}(m(\text{epi}(F));m(\bar{x},\bar{y}))\\
&= m(\mathbb{T}(\text{epi}(F);(\bar{x},\bar{y}))\\
&= m(\text{epi}(D^e_{\mathbb{T}}F(\bar{x},\bar{y}))\\
&= \{(x,\lambda\tilde{y}) : x\in X,\ \tilde{y}\in D^e_{\mathbb{T}}F(\bar{x},\bar{y})+C\}\\
&= \{(x,y) : x\in X,\ y\in \lambda D^e_{\mathbb{T}}F(\bar{x},\bar{y})+C\}\\
&= \text{epi}(\lambda D^e_{\mathbb{T}}F(\bar{x},\bar{y})). \qquad \blacksquare
\end{aligned}$$

4. Optimality Conditions in Set Optimization

Now we apply the concept of $\mathbb{T}$-epiderivatives to optimality conditions in set optimization. Our main focus is on sufficient conditions which cannot be handled by contingent derivatives in an appropriate way.

Let C be a closed convex pointed cone in Y with $\text{int } C \neq \emptyset$ (i.e. C is solid) and A be a nonempty subset of Y. An element $a \in A$ is called a *minimal element* (respectively, *weak minimal element*) of A if

$$(\{a\}-C)\cap A \subset \{a\}+C \quad \text{or equivalently,} \quad (\{a\}-C)\cap A = \{a\}.$$
$$\Big(\text{respectively, } (A-\{a\})\cap(-\text{int } C)=\emptyset \quad \text{or equivalently,} \quad (\{a\}-\text{int } C)\cap A=\emptyset\Big).$$

Consider the set-valued vector optimization problem (in short, VOP)

$$\min_{x\in K} F(x),$$

where K is a nonempty subset of X and $F : K \to 2^Y$ is a set-valued map with nonempty values.

A pair $(\bar{x},\bar{y})$ with $\bar{x}\in K$ and $\bar{y}\in F(\bar{x})$ is called a *minimizer* (respectively, *weak minimizer*) of VOP if $\bar{y}$ is a minimal element (respectively, weak minimal element) of the set $F(K) = \bigcup_{x\in K} F(x)$, that is,

$$(\{\bar{y}\}-C)\cap F(K) \subset \{\bar{y}\}+C \quad \text{or equivalently,} \quad (\{\bar{y}\}-C)\cap F(K) = \{\bar{y}\}.$$
$$\Big(\text{respectively, } (F(K)-\{\bar{y}\})\cap(-\text{int } C)=\emptyset \text{ or equivalently, } (\{\bar{y}\}-\text{int } C)\cap F(K)=\emptyset\Big).$$

More precisely, $(\bar{x},\bar{y})\in \text{graph}(F)$ is a weak minimizer of VOP if

$$F(x)-\{\bar{y}\} \subset Y\setminus(-\text{int } C) \quad \text{for all } x\in K.$$

We present an existence result for a minimal element of a nonempty noncompact set.

Theorem 4.1. *Let Y be a topological vector space and A be a nonempty subset of Y. Assume that there exist nonempty compact subsets B and D of A such that for each $z \in A \setminus D$, there exists $\hat{y} \in B$ satisfying $\hat{y} \in A \cap (z - \text{int } C)$. Then, A has a minimal element.*

Proof. (Thanks to one of the referees to provide this short proof). Since B and D are compact, $(B \cup D)$ is compact as well. By Corollary 3.8 in [21], we can find $\bar{y} \in B \cup D$ satisfying

$$(B \cup D) \cap (\bar{y} - C) = \{\bar{y}\}. \tag{4.1}$$

We will prove that $\bar{y}$ is a minimal point of A. Arguing by contradiction, assume that it is not the case, then there is some $y \in A$ with $y \neq \bar{y}$ satisfying

$$y \in \bar{y} - C. \tag{4.2}$$

Thus, $y \notin D$ by (4.1). By the imposed coercivity condition, we find $\hat{y} \in B$ such that $\hat{y} \in A \cap (y - \text{int } C)$, and thus

$$\hat{y} \in y - \text{int } C. \tag{4.3}$$

Combining (4.2) and (4.3), and taking into account that $C + \text{int } C = \text{int } C$ due to the convexity and conical property of C we have

$$\hat{y} \in y - \text{int } C \subset \bar{y} - C - \text{int } C = \bar{y} - \text{int } C \subset \bar{y} - C \setminus \{0\}.$$

hence, $\hat{y} \in B \cap (\bar{y} - C)$ and $\hat{y} \neq \bar{y}$, which contradicts the relation (4.1). This contradiction justifies the minimality of $\bar{y}$ to A. The proof is complete. ∎

Remark 4.1. The coercivity condition "Assume that there exist nonempty compact subsets B and D of A such that for each $z \in A \setminus D$, there exists $\hat{y} \in B$ satisfying $\hat{y} \in A \cap (z - \text{int } C)$" in Theorem 4.1 is satisfied if the set A is compact.

Corollary 4.1. *Let X and Y be topological vector spaces, K be a nonempty subset of X and $F : K \to 2^Y$ be a set-valued map with nonempty values. Assume that there exist a nonempty compact subsets B and D of $F(K)$ such that for each $z \in F(K) \setminus D$, there exists $\hat{y} \in B$ satisfying $\hat{y} \in F(K) \cap (z - \text{int } C)$. Then, there exists a minimizer of VOP.*

We consider the following generalized vector $\mathbb{T}$-variational inequality problem (for short, $\mathbb{T}$-GVVIP): Find $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ such that

$$D^e_{\mathbb{T}} F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\text{int } C \quad \text{for all } x \in K.$$

Definition 4.1. Let $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ be given. A set-valued map $F : K \to 2^Y$ is called $\mathbb{T}$*-pseudoconvex at* $(\bar{x}, \bar{y})$ if

$$D^e_{\mathbb{T}}F(\bar{x}, \bar{y})(x-\bar{x}) \notin -\text{int}\, C \text{ for all } x \in K \quad \Rightarrow \quad y-\bar{y} \notin -\text{int}\, C \text{ for all } y \in F(x).$$

It is called *G-pseudoconvex at* $(\bar{x}, \bar{y})$ if we replace $D^e_{\mathbb{T}}F(\bar{x}, \bar{y})$ by $D^e_G F(\bar{x}, \bar{y})$ in the above relation.

As a simple example we consider the set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}}$ with $F(x) = \{-\frac{1}{1+x^2}\} + \mathbb{R}_+$ for all $x \in \mathbb{R}$. For $\bar{x} = 0$ and $\bar{y} = -1$ we have by Theorem 3.4 for the contingent epiderivative $D^e_C F(0, -1)(x) = 0$ for all $x \in \mathbb{R}$. Since for every $y \in F(x)$ with $x \in \mathbb{R}$

$$y - \bar{y} = y + 1 \geq \underbrace{-\frac{1}{1+x^2}}_{\geq -1} + 1 \geq 0,$$

the set-valued map F is C-pseudoconvex at $(0, -1)$.

Theorem 4.2. *Let* $(\bar{x}, \bar{y}) \in$ *graph*(F) *be given. If* $D^e_{\mathbb{T}}F(\bar{x}, \bar{y})$ *exists,* F *is* $\mathbb{T}$*-pseudoconvex at* $(\bar{x}, \bar{y})$*, then every solution* $(\bar{x}, \bar{y})$ *of* $\mathbb{T}$*-GVVIP is a weak minimizer of VOP.*

Proof. It directly follows from the definition of $\mathbb{T}$-pseudoconvexity of F. ■

Proposition 4.1. *Let* $D^e_G F(\bar{x}, \bar{y})$ *and* $D^e_{\mathbb{T}}F(\bar{x}, \bar{y})$ *be exist,* $G\,(\textit{epi}(F); (\bar{x}, \bar{y})) \subset \mathbb{T}(\textit{epi}(F); (\bar{x}, \bar{y}))$*, and* C *be a closed pointed solid convex cone in* Y*. If* F *is G-pseudoconvex at* $(\bar{x}, \bar{y})$*, then it is* $\mathbb{T}$*-pseudoconvex at* $(\bar{x}, \bar{y})$*, that is, if*

$$D^e_G F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\text{int}\, C \textit{ for all } x \in K \Rightarrow y - \bar{y} \notin -\text{int}\, C \textit{ for all } y \in F(x)$$

then

$$D^e_{\mathbb{T}}F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\text{int}\, C \text{ for all } x \in K \Rightarrow y - \bar{y} \notin -\text{int}\, C \text{ for all } y \in F(x).$$

Proof. Assume that for all $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$, we have

$$D^e_{\mathbb{T}}F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\text{int}\, C \quad \text{for all } x \in K \tag{4.4}$$

and there exists $y \in F(x)$ such that $y - \bar{y} \in -\text{int}\, C$. Then, by G-pseudoconvexity of F at $(\bar{x}, \bar{y})$, there exists $z \in K$ such that

$$D^e_G F(\bar{x}, \bar{y})(z - \bar{x}) \in -\text{int}\, C. \tag{4.5}$$

By Theorem 3.6,

$$D_G^e F(\bar{x},\bar{y})(z-\bar{x}) \in \{D_{\mathbb{T}}^e F(\bar{x},\bar{y})(z-\bar{x})\} + C. \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$D_{\mathbb{T}}^e F(\bar{x},\bar{y})(z-\bar{x}) \in \{D_G^e F(\bar{x},\bar{y})(z-\bar{x})\} - C \subset -\text{int } C - C = -\text{int } C$$

which is a contradiction of (4.4). ∎

Proposition 4.2. *Let $F : K \to 2^Y$ be a set-valued map and $(\bar{x},\bar{y}) \in \text{graph}(F)$ for which $D_{\mathbb{T}}^e F(\bar{x},\bar{y})$ exists. If*

$$F(x) - \{\bar{y}\} \subset \{D_{\mathbb{T}}^e F(\bar{x},\bar{y})(x-\bar{x})\} + C \quad \textit{for all} \quad x \in K \tag{4.7}$$

then F is $\mathbb{T}$-pseudoconvex at $(\bar{x},\bar{y})$.

Proof. Let

$$D_{\mathbb{T}}^e F(\bar{x},\bar{y})(x-\bar{x}) \notin -\text{int } C \quad \text{for all } x \in K. \tag{4.8}$$

Since (4.7) holds for all $x \in K$, we have

$$y - \bar{y} \in \{D_{\mathbb{T}}^e F(\bar{x},\bar{y})(x-\bar{x})\} + C \quad \text{for all} \quad y \in F(x). \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$y - \bar{y} \notin -\text{int } C \quad \text{for all } y \in F(x).$$

Hence, F is $\mathbb{T}$-pseudoconvex at $(\bar{x},\bar{y})$.

Proposition 4.3. *Let $F : K \to 2^Y$ be a set-valued map and $(\bar{x},\bar{y}) \in \text{graph}(F)$ for which $D_R^e F(\bar{x},\bar{y})$ and $D_{\mathbb{T}}^e F(\bar{x},\bar{y})$ exist and $R\,(\textit{epi}(F);(\bar{x},\bar{y})) \subset \mathbb{T}\,(\textit{epi}(F);(\bar{x},\bar{y}))$. Then (4.4) holds and hence F is $\mathbb{T}$-pseudoconvex at $(\bar{x},\bar{y})$.*

Proof. Let $y \in F(x) - \{\bar{y}\}$ be arbitrarily chosen. Then $\bar{y} + y \in F(x) \subset F(x) + C$. This implies that $\bar{y} + t_n y_n \in F(\bar{x} + t_n x_n) + C$ with $t_n = 1$, $y_n = y$ and $x_n = x - \bar{x}$. By the definition of the radial epiderivative, we have $y \in D_R^e F(\bar{x},\bar{y})(x-\bar{x}) + C$.
By Theorem 3.6,

$$y \in D_R^e F(\bar{x},\bar{y})(x-\bar{x}) + C \subseteq \{D_{\mathbb{T}}^e F(\bar{x},\bar{y})(x-\bar{x})\} + C + C$$

$$= \{D_{\mathbb{T}}^e F(\bar{x},\bar{y})(x-\bar{x})\} + C.$$

Thus, $y \in \{D_{\mathbb{T}}^e F(\bar{x},\bar{y})(x-\bar{x})\} + C$ and hence $F(x) - \{\bar{y}\} \subseteq \{D_{\mathbb{T}}^e F(\bar{x},\bar{y})(x-\bar{x})\} + C$. ∎

Remark 4.2.

(a) It is clear from Lemma 3 in [16] and Theorem 2.3 in [8] that (4.4) holds for $D^e_C F(\bar{x}, \bar{y})$ and $D^e_A F(\bar{x}, \bar{y})$ if K is a convex set and $F : K \to 2^Y$ is a C-convex set-valued map. In this case, F is $\mathbb{T}$-pseudoconvex at $(\bar{x}, \bar{y})$.

(b) Let C be a pointed proper convex cone with $\text{int } C \neq \emptyset$ and $(\bar{x}, \bar{y}) \in \text{graph}(F)$. If $D^e_R F(\bar{x}, \bar{y})$ exists, then it is proved by Flores-Bazán [10] that

$$D^e_R F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\text{int } C \quad \text{for all} \quad x \in K$$

if and only if $(\bar{x}, \bar{y})$ is a weak minimizer of VOP.

Definition 4.2. Let $(\bar{x}, \bar{y}) \in \text{graph}(F)$ be given. A $\mathbb{T}$-epiderivative is called a $\mathbb{T}$-*variation of* F *at* $(\bar{x}, \bar{y})$ if

$$D^e_{\mathbb{T}} F(\bar{x}, \bar{y})(x - \bar{x}) \in -\text{int } C \text{ for some } x \in K \;\Rightarrow\; \exists\, y \in F(K) \text{ such that } y - \bar{y} \in -\text{int } C.$$

Remark 4.3. This means that a negative $\mathbb{T}$-epiderivative of F at $(\bar{x}, \bar{y})$ implies the existence of a point strictly less than $\bar{y}$. This implication extends the known standard result that a negative directional derivative implies that function values locally decrease on a ray. In the case of a $\mathbb{T}$-variation, we do not consider a short ray but only one point. By the proof of Thm. 7 in [16] the contingent epiderivative is a $\mathbb{T}$-variation.

Theorem 4.3. *Let $(\bar{x}, \bar{y}) \in$ graph(F) be given. If $(\bar{x}, \bar{y})$ is a weak minimizer of VOP and $D^e_{\mathbb{T}} F(\bar{x}, \bar{y})$ is a $\mathbb{T}$-variation of F at $(\bar{x}, \bar{y})$, then $(\bar{x}, \bar{y})$ is a solution of $\mathbb{T}$-GVVIP.*

Proof. Suppose that there is an $x \in K$ such that

$$D^e_{\mathbb{T}} F(\bar{x}, \bar{y})(x - \bar{x}) \in -\text{int } C.$$

Since $D^e_{\mathbb{T}} F(\bar{x}, \bar{y})$ is a $\mathbb{T}$-variation of F at $(\bar{x}, \bar{y})$, there exists a $y \in F(x)$ such that $y - \bar{y} \in -\text{int } C$, that is, $(\bar{x}, \bar{y})$ is not a weak minimizer of VOP. This completes the proof. ■

Theorem 4.4. *Let $(\bar{x}, \bar{y}) \in \text{graph}(F)$ be given. If $(\bar{x}, \bar{y})$ is a weak minimizer of VOP and the S-epiderivative $D^e_S F(\bar{x}, \bar{y})$ exists, then*

$$D^e_S F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\text{int } C \quad \textit{for all} \quad x \in K.$$

Proof. Suppose that there is an $x \in K$ such that

$$y := D_S^e F(\bar{x}, \bar{y})(x - \bar{x}) \in -\text{int } C. \tag{4.7}$$

By definition of the S-epiderivative $(x-\bar{x}, y) \in \text{epi}(D_S^e F(\bar{x}, \bar{y})) = S(\text{epi}(F); (\bar{x}, \bar{y}))$. Then, there exist sequences $\{t_n\} \subset (0, \infty)$, $\{x_n\} \subset K$ with $x_n \to \bar{x}$, and $y_n \in F(x_n) + C$ such that

$$t_n((x_n, y_n) - (\bar{x}, \bar{y})) \to (x - \bar{x}, y).$$

Hence, we have

$$t_n(y_n - \bar{y}) \to y \in -\text{int } C.$$

Therefore, there exists an integer $N \in \mathbb{N}$ such that

$$t_n(y_n - \bar{y}) \in -\text{int } C \quad \text{for all} \quad n \geq N$$

resulting in

$$y_n - \bar{y} \in -\text{int } C \quad \text{for all} \quad n \geq N$$

which contradicts that $(\bar{x}, \bar{y})$ is a weak minimizer of VOP. ■

5. Existence of Solutions of Generalized Vector $\mathbb{T}$-Variational Inequalities

We shall use the following particular forms of Corollaries 3.2 and 4.1 in [23] to establish the existence results for solutions of $\mathbb{T}$-GVVIP.

Theorem A. ([23]). *Let $\mathcal{X}$ be a nonempty convex subset of a Hausdorff topological vector space $\mathcal{E}$. Let $\mathcal{S} : \mathcal{X} \to 2^{\mathcal{X}}$ be a set-valued map such that the following conditions hold:*

- (i) *For all $x \in \mathcal{X}$, $x \notin \mathcal{S}(x)$ and $\mathcal{S}(x)$ is convex;*
- (ii) *For all $y \in \mathcal{X}$, $\mathcal{S}^{-1}(y) = \{x \in \mathcal{X} : y \in \mathcal{S}(x)\}$ is open in $\mathcal{X}$;*
- (iii) *There exist a nonempty compact convex subset $\mathcal{C} \subseteq \mathcal{X}$ and a nonempty compact subset $\mathcal{K}$ of $\mathcal{X}$ such that for each $x \in \mathcal{X} \setminus \mathcal{K}$, there exists $\hat{y} \in \mathcal{C}$ satisfying $x \in \mathcal{S}^{-1}(\hat{y})$.*

Then, there exists a point $\bar{x} \in \mathcal{X}$ such that $\mathcal{S}(\bar{x}) = \emptyset$.

Definition 5.1. ([24]). Let $\mathcal{E}$ be a topological vector space and let $\mathcal{C}$ be a lattice with a minimal element, denoted by **0**. A mapping $\Phi : 2^{\mathcal{E}} \to \mathcal{C}$ is called a *measure of noncompactness* provided that the following conditions hold for any $M, N \in 2^{\mathcal{E}}$:

- (a) Φ ($\overline{\text{co}}M) = \Phi(M)$, where $\overline{\text{co}}M$ denotes the closed convex hull of M.

(b) $\Phi(M) = \mathbf{0}$ if and only if M is precompact.

(c) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

Definition 5.2 ([24]). Let $\mathcal{E}$ be a topological vector space, $\mathcal{X} \subseteq \mathcal{E}$, and let Φ be a measure of noncompactness on $\mathcal{E}$. A set-valued map (correspondence) $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{E}}$ is called Φ-*condensing* provided that if $M \subseteq \mathcal{X}$ with $\Phi(\mathcal{T}(M)) \geq \Phi(M)$ then M is relative compact, that is, $\overline{M}$ is compact.

Remark 5.1. ([23]). Note that every set-valued map defined on a compact set is Φ-condensing for any measure of noncompactness Φ. If $\mathcal{E}$ is locally convex, then a compact set-valued map (that is, $\mathcal{T}(\mathcal{X})$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $\mathcal{S} : \mathcal{X} \to 2^{\mathcal{E}}$ is Φ-condensing and $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{E}}$ satisfies $\mathcal{T}(x) \subseteq \mathcal{S}(x)$ for all $x \in \mathcal{X}$, then $\mathcal{T}$ is also Φ-condensing.

Theorem B. *Let $\mathcal{X}$ be a nonempty closed convex subset of a Hausdorff topological vector space $\mathcal{E}$ and let Φ be a measure of noncompactness on $\mathcal{E}$. Let $\mathcal{S} : \mathcal{X} \to 2^{\mathcal{X}}$ be a set-valued map such that the following conditions hold:*

(i) *For all $x \in \mathcal{X}$, $x \notin \mathcal{S}(x)$ and $\mathcal{S}(x)$ is convex;*

(ii) *For all $y \in \mathcal{X}$, $\mathcal{S}^{-1}(y) = \{x \in \mathcal{X} : y \in \mathcal{S}(x)\}$ is open in $\mathcal{X}$;*

(iii) *$\mathcal{S}$ is Φ-condensing.*

Then, there exists a point $\bar{x} \in \mathcal{X}$ such that $\mathcal{S}(\bar{x}) = \emptyset$.

We establish some existence results for solutions of $\mathbb{T}$-GVVIP.

Theorem 5.1. *Let X and Y be Hausdorff topological vector spaces, K be a nonempty convex subset of X and $F : K \to 2^Y$ be a C-convex set-valued map with nonempty convex values such that for all $y \in Y$, $F^{-1}(y) = \{x \in K : y \in F(x)\}$ is open in K and the graph of F, graph$(F) = \{(x, y) \in K \times Y : y \in F(x)\}$ is closed in $K \times Y$. Let the Condition A be satisfied. For each $u \in K$, let $\{(x, y) \in K \times Y : D^e_{\mathbb{T}}F(x, y)(u - x) \notin -\text{int } C\}$ be a closed subset of $K \times Y$. Assume that there exist a nonempty compact convex subset $B_1 \times B_2 \subseteq K \times Y$ and a nonempty compact subset $D_1 \times D_2$ of $K \times Y$ such that for each $(\hat{u}, \hat{v}) \in K \times Y \setminus D_1 \times D_2$, there exists $(\hat{y}_1, \hat{y}_2) \in B_1 \times B_2$ satisfying $D^e_{\mathbb{T}}F(\hat{u}, \hat{v})(\hat{y}_1 - \hat{u}) \in -\text{int } C$ and $\hat{y}_2 \in F(\hat{v})$. Then, there exists a solution $(\bar{x}, \bar{y}) \in K \times Y$ with $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ of $\mathbb{T}$-GVVIP.*

Proof. Define a set-valued map $\mathcal{P} : K \times Y \to 2^K$ as

$$\mathcal{P}(x, y) = \{u \in K : D^e_{\mathbb{T}}F(x, y)(u - x) \in -\text{int } C\} \quad \text{for all } (x, y) \in K \times Y.$$

Then, $x \notin \mathcal{P}(x, y)$ for all $(x, y) \in K \times Y$. If $x \in \mathcal{P}(x, y)$, then

$$D^e_{\mathbb{T}}F(x, y)(0) \in -\text{int } C.$$

Since the $\mathbb{T}$-epiderivative is positively homogeneous and subadditive (Theorem 3.3), we have

$$0 = D^e_{\mathbb{T}} F(x,y)(0) \in -\text{int } C,$$

a contradiction.

Again, by using the positive homogeneity and subadditivity of the $\mathbb{T}$-epiderivative, it is easy to see that $\mathcal{P}(x,y)$ is convex for all $(x,y) \in K \times Y$.

By the assumption, the complement $[\mathcal{P}^{-1}(u)]^c = \{(x,y) \in K \times Y : D^e_{\mathbb{T}} F(x,y)(u-x) \notin -\text{int } C\}$ of $\mathcal{P}^{-1}(u) = \{(x,y) \in K \times Y : D^e_{\mathbb{T}} F(x,y)(u-x) \in -\text{int } C\}$ is closed in $K \times Y$.

Now, we define another set-valued map $\mathcal{S} : K \times Y \to 2^{K \times Y}$ as

$$\mathcal{S}(x,y) = \begin{cases} \mathcal{P}(x,y) \times F(x), & \text{if } (x,y) \in \text{graph}(F) \\ K \times F(x), & \text{if } (x,y) \notin \text{graph}(F). \end{cases}$$

Then, clearly $(x,y) \notin \mathcal{S}(x,y)$ for all $(x,y) \in K \times Y$ and $\mathcal{S}(x,y)$ is convex because $\mathcal{P}(x,y)$, $F(x)$ and K are convex. For all $(u,v) \in K \times Y$,

$$\begin{aligned} \mathcal{S}^{-1}(u,v) = & \left[\mathcal{P}^{-1}(u) \cap (K \times Y) \cap (F^{-1}(v) \times Y)\right] \\ & \bigcup \left[(K \times Y \setminus \text{graph}(F)) \cap (K \times Y) \cap (F^{-1}(v) \times Y)\right] \\ = & \left[\mathcal{P}^{-1}(u) \cap (F^{-1}(v) \times Y)\right] \bigcup \left[(K \times Y \setminus \text{graph}(F)) \cap (F^{-1}(v) \times Y)\right]. \end{aligned}$$

Since $\mathcal{P}^{-1}(u)$ is open in $K \times Y$, $F^{-1}(v)$ is open in K and $K \times Y \setminus \text{graph}(F)$ is open in $K \times Y$, we have $\mathcal{S}^{-1}(u,v)$ is open in $K \times Y$.

Then all the conditions of Theorem A (where $\mathcal{X} = K \times Y$ is nonempty convex subset of $X \times Y$) are satisfied. Hence, there exists $(\bar{x}, \bar{y}) \in K \times Y$ such that $\mathcal{S}(\bar{x}, \bar{y}) = \emptyset$. If $(\bar{x}, \bar{y}) \notin \text{graph}(F)$, then $K \times F(\bar{x}) = \emptyset$ which implies that either $K = \emptyset$ or $F(\bar{x}) = \emptyset$, a contradiction to our assumption that K is nonempty and $F(x)$ is nonempty for all $x \in K$. Therefore, $(\bar{x}, \bar{y}) \in \text{graph}(F)$ and so $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ such that

$$\mathcal{P}(\bar{x}, \bar{y}) \times F(\bar{x}) = \emptyset.$$

Since $\bar{y} \in F(\bar{x}) \neq \emptyset$, we have $\mathcal{P}(\bar{x}, \bar{y}) = \emptyset$. Thus,

$$D^e_{\mathbb{T}} F(\bar{x}, \bar{y})(u - \bar{x}) \notin -\text{int } C \quad \text{for all } u \in K.$$

This completes the proof. ∎

Remark 5.2.

(a) For all $y \in Y$, $F^{-1}(y) = \{x \in K : y \in F(x)\}$ is open in K if F is lower semicontinuous on Y.

(b) The graph of F, $\text{graph}(F) = \{(x, y) \in K \times Y : y \in F(x)\}$ is closed in $K \times Y$ if F is upper semicontinuous with nonempty compact values.

(c) The last assumption (coercivity condition) in the previous theorem holds if K is a compact convex subset of X and Y is a compact space.

Let $\mathcal{X}$ and $\mathcal{Z}$ be topological vector spaces and let U be a nonempty subset of $\mathcal{X}$. Let $G : U \to 2^{\mathcal{Z}} \setminus \{\emptyset\}$ be a set-valued map and $g : U \to \mathcal{Z}$ be a single-valued map. Recall that g is called a *selection* of G on U if $g(x) \in G(x)$ for all $x \in U$. Furthermore, the function g is called a *continuous selection* of G on U if it is a selection of G and continuous on U.

Theorem 5.2. *Let X and Y be Hausdorff topological vector spaces, K be a nonempty convex subset of X and $F : K \to 2^Y$ be a C-convex set-valued map with nonempty values such that it has a selection $f : K \to Y$. Let the Condition A be satisfied. For each $u \in K$, let $\{(x, y) \in K \times Y : D^e_{\mathbb{T}} F(x, f(x))(u - x) \notin -\text{int } C\}$ be a closed subset of $K \times Y$. Assume that there exist a nonempty compact convex subset $B \subseteq K$ and a nonempty compact subset D of K such that for each $\hat{u} \in K \setminus D$, there exists $\hat{y} \in B$ satisfying $D^e_{\mathbb{T}} F(\hat{u}, f(\hat{u}))(\hat{y} - \hat{u}) \in -\text{int } C$. Then, there exists a solution $(\bar{x}, \bar{y}) \in K \times Y$ with $\bar{x} \in K$ and $\bar{y} = f(\bar{x}) \in F(\bar{x})$ of $\mathbb{T}$-GVVIP.*

Proof. For each $x \in K$, we define a set-valued map $\mathcal{S} : K \to 2^K$ as

$$\mathcal{S}(x) = \{u \in K : D^e_{\mathbb{T}} F(x, f(x))(u - x) \in -\text{int } C\}.$$

Then, it is clear that $x \notin \mathcal{S}(x)$ for all $x \in K$. It can be easily seen that $\mathcal{S}(x)$ is convex for all $x \in K$. By our assumption

$$[\mathcal{S}^{-1}(u)]^c = \{(x, y) \in K \times Y : D^e_{\mathbb{T}} F(x, f(x))(u - x) \notin -\text{int } C\}$$

is closed in K. Thus all the conditions of Theorem A are satisfied and hence there exists $\bar{x} \in K$ such that $\mathcal{S}(\bar{x}) = \emptyset$, that is,

$$D^e_{\mathbb{T}} F(\bar{x}, f(\bar{x}))(u - \bar{x}) \notin -\text{int } C \quad \text{for all } u \in K.$$

Let $\bar{y} = f(\bar{x}) \in F(\bar{x})$. Then, there exists $\bar{x} \in K$ and $\bar{y} = f(\bar{x}) \in F(\bar{x})$ such that

$$D^e_{\mathbb{T}} F(\bar{x}, \bar{y})(u - \bar{x}) \notin -\text{int } C \quad \text{for all } u \in K.$$

This completes the proof. ■

Theorem 5.3. *Let X and Y be Hausdorff topological vector spaces, K be a nonempty closed convex subset of X and $F : K \to 2^Y$ be a C-convex set-valued map with nonempty convex values such that for all $y \in Y$, $F^{-1}(y) = \{x \in K :$*

$y \in F(x)\}$ is open in K and the graph of F, graph$(F) = \{(x,y) \in K \times Y : y \in F(x)\}$ is closed in $K \times Y$. Let the Condition A be satisfied. For each $u \in K$, let $\{(x,y) \in K \times Y : D^e_{\mathbb{T}}F(x,y)(u-x) \notin -\mathrm{int}\, C\}$ be a closed subset of $K \times Y$. Let Φ be a measure of noncompactness on $X \times Y$ and the set-valued map $T : (x,y) \mapsto K \times F(x)$ from $K \times Y$ to itself be Φ-condensing. Then, there exists a solution $(\bar{x},\bar{y}) \in K \times Y$ with $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ of $\mathbb{T}$-GVVIP.

Proof. Let $\mathcal{P}$ and $\mathcal{S}$ be the same as defined in the proof of Theorem 5.1. In view of Theorem B, it is sufficient to show that the set-valued map $\mathcal{S}$ is Φ-condensing. By the definition of $\mathcal{S}$, we have

$$\mathcal{S}(x,y) \subseteq T(x,y) = K \times F(x) \quad \text{for all } (x,y) \in K \times Y.$$

Since T is Φ-condensing, by Remark 5.2, $\mathcal{S}$ is Φ-condensing. This completes the proof. ■

By using the same argument as in the proof of Theorem 5.3, we can easily derive the following result.

Theorem 5.4. *Let X and Y be Hausdorff topological vector spaces, K be a nonempty closed convex subset of X and $F : K \to 2^Y$ be a C-convex set-valued map with nonempty values such that it has a selection $f : K \to Y$. For each $u \in K$, let $\{(x,y) \in K \times Y : D^e_{\mathbb{T}}F(x,f(x))(u-x) \notin -\mathrm{int}\, C\}$ be a closed subset of $K \times Y$. Let the Condition A be satisfied. Let Φ be a measure of noncompactness on $X \times Y$ and the set-valued map $T : (x,y) \mapsto K \times F(x)$ from $K \times Y$ to itself be Φ-condensing. Then, there exists a solution $(\bar{x},\bar{y}) \in K \times Y$ with $\bar{x} \in K$ and $\bar{y} = f(\bar{x}) \in F(\bar{x})$ of $\mathbb{T}$-GVVIP.*

6. Conclusion

This paper presents a unified approach to epiderivatives and its use in set optimization. Since the concept of an epiderivative depends on the used notion of a tangent cone, the known epiderivatives are quite different. In this article we work out the basic mathematical structure being necessary for epiderivatives in optimization. It turns out that one needs only some few properties of tangent cones for an efficient use of epiderivatives in optimization, as for optimality conditions and variational inequalities.

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