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MEAN SQUARE ERROR SYNCHRONIZATION IN NETWORKS WITH RING STRUCTURE

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Abstract. A class of stochastic networks with ring structure is considered in which the flow of information depends on a Markov chain. We find sufficient conditions such that the mean square differences between the even or odd or all units in the network will eventually tend to zero. Such probabilistic synchronization then leads to pattern formation in our networks. Although we have discussed only one or two colored patterns, it is hoped that our investigations will lead to more interesting patterns in stochastic networks in the future.

1. Introduction

Patterns in nature are of great interests to many people. Therefore building mathematical models that exhibits (statistically) orderly outcomes are important issues in *neuromorphic engineering*. Some of the well known pattern-forming models are partial differential systems of the form

$$\frac{\partial u}{\partial t} = F(u, t),$$

where F is generically a differential operator and $u = u(x, t)$ are sought as its solutions, that exhibit the desired orderly distributions.

There are also models based on discrete time coupled dynamical systems (see e.g. [12-14]). Indeed, the Game of Life is a cellular automaton and it is a well known example which exhibits many interesting patterns. In [1], a network model is built and the concept of synchronization is introduced to explain the formation of patterns over time. Roughly, we imagine there is a network of (finitely many or infinitely

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many) compartments (patches, colonies, neural units, etc.) $..., u_1, u_2, ..., u_n, ...$. Each compartment u_i has a state value depending on the time period it is in. Therefore we may denote the state value by $u_i^{(t)}$ where t is a nonnegative integer. The compartments interact with each other in a given manner at the time period t and subsequently the state values changes. Such interaction can therefore be expressed as

$$u^{(t+1)} = G\left(u^{(t)}, t\right),$$

where $u^{(t)}$ is the vector $\left(..., u_1^{(t)}, ..., u_n^{(t)}, ...\right)^\dagger$ and G is generically a function. Given an initial distribution $u^{(0)}$, we may find $u^{(1)}, u^{(2)}, \ldots$, from the above governing equation and ultimately $u^{(0)}$ may evolve into one $u^{(\infty)}$ with orderly distribution (e.g. $(..., 1, 0, 0, 1, 0, 0, ...)^\dagger$, where the dagger denotes the transposition). If this is the case, we may say that a pattern formation is observed. There are perhaps different manners to describe the mechanism of pattern formation, but one plausible idea is to attribute pattern formation to synchronization and stability. More precisely, we may try to show that different groups of neural units synchronize in the sense that any two u_i, u_j in the same group behaves like $u_i^{(t)} \approx u_j^{(t)}$ for all large t, or more precisely, $\lim_{t\to\infty} \left|u_i^{(t)} - u_j^{(t)}\right| = 0$. Then additional stability conditions that guarantee convergence to a steady state will induce pattern formation.

In this paper, we are interested in networks consisting of n units $u_1, u_2, ..., u_n$ placed in a clockwise manner on the vertices of a regular n-gon so that each unit interacts with its two immediate neighbors. To be specific, let $u_2^{(t)}$ and $u_2^{(t+1)}$ be the state values of the second unit at two consecutive time periods. If $u_1^{(t)} > u_2^{(t)}$, then part of the information content of u_1 flows into u_2, so that

$$u_2^{(t+1)} - u_2^{(t)} = a\left(u_1^{(t)} - u_2^{(t)}\right).$$

Similarly, part of the information content of u_3 is shifted to u_2 if $u_3^{(t)} > u_2^{(t)}$. By superposition, we may then assume that

$$u_2^{(t+1)} - u_2^{(t)} = a\left(u_1^{(t)} - u_2^{(t)}\right) + a\left(u_3^{(t)} - u_2^{(t)}\right) = a\left(u_1^{(t)} - 2u_2^{(t)} + u_3^{(t)}\right). \tag{1}$$

Evidently, we have analyzed above only the case where the information contained by the neighbors u_1 and u_3 of u_2 is greater than the one in u_2. We notice that (1) remains valid in the other cases. For example, if $u_2^{(t)} > u_1^{(t)}$ and $u_3^{(t)} > u_2^{(t)}$ it is clear that some information will flow from u_3 into u_2 and from u_2 into u_1, respectively. Therefore $u_2^{(t+1)} = u_2^{(t)} + a\left(u_3^{(t)} - u_2^{(t)}\right) - a\left(u_2^{(t)} - u_1^{(t)}\right)$ and we get (1).

The parameter a is naturally called the 'diffusion constant'. In [1], we assume that a is a 'deterministic constant' (see also [2], [3], [10] for other deterministic models). However, in general, this parameter may be subject to random fluctuations and depends on the time period t. In this paper, we will consider a relatively simple case in which

$$a = a(t, r(t)),$$

satisfies:

(H1) $\{r(t)\}_{t=0}^{\infty}$ is a homogeneous Markov chain with the state space $\mathbf{Z}$, the set of integers, and the time space $\mathbf{N}$, the set of nonnegative integers, and the infinite transition matrix

$$Q = (q_{i,j})$$

defined by

$$q_{i,j} = P\{r(t+1) = j | r(t) = i\};$$

(H2) the transition matrix $Q = (q_{i,j})$ satisfies $q_{i,j} = 0$ for $j \in \mathbf{Z} \backslash \{i - m_0, ..., i, ..., i + m_0\}$, where $m_0 \in \mathbf{N}$;

(H3) for each $t \in \mathbf{N}$, $\{a(t,i)\}_{i \in \mathbf{Z}} \subset \mathbf{R}_+ = [0, \infty)$ is bounded.

Thus we may now write

$$u_2^{(t+1)} = u_2^{(t)} + a(t, r(t)) \left(u_1^{(t)} - 2u_2^{(t)} + u_3^{(t)} \right).$$

In view of the techniques that will be used to handle our network, we may go one step further if we assume that

$$u_2^{(t+1)} = f\left(t, u_2^{(t)}\right) + a(t, r(t)) \left(f\left(t, u_1^{(t)}\right) - 2f\left(t, u_2^{(t)}\right) + f\left(t, u_3^{(t)}\right) \right),$$

where $f : \mathbf{N} \times \mathbf{R} \to \mathbf{R}$ is a Lipschitz function which satisfies

$$|f(t,u) - f(t,v)| \le \Gamma(t) |u - v|, \ u, v \in \mathbf{R}, t \in \mathbf{N} \tag{2}$$

for some positive function $\Gamma : \mathbf{N} \to (0, \infty)$.

Finally, we also assume that the activation of each unit is uniform, so that the following dynamical system holds:

$$\begin{aligned}
u_1^{(t+1)} &= f\left(t, u_1^{(t)}\right) + a(t, r(t)) \left(f\left(t, u_n^{(t)}\right) - 2f\left(t, u_1^{(t)}\right) + f\left(t, u_2^{(t)}\right) \right), \\
u_2^{(t+1)} &= f\left(t, u_2^{(t)}\right) + a(t, r(t)) \left(f\left(t, u_1^{(t)}\right) - 2f\left(t, u_2^{(t)}\right) + f\left(t, u_3^{(t)}\right) \right), \\
&\vdots \\
u_n^{(t+1)} &= f\left(t, u_n^{(t)}\right) + a(t, r(t)) \left(f\left(t, u_{n-1}^{(t)}\right) - 2f\left(t, u_n^{(t)}\right) + f\left(t, u_1^{(t)}\right) \right).
\end{aligned} \tag{3}$$

If the coefficient $a(t,i)$ does not depend on i, that is $a(t,i) = a(t), t \in \mathbf{N}, i \in \mathbf{Z}$, then (3) is deterministic and, given the initial distribution $u^{(0)} = \left(u_1^{(0)}, ..., u_n^{(0)}\right)^\dagger$, we can generate the sequence $\left\{u^{(0)}, u^{(1)}, u^{(2)}, ...\right\}$ in a unique manner. We can also associate with u_i and u_j the state value sequences $\left\{u_i^{(0)}, u_i^{(1)}, u_i^{(t)}, ...\right\}$ and $\left\{u_j^{(0)}, u_j^{(1)}, u_j^{(2)}, ...\right\}$, respectively. If

$$\lim_{t\to\infty}\left|u_i^{(t)} - u_j^{(t)}\right| = 0,$$

the units u_i and u_j are said to be "completely in synchronization".[1] More generally, let Λ be a subset of $\{u_1, u_2, ..., u_n\}$. We say that the units $u_i, i \in \Lambda$ are in synchronization if all the mutually distinct units in Λ are in synchronization. In case $\Lambda = \{u_1, u_2, ..., u_n\}$, we also say that our network is in synchronization. In [1], conditions are obtained that guarantee synchronization in the deterministic case.

However, if the random walk assumption is general, the concept of synchronization will need modifications. In this paper, we will consider the mean square error synchronization. We will also derive several synchronization criteria for all the odd units $u_1, u_3, ...$, or all the even units $u_2, u_4, ...$, or all units to be in synchronization in our stochastic model:

$$\begin{array}{cccc} & A & B & \\ B & & & A \\ A & & & B \\ & B & A & \end{array}$$

A two colored pattern in a ring with 8 nodes

$$\begin{array}{cccc} & C & C & \\ C & & & C \\ C & & & C \\ & C & C & \end{array}$$

A one colored pattern in a ring with 8 nodes

We remark that complete synchronization of coupled neurons with ring structure or star-shaped structure have been studied. See e.g. [12] in which a continuous time model with ring structure is studied numerically; and [13] in which a discrete

[1] We may treat (3) as a system of coupled units each behaves in identical manner. Hence the terminolgy "complete synchronization" can be adopted from [12]. For simplicity, we will however use the more simple term "synchronization" in later discussions.

time model with star-shaped structure is studied. But to the best of our knowledge, probabilistic synchronization of coupled systems is new. Besides, in this paper, we emphasize the concept of complete synchronization in a subgroup of the units in the network which is essential in explaining the formation of patterns in ring structures.

2. Mean Square Error Synchronization

We need some preparatory terminologies to proceed and to explain the concept of mean square error synchronization.

Let H be a real separable Hilbert space and $L(H)$ the set of all bounded linear operators on H. We write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements in H and of operators belonging to $L(H)$. The adjoint of an operator $V \in L(H)$ will be denoted by V^* as usual. We will say that $S \in L(H)$ is nonnegative (and we will write $S \geq 0$) if S is self-adjoint and $\langle Sx, x \rangle \geq 0, x \in H$. We will also denote by $\mathcal{E}$ the Banach subspace of $L(H)$ formed by all self-adjoint operators, by $L^+(H)$ the cone of all nonnegative operators of $\mathcal{E}$ and by I_H (or simply I) the identity operator on H. For any $S \in L(H)$ we will denote by $\rho(S)$ the spectral radius of S.

Now, consider B a real Banach space endowed with the norm $\|\cdot\|$. A sequence $\{g_n\}_{n\in\mathbf{N}} \subset B$ is bounded if there exists $M > 0$ such that $\|g_n\| \leq M$ for all $n \in \mathbf{N}$. We will denote by l_B^∞ the set of all infinite sequences $g = \{g_i\}_{i\in\mathbf{Z}}, g_i \in B$ satisfying $\|g\|_\infty = \sup_{i\in\mathbf{Z}} \|g_i\| < \infty$. It is easy to see that l_B^∞ is a real linear space with the usual termwise addition and (real) scalar multiplication. Moreover, l_B^∞ is a Banach space with the norm $\|.\|_\infty$. In particular, if B is the Banach space $L(H)$ (respectively $\mathcal{E}$), we will denote by $l_{L(H)}^\infty$ (respectively $l_\mathcal{E}^\infty$) the Banach space l_B^∞.

If H is the Hilbert space $\mathbf{R}^n, n \in \mathbf{N}, n \geq 2$, endowed with the inner product $\langle x, y \rangle = x_1y_1 + x_2y_2 + ... + x_ny_n$, $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n), x, y \in \mathbf{R}^n$ and the norm $\|x\| = \sqrt{\langle x, x \rangle}$, then the more suggestive notation $SL(\mathbf{R}^n)$ is used for the Banach subspace $\mathcal{E} \subset L(H)$. Correspondingly, the Banach space $l_\mathcal{E}^\infty$ will be denoted by $l_{SL(\mathbf{R}^n)}^\infty$.

We say that the element $X \in l_\mathcal{E}^\infty$ is nonnegative iff $X(i) \in L^+(H)$ for all $i \in \mathbf{Z}$ and we write $X \geq 0$. Let us denote by $\mathcal{K}^\infty$ the cone of all nonnegative elements of $l_\mathcal{E}^\infty$. The cone $\mathcal{K}^\infty$ induces the following order on $l_\mathcal{E}^\infty$: $X \geq Y$ iff $X - Y \in \mathcal{K}^\infty$. An element $X \in \mathcal{K}^\infty$ is called positive if there exists $\gamma > 0$ such that $X \geq \gamma\Phi$, where $\Phi = (..., I_H, I_H, I_H, ...) \in l_\mathcal{E}^\infty$. A sequence $\{X_n\}_{n\in\mathbf{N}} \subset \mathcal{K}^\infty$ is *uniformly positive* iff there exists $\gamma > 0$ such that $X_n \geq \gamma\Phi$ for all $n \in \mathbf{N}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. If ξ is a random variable, then we will denote by $E(\xi)$ the expectation (mean) of ξ. For any σ algebra $\mathcal{G}$ of subsets of $\mathcal{F}$, $\mathcal{G} \subset \mathcal{F}$, we denote by $E[\xi|\mathcal{G}]$ the conditional expectation (mean) of ξ with respect to $\mathcal{G}$. If $\mathcal{G}_\eta$ is the σ-algebra generated by the random variable η, then we also use the notation $E[\xi|\eta]$ for the conditional expectation of ξ with respect to $\mathcal{G}_\eta$. We recall that $E\left[\xi|\eta=x\right]$, the conditional expectation on the event $\eta = x$, is defined as it follows:

$$E[\xi|\eta=x] = \frac{1}{P\{\eta(\omega)=x\}}\int \chi_{\{\eta(\omega)=x\}}\xi(\omega)P(d\omega), \tag{4}$$

if $P\{\omega|\eta(\omega)=x\}>0$ and $E\left[\xi|\eta=x\right]=0$, if $P\{\omega|\eta(\omega)=x\}=0$.

Assume that (H1) and (H2) hold and let us denote $P_j^t = P\left(r(t)=j\right)$ for $t\in\mathbf{N}$,$j\in\mathbf{Z}$. It is known (see [5] and [6]) that if ξ is an integrable, $\mathbf{R}^n$ valued random variable on $(\Omega,\mathcal{F},P)$, then

$$E[\xi\left(\omega\right)] = \sum_{j=-\infty}^{\infty} P_j^t E[\xi\left(\omega\right)|r(t)=j]. \tag{5}$$

Given $x\in\mathbf{R}^n$ and $k\in\mathbf{N}$ the system (3) with the initial condition

$$u^{(k)} = \left(u_1^{(k)},...,u_n^{(k)}\right)^\dagger = x, \tag{6}$$

will generate a sequence $\left\{u^{(t)}\right\}_{t\ge k}$. This sequence is called a solution of (3), (6).

Definition 1. If for any $l\in\mathbf{Z}$, $k\in\mathbf{N}$ and $x\in\mathbf{R}^n$, the components $u_i = \{u_i^{(t)}\}_{t\ge k}$ and $u_j = \{u_j^{(t)}\}_{t\ge k}$ of the solution $\left\{u^{(t)}\right\}_{t\ge k}$ of (3), (6) satisfy

$$\lim_{t\to\infty} E\left[\left|u_i^{(t)}-u_j^{(t)}\right|^2 | \, r(k)=l\right]=0,$$

then we will say that u_i and u_j are synchronized in conditional mean. If Λ is a subset of $\{u_1,...,u_n\}$ and u_i and u_j are synchronized in conditional mean for all $u_i,u_j\in\Lambda$, then we will say that system (3),(6) is Λ-synchronized in conditional mean. If u_i and u_j are synchronized in conditional mean for all $u_i,u_j\in\{u_1,u_2,...,u_n\}$, then we will say that system (3), (6) is synchronized in conditional mean.

Definition 2. If for any $k\in\mathbf{N}$ and $x\in\mathbf{R}^n$, the components u_i,u_j of the solution $\left\{u^{(t)}\right\}_{t\ge k}$ of (3), (6) satisfy

$$\lim_{t\to\infty} E\left|u_i^{(t)}-u_j^{(t)}\right|^2=0,$$

then u_i and u_j are said to be synchronized in mean square error.

As in Definition 1 we can define the notions of Λ-synchronization in mean square error and synchronization in mean square error, respectively.

3. Asymptotic Behavior of Backward Discrete Time Systems

Throughout this section we assume that (H1)-(H3) hold. Let $\{V_t\}_{t\in\mathbf{N}} \subset l^\infty_{L(H)}$. For any $t \in \mathbf{N}$, we introduce the function $\mathcal{V}_t : l^\infty_{\mathcal{E}} \to l^\infty_{\mathcal{E}}$,

$$\mathcal{V}_t\left(S\right)\left(i\right) = V_t^*\left(i\right) \sum_{j=i-m_0}^{i+m_0} q_{ij} S\left(j\right) V_t\left(i\right), T \in l^\infty_{\mathcal{E}}, i \in \mathbf{Z} \tag{7}$$

where $V_t\left(i\right)$ is the i-th component of $V_t \in l^\infty_{L(H)}$. It is clear that $\mathcal{V}_t$ is a well defined, linear and bounded operator. Moreover, we see that $\mathcal{V}_t\left(\mathcal{K}^\infty\right) \subset \mathcal{K}^\infty$. For any fixed $N_0 \in \mathbf{N}^* := \mathbf{N}/\{0\}$, we consider the following backward difference equation

$$Y_{N_0,t} = \mathcal{V}_t\left(Y_{N_0,t+1}\right), t < N_0 \tag{8}$$

$$Y_{N_0,N_0} = \Phi, \tag{9}$$

Let us define the following linear operator $T\left(t,k\right), t,k \in \mathbf{N}, t \geq k$:

$$\begin{aligned} T\left(t,k\right) &= \mathcal{V}_k \cdots \mathcal{V}_{t-1},\ t > k \\ T\left(k,k\right) &= I_{l^\infty_{\mathcal{E}}}, k \in \mathbf{N}. \end{aligned} \tag{10}$$

(Here $I_{l^\infty_{\mathcal{E}}}$ is the identity operator on $l^\infty_{\mathcal{E}}$). Clearly $T\left(t,k\right)\left(\mathcal{K}^\infty\right) \subset \mathcal{K}^\infty$. We see that $T\left(t,k\right) T\left(m,t\right) = T\left(m,k\right)$ for all $m,t,k \in \mathbf{N}$, $m \geq t \geq k$ and $T\left(t,k\right)$ is *monotone*, that is $T\left(t,k\right)\left(R\right) \leq T\left(t,k\right)\left(S\right)$ for all $R,S \in l^\infty_{\mathcal{E}}$ satisfying $R \leq S$.

It is not difficult to see that for any fixed $N_0 \in \mathbf{N}\backslash\{0\}$, the difference equation (8), (9) has a unique solution

$$Y_{N_0,t} = T\left(N_0,t\right)\left(\Phi\right). \tag{11}$$

The solution $\{Y_{N_0,t}\}_{t\leq N_0}$ of (8),(9) is said to possess the property (E) if there are $\beta > 1, \alpha \in (0,1)$ such that for all $N_0, t \in \mathbf{N}, t \leq N_0$,

$$\left\|Y_{N_0,t}\right\|_\infty \leq \beta\alpha^{N_0-t}.$$

The following result is known (see [11], [8], [9]).

Lemma 1. *If* $\Gamma \in L\left(l^\infty_{\mathcal{E}}\right)$ *and* $\Gamma\left(\mathcal{K}^\infty\right) \subset \mathcal{K}^\infty$ *then* $\left\|\Gamma\right\| = \left\|\Gamma\left(\Phi\right)\right\|_\infty$.

An infinite dimensional version of Theorem 3.4 in [4] is the following:

Theorem 1. *The following statements are equivalent:*

(a) *The solution* $\{Y_{N_0,k}\}_{k\leq N_0}$ *of (8), (9) has the property (E).*

(b) *There are* $\beta > 1$ *and* $\alpha \in (0,1)$ *such that*

$$\|T(t,k)\| \leq \beta\alpha^{t-k}.$$

for all $t,k \in \mathbf{N}$,$t \geq k$. (*Here* $\|T(t,k)\| = \sup_{\|S\|_\infty = 1, S \in l^\infty_{\mathcal{E}}} \|T(t,k)(S)\|_\infty$.)

(c) *The equation*

$$Z_t = \mathcal{V}_t(Z_{t+1}) + \Phi \tag{12}$$

has a unique, bounded on $\mathbf{N}$ *and uniformly positive solution* $\{Z_t\}_{t\in\mathbf{N}} \subset \mathcal{K}^\infty$, *i.e. there exists* $M > 0$, *such that (12) has a unique solution which satisfies*

$$\Phi \leq Z_t \leq M\Phi, t \in \mathbf{N}. \tag{13}$$

The proof is quite similar to that provided in [4, Theorem 3.4]. However we note that, unlike our case, the ordered Banach space considered in [4] is finite dimensional and the cone of its nonnegative elements satisfies a regularity condition. This regularity condition was required to ensure the convergence of several series. In our situation the ordered Banach space $l^\infty_{\mathcal{E}}$ is infinite dimensional and the cone $\mathcal{K}^\infty$ is not regular (that is a monotone decreasing and bounded below sequence from $l^\infty_{\mathcal{E}}$, $x_1 \geq x_2 \geq ... \geq x_t \geq ... \geq x$ is not necessarily convergent in $l^\infty_{\mathcal{E}}$).

For this reason, we sketch the proof of the result stated above and we will see that the series convergence can be obtained when the regularity condition is missing.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from the properties of the operator $T(t,m)$, $m,t \in \mathbf{N}$, $m \geq t$, (11) and from Lemma 1.

(b) $\Rightarrow$ (c): Assume that (b) holds. Then

$$\sum_{N_0=t}^{\infty} \|T(N_0,t)(\Phi)\|_\infty \leq \sum_{N_0=t}^{\infty} \beta\alpha^{N_0-t} = \beta\frac{1}{1-\alpha}$$

and $Z_t = \sum_{N_0=t}^{\infty} T(N_0,t)(\Phi)$ is convergent in $l^\infty_{\mathcal{E}}$. It is easy to see that $Z_t \in \mathcal{K}^\infty$.

Since $\mathcal{V}_t$ is continuous on $l^\infty_{\mathcal{E}}$, it follows that Z_t verifies (12) and (13) is satisfied for $M = \beta/(1-\alpha)$ and any $t \in \mathbf{N}$.

Assuming that $Z_t, W_t \in \mathcal{K}^\infty$ are two uniformly positive and bounded on $\mathbf{N}$ solutions of (12) and iterating, we deduce that

$$\|Z_t - W_t\|_\infty \leq (\|Z_m\|_\infty + \|W_m\|_\infty)\|T(m,t)(\Phi)\|_\infty \leq 2M\beta\alpha^{m-t}$$

for all $m \geq t, m,t \in \mathbf{N}$. As $m \to \infty$, we obtain $\|Z_t - W_t\|_\infty = 0$ for all $t \in \mathbf{N}$. It follows the uniqueness of the solution.

(c) $\Rightarrow$ (b): Assume that (12) has a solution $Z_t \in \mathcal{K}^\infty$, which satisfies the condition (13). We set $f_{N_0,t} = T(N_0,t)(Z_{N_0}), N_0 \geq t, t, N_0 \in \mathbf{N}$. By (12) and (13) we obtain $\left(1-\frac{1}{M}\right) f_{N_0,t} \geq f_{N_0+1,t}$. Iterating we get $f_{N_0,t} \leq \left(1-\frac{1}{M}\right)^{N_0-t} Z_t$, for all $N_0 \geq t, t, N_0 \in \mathbf{N}$. Using (13) and the monotonicity of $T(t,m)$ we deduce that (b) holds.

In the time invariant case, where $V_t = V \in l^\infty_{L(H)}$ for any $t \in \mathbf{N}$, we use the notation $\mathcal{V}$ for $\mathcal{V}_t$ and we obtain the following result:

Proposition 1. *The following statements are equivalent:*

(a) *The solution* $\{Y_{N_0,t}\}_{t\leq N_0}$ *of* (8), (9) *has the property (E).*

(b) $\rho(\mathcal{V}) < 1$, *where* $\rho(\mathcal{V})$ *denotes the spectral radius of* $\mathcal{V}$.

(c) *There exists a unique solution* $Z \in \mathcal{K}^\infty$ *of the equation*

$$Z = \mathcal{V}(Z) + \Phi \tag{14}$$

Proof. (a) $\Leftrightarrow$ (b): First we note that, in the time invariant case, we have $T(t,m) = \mathcal{V}^{t-m}$. In view of Theorem 1, statement (a) is equivalent to "there are $\beta > 1$ and $\alpha \in (0,1)$ such that $\left\|\mathcal{V}^{t-m}\right\| \leq \beta\alpha^{t-m}$ for all $t, m \in \mathbf{N}, t \geq m$". Hence, if (a) holds, then $\rho(\mathcal{V}) = \lim_{t\to\infty} \sqrt[t]{\|\mathcal{V}^t\|} \leq \alpha < 1$ and (b) is clearly true. The converse follows immediately in the same manner.

(a) $\Leftrightarrow$ (c): Since $Z_t = \sum\limits_{N_0=t}^{\infty} T(N_0,t)(\Phi) = \sum\limits_{k=1}^{\infty} \mathcal{V}^k(\Phi) + \Phi$ does not depend on t, the equivalence follows from the proof of Theorem 1.

The above proposition was proved in [8] for $H = \mathbf{R}$.

Remark 1. From the proof of Theorem 1 (respectively of Proposition 1), it follows easily that if equation (12) (respectively (14)) has a bounded on $\mathbf{N}$ and uniformly positive (respectively positive) solution, then such a solution is unique.

4. The Special Case $n = 6$

To motivate the main results, let us consider system (3) in the case $n = 6$. We may easily check that, for all $t \geq k$,

$$\begin{pmatrix} u_1^{(t+1)} - u_5^{(t+1)} \\ u_2^{(t+1)} - u_4^{(t+1)} \end{pmatrix} = A_1(t, r(t)) \begin{pmatrix} f\left(t, u_1^{(t)}\right) - f\left(t, u_5^{(t)}\right) \\ f\left(t, u_2^{(t)}\right) - f\left(t, u_4^{(t)}\right) \end{pmatrix}, \tag{15}$$

$$\begin{pmatrix} u_2^{(t+1)} - u_6^{(t+1)} \\ u_3^{(t+1)} - u_5^{(t+1)} \end{pmatrix} = A_1 (t, r(t)) \begin{pmatrix} f\left(t, u_2^{(t)}\right) - f\left(t, u_6^{(t)}\right) \\ f\left(t, u_3^{(t)}\right) - f\left(t, u_5^{(t)}\right) \end{pmatrix}, \tag{16}$$

$$\begin{pmatrix} u_3^{(t+1)} - u_1^{(t+1)} \\ u_4^{(t+1)} - u_6^{(t+1)} \end{pmatrix} = A_1 (t, r(t)) \begin{pmatrix} f\left(t, u_3^{(t)}\right) - f\left(t, u_1^{(t)}\right) \\ f\left(t, u_4^{(t)}\right) - f\left(t, u_6^{(t)}\right) \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} u_1^{(t+1)} - u_2^{(t+1)} \\ u_6^{(t+1)} - u_3^{(t+1)} \\ u_5^{(t+1)} - u_4^{(t+1)} \end{pmatrix} = B_1 (t, r(t)) \begin{pmatrix} f\left(t, u_1^{(t)}\right) - f\left(t, u_2^{(t)}\right) \\ f\left(t, u_6^{(t)}\right) - f\left(t, u_3^{(t)}\right) \\ f\left(t, u_5^{(t)}\right) - f\left(t, u_4^{(t)}\right) \end{pmatrix}, \tag{18}$$

where

$$A_1 (t, i) = \begin{pmatrix} 1 - 2a(t, i) & a(t, i) \\ a(t, i) & 1 - 2a(t, i) \end{pmatrix}, t \in \mathbf{N}, i \in \mathbf{Z},$$

$$B_1 (t, i) = \begin{pmatrix} 1 - 3a(t, i) & a(t, i) & 0 \\ a(t, i) & 1 - 2a(t, i) & a(t, i) \\ 0 & a(t, i) & 1 - 3a(t, i) \end{pmatrix}, t \in \mathbf{N}, i \in \mathbf{Z}.$$

Let us discuss (15). The initial distribution (at the time period k) of the system (15) is $w = \begin{pmatrix} x_1 - x_5 \\ x_2 - x_4 \end{pmatrix}$, where $x = (x_1, x_2, x_3, x_4, x_5, x_6)^\dagger$ is the initial distribution of the system (3) at the moment $t = k$. Obviously,

$$\begin{aligned} &\begin{pmatrix} \left|u_1^{(t+1)} - u_5^{(t+1)}\right| \\ \left|u_2^{(t+1)} - u_4^{(t+1)}\right| \end{pmatrix} \\ &\leq \begin{pmatrix} |1-2a(t, r(t))| & |a(t, r(t))| \\ |a(t, r(t))| & |1-2a(t, r(t))| \end{pmatrix} \cdot \begin{pmatrix} \left|f\left(t, u_1^{(t)}\right) - f\left(t, u_5^{(t)}\right)\right| \\ \left|f\left(t, u_2^{(t)}\right) - f\left(t, u_4^{(t)}\right)\right| \end{pmatrix} \end{aligned} \tag{19}$$

We adopt the following notation $|w| = \begin{pmatrix} |x_1 - x_5| \\ |x_2 - x_4| \end{pmatrix}$,

$$U(t+1) = \begin{pmatrix} \left|u_1^{(t+1)} - u_5^{(t+1)}\right| \\ \left|u_2^{(t+1)} - u_4^{(t+1)}\right| \end{pmatrix},$$

$$F(t,U(t)) = \begin{pmatrix} \left|f\left(t,u_1^{(t)}\right) - f\left(t,u_5^{(t)}\right)\right| \\ \left|f\left(t,u_2^{(t)}\right) - f\left(t,u_4^{(t)}\right)\right| \end{pmatrix}, \text{ and}$$

$$A(t,i) = \begin{pmatrix} |1-2a(t,i)| & |a(t,i)| \\ |a(t,i)| & |1-2a(t,i)| \end{pmatrix}.$$

Then (19) may be written as

(20) $$U(t+1) \leq A(t,r(t))\, F(t,U(t))$$

and

(21) $$F(t,U(t)) \leq \Gamma(t)\, U(t)$$

for all $t \in \mathbf{N}$.

In what follows we will use the following order on $\mathbf{R}^n$: $x \geq y$ iff $x_i \geq y_i$ for $i = 1,2,..,n$.

Definition 3. We say that $S \in SL(\mathbf{R}^n)$ has the property **(P1)** iff $\langle Su,v\rangle \geq 0$ for all $u,v \in \mathbf{R}^n$, $u,v \geq 0$.

Remark 2. If $S,L \in SL(\mathbf{R}^n)$ have the property **(P1)**, then $\langle L^*SLu,u\rangle \leq \langle L^*SLv,v\rangle$ for all $u,v \in \mathbf{R}^n$, $0 \leq u \leq v$. Indeed, it is clear that $\langle Su,u\rangle \leq \langle Sv,v\rangle$ for all $u,v \in \mathbf{R}^n$ such that $0 \leq u \leq v$. Since $Lu \leq Lv$ for all $0 \leq u \leq v$, we get $\langle SLu,Lu\rangle \leq \langle SLv,Lv\rangle$ and the conclusion follows.

Let $h \in l^\infty_{SL(\mathbf{R}^2)}$ such that $h(i)$ has the property **(P1)** for any $i \in \mathbf{Z}$. Evidently, $A(t,i)$ has the property **(P1)** for any $t \in \mathbf{N}$ and $i \in \mathbf{Z}$. Using Remark 2, (20) and (21), we get

$$\begin{aligned} & E\left[\langle h(r(t+1))\,U(t+1), U(t+1)\rangle \,|r(k)=l\right] \\ \leq\ & E\left[\langle A^*(t,r(t))\,h(r(t+1))\,A(t,r(t))\,F(t,U(t)), F(t,U(t))\rangle \,|r(k)=l\right] \\ \leq\ & \Gamma(t)^2\, E\left[\langle A^*(t,r(t))\,h(r(t+1))\,A(t,r(t))\,U(t), U(t)\rangle \,|r(k)=l\right] \end{aligned}$$

for all $l \in \mathbf{Z}$. Applying the properties of a Markov chain (see also the proof of Theorem 1 in [8]), we see that the last term is just

$$\Gamma(t)^2 \sum_{j=-\infty}^{\infty} E\left[\left\langle \left[q_{r(t),j} A^*(t,r(t))\,h(j)\,A(t,r(t))\right] U(t), U(t)\right\rangle |r(k)=l\right].$$

Let us consider the mapping $\mathcal{L}_t : l^\infty_{SL(\mathbf{R}^2)} \to l^\infty_{SL(\mathbf{R}^2)}$ defined by

(22) $$(\mathcal{L}_t h)(i) = \Gamma(t)^2 \sum_{j=-\infty}^{\infty} q_{i,j} A^*(t,i)\,h(j)\,A(t,i),\ i \in \mathbf{Z}, h \in l^\infty_{SL(\mathbf{R}^2)}.$$

By (H1)-(H3), it follows that $\mathcal{L}_t \in L\left(l^{\infty}_{SL(\mathbf{R}^2)}\right)$. We note that if $h(i)$ has the property (**P1**) for any $i \in \mathbf{Z}$, then $\mathcal{L}_t(h)(i)$ has the property (**P1**) for any $i \in \mathbf{Z}$ and $t \in \mathbf{N}$. Moreover, $\mathcal{L}_t(\mathcal{K}^{\infty}) \subset \mathcal{K}^{\infty}$ ($\mathcal{L}_t$ is a nonnegative operator on the ordered Banach space $l^{\infty}_{SL(\mathbf{R}^2)}$). Hence

$$\begin{aligned} &E\left[\langle h(r(t+1))U(t+1), U(t+1)\rangle \,|r(k)=l\right] \\ &\leq E\left[\langle \mathcal{L}_t(h)(r(t))U(t), U(t)\rangle \,|r(k)=l\right]. \end{aligned}$$

Now it is easy to verify by induction that

$$\begin{aligned} &E\left[\langle h(r(t+1))U(t+1), U(t+1)\rangle \,|r(k)=l\right] \\ &\leq \langle \mathcal{L}_k \cdots \mathcal{L}_{t-1}\mathcal{L}_t(h)(l)|w|, |w|\rangle. \end{aligned} \tag{23}$$

We just proved the following theorem.

Theorem 2. *Assume that (H1)-(H3) hold. If* $u = \left\{u^{(t)}\right\}_{t \geq k}$ *is the solution of (3),(6) and* $U, |w|$ *are defined as above, then for all* $l \in \mathbf{Z}$, $k \in \mathbf{N}$ *and* $x \in \mathbf{R}^6$,

$$E\left[\|U(t+1)\|^2 \,|r(k)=l\right] \leq \langle \mathcal{L}_k \cdots \mathcal{L}_{t-1}\mathcal{L}_t(\Phi)(l)|w|, |w|\rangle. \tag{24}$$

Obviously (24) is obtained by taking $h = \Phi$ in (23) (i.e. $h(i) = I_2, i \in \mathbf{Z}$).
A direct consequence of Theorems 1 and 2 is the following result.

Theorem 3. *Assume that the hypotheses of the above theorem hold. If there exists a unique uniformly positive and bounded on* $\mathbf{N}$ *solution* $\{Z_t\}_{t \in \mathbf{N}}$ *of the equation*

$$Z_t = \mathcal{L}_t Z_{t+1} + \Phi, \tag{25}$$

then there exist $\beta > 1$ *and* $\alpha \in (0,1)$ *such that for all* $l \in \mathbf{Z}$, $x \in \mathbf{R}^6$ *and* $t, k \in \mathbf{N}$, $t \geq k$,

$$E\left[\|U(t)\|^2 \,|r(k)=l\right] \leq \beta\alpha^{t-k}\|w\|^2. \tag{26}$$

Proof. Assume that there is a unique, uniformly positive and bounded on $\mathbf{N}$ solution $\{Z_t\}_{t \in \mathbf{N}}$ of (25). By taking $\mathcal{E} = SL\left(\mathbf{R}^2\right)$, $V_t(i) = A(t,i)$, $i \in \mathbf{Z}$, $t \in \mathbf{N}$ in Theorem 1, we see that $\mathcal{V}_t = \mathcal{L}_t, t \in \mathbf{N}$ and for all $l \in \mathbf{Z}$,

$$\begin{aligned} &\langle \mathcal{L}_k \cdots \mathcal{L}_{t-1}\mathcal{L}_{t-1}(\Phi)(l)|w|, |w|\rangle = \langle T(t,k)(\Phi)(l)|w|, |w|\rangle \\ &\leq \|T(t,k)(\Phi)\|_{\infty}\|w\|^2 \leq \|T(t,k)\|\,\|w\|^2 \leq \beta\alpha^{t-k}\|w\|^2. \end{aligned}$$

The conclusion follows from (24).

We note that under the hypotheses of Theorem 3, it follows that the system (3), (6) is $\{u_1, u_5\}$-, $\{u_2, u_4\}$- synchronized in conditional mean.

Since the operator $T(t,k)$ does not depend on the components $u_i^{(t)}, i \in \{1, .., 6\}$, of the solution $u^{(t)}$, we get the following result.

Corollary 1. *If the hypotheses of the above theorem hold, then, for any* $x \in \mathbf{R}^6$, *the system (3), (6) is* $\{u_1, u_5\}$-, $\{u_2, u_4\}$-, $\{u_2, u_6\}$-, $\{u_3, u_5\}$-, $\{u_3, u_1\}$- *and* $\{u_4, u_6\}$-*synchronized both in conditional mean and mean error.*

Proof. Applying Theorem 3, it follows that there are $\beta > 1$ and $\alpha \in (0,1)$ such that, for all $x \in \mathbf{R}^6, l \in \mathbf{Z}, \{i,j\} \in \{\{1,5\},\{2,4\}\}$ and $t, k \in \mathbf{N}, t \geq k$, we have

$$E\left[\left|u_i^{(t)} - u_j^{(t)}\right|^2 | \, r(k) = l\right] \leq \beta\alpha^{t-k} \|w\|^2 .$$

From (5), we get

$$E\left[\left|u_i^{(t)} - u_j^{(t)}\right|^2\right] = \sum_{l=-\infty}^{\infty} P_l^k E\left[\left|u_i^{(t)} - u_j^{(t)}\right|^2 | \, r(k) = l\right] \leq \beta\alpha^{t-k} \|w\|^2 .$$

Thus $\lim_{t\to\infty} E\left[\left|u_i^{(t)} - u_j^{(t)}\right|^2 | \, r(k) = l\right] = 0$ for any $l \in \mathbf{Z}$, respectively $\lim_{t\to\infty} E\left[\left|u_i^{(t)} - u_j^{(t)}\right|^2\right] = 0$ for $k \in \mathbf{N}$, $x \in \mathbf{R}^n$ and $\{i,j\} \in \{\{1,5\},\{2,4\}\}$. Hence system (3), (6) is $\{u_1, u_5\}$-, $\{u_2, u_4\}$-synchronized both in conditional mean and mean error. The conclusions of Theorem 3 remain true for systems (16),(17) and, reasoning as above, we see that (3),(6) is $\{u_2, u_6\}$-, $\{u_3, u_5\}$-, $\{u_3, u_1\}$- and $\{u_4, u_6\}$-synchronized both in conditional mean and mean error for any $x \in \mathbf{R}^6$.

Since synchronization is a transitive and reflexive relation, it is clear that if (3), (6) is $\{u_2, u_6\}$-, $\{u_3, u_5\}$-, $\{u_3, u_1\}$- and $\{u_4, u_6\}$-synchronized both in conditional mean and mean error, then we may come up with our *first important observation*, that it is $\{u_1, u_3, u_5\}$- and $\{u_2, u_4, u_6\}$-synchronized both in conditional mean and mean error (cf. [1]).

Example 1. Let $\xi_t, t \in \mathbf{N}$, be a sequence of independent random variables, with the same distribution function. We assume that they have the Binomial Distribution

$$B_n^p := \sum_{k=1}^{n} C_n^k p^k (1-p)^{n-k} \varepsilon_k ,$$

with parameters $n = 3$ and $p = 1/2$. Here n is the number of independent trials, p is the probability of success and ε_k is the Dirac measure on $\mathbf{Z}$ associated with $k \in \mathbf{Z}, \varepsilon_k(A) = \chi_A(k)$ for any $A \subset \mathbf{Z}$.

Now, let us consider the system (3), (6) for $n = 6$, where $f : \mathbf{N} \times \mathbf{R} \to \mathbf{R}$,$f(t,x) = x$, $a(t,i) = a(i) = 1/2\left(1 - \frac{1}{|i|+2}\right)$ and $\{r(t)\}_{t\in\mathbf{N}}$ is the random walk defined by

((*)) $$r(t) = \xi_0 + ... + \xi_t, t \in \mathbf{N}.$$

It is well known that $\{r(t)\}_{t\in\mathbf{N}}$ is a Markov process having the transition matrix $Q = (q_{ij})_{i,j\in\mathbf{Z}}$ defined by

$$q_{ij} = B_3^{1/2}(\{j-i\}) = \left(\frac{1}{2}\right)^3 \sum_{k=0}^{3} C_3^k \varepsilon_k(\{j-i\}).$$

A simple computation shows that $q_{ij} = 0$ if $i < j$ or $j > i+3$ and $q_{ii} = q_{ii+3} = 1/8$, $q_{ii+1} = q_{ii+2} = 3/8$. Hence the transition matrix is

$$Q = \begin{array}{c c} & \begin{array}{cccccc} & & & i & i+1 & i+2 \end{array} \\ i & \begin{pmatrix} . & . & . & . & . & . \\ 1/8 & 3/8 & 3/8 & 1/8 & 0 & ... \\ 0 & 1/8 & 3/8 & 3/8 & 1/8 & 0 \\ 0 & 0 & 1/8 & 3/8 & 3/8 & 1/8 \\ ... & 0 & 0 & 1/8 & 3/8 & 3/8 \\ .. & 0 & 0 & 0 & 1/8 & 3/8 \\ .. & .. & .. & .. & .. & .. \end{pmatrix} \end{array}.$$

Evidently, hypotheses (H1)-(H3) are satisfied. We will prove that (3), (6) is $\{u_1, u_3, u_5\}$- and $\{u_2, u_4, u_6\}$-synchronized both in conditional mean and mean error. First, we see that

$$A(i) = A(t,i) = \begin{pmatrix} \frac{1}{|i|+2} & 1/2\left(1 - \frac{1}{|i|+2}\right) \\ 1/2\left(1 - \frac{1}{|i|+2}\right) & \frac{1}{|i|+2} \end{pmatrix}.$$

Evidently

$$\|A(i)\| = \sup_{\|x\|=1, x=(x_1,x_2)\in\mathbf{R}^2} |\langle A(i)x, x\rangle| \le \frac{1}{|i|+2} + 1/2\left(1 - \frac{1}{|i|+2}\right) < \frac{3}{4}.$$

On the other hand, we notice that, under the above settings, the operator defined by (22) and denoted by $\mathcal{L}$ (because it does not depend on t) has the property

$$\|(\mathcal{L}h)(i)\| = \left\| \sum_{j=-\infty}^{\infty} q_{i,j} A^*(t,i)\, h(j)\, A(t,i) \right\| \le \left(\frac{3}{4}\right)^2 \sum_{j=-\infty}^{\infty} q_{i,j} \|h(j)\|.$$

This implies $\|\mathcal{L}\Phi\| = \sup_{i\in\mathbf{Z}} \|(\mathcal{L}\Phi)(i)\| \leq \left(\frac{3}{4}\right)^2$. Since $\mathcal{L}(\mathcal{K}^\infty) \subset \mathcal{K}^\infty$, it follows from Lemma 1 that $\|\mathcal{L}\| = \|\mathcal{L}\Phi\| \leq \left(\frac{3}{4}\right)^2$. Consequently, $r(\mathcal{L}) < 1$. In view of Proposition 1, Theorem 3 and Remark 1, system (3),(6) is $\{u_1, u_3, u_5\}$- and $\{u_2, u_4, u_6\}$-synchronized both in conditional mean and mean error.

The following computations illustrate the theoretical results. Generating randomly the values of the Markov process considered in this example, we may obtain the sequence $r(0) = 1, r(1) = 2, r(2) = 4, r(3) = 6, r(4) = 9, r(5) = 9, ..., r(13) = 21, \dots$.

The information contained by the six units of our network, at different time periods, are given below

$$\begin{array}{ccc} & \overset{t=0}{u_1^{(0)}\quad u_2^{(0)}} & \\ u_6^{(0)} & & u_3^{(0)} \\ & u_5^{(0)}\quad u_4^{(0)} & \end{array} = \begin{array}{ccc} & \overset{t=0}{1\quad 2} & \\ 6 & & 3 \\ & 5\quad 4 & \end{array} \to \begin{array}{ccc} & \overset{t=1}{3\quad 2} & \\ 4 & & 3 \\ & 5\quad 4 & \end{array} \to \dots$$

$$\to \begin{array}{ccc} & \overset{t=4}{3.3216\quad 3.2682} & \\ 3.6783 & & 3.3216 \\ & 3.7317\quad 3.6783 & \end{array} \to \dots \to \begin{array}{ccc} & \overset{t=14}{3.4882\quad 3.5109} & \\ 3.5117 & & 3.4882 \\ & 3.4890\quad 3.5117 & \end{array} \to \dots$$

For $t = 14$ and beyond, the $\{u_1, u_3, u_5\}$- and $\{u_2, u_4, u_6\}$-synchronization becomes “visible” since the values of the units u_1, u_3, u_5 are approximately 3.51 and the ones of u_2, u_4, u_6 are very close to 3.48.

Now we consider system (18) with the initial distribution $v = (x_1 - x_2, x_6 - x_3, x_5 - x_4)^\dagger$ (at the moment $t = k$), where $x = (x_1, x_2, x_3, x_4, x_5, x_6)^\dagger$ is the initial value of (3) . We denote

$$B(t,i) = \begin{pmatrix} |1 - 3a(t,i)| & |a(t,i)| & 0 \\ |a(t,i)| & |1 - 2a(t,i)| & |a(t,i)| \\ 0 & |a(t,i)| & |1 - 3a(t,i)| \end{pmatrix}$$

and we introduce the function $\mathcal{M}_t : l^\infty_{SL(\mathbf{R}^3)} \to l^\infty_{SL(\mathbf{R}^3)}$ defined by

$$(27) \qquad (\mathcal{M}_t h)(i) = \Gamma(t)^2 \sum_{j=-\infty}^{\infty} q_{i,j} B^*(t,i)\, h(j)\, B(t,i),\ i \in \mathbf{Z}, h \in l^\infty_{SL(\mathbf{R}^3)}.$$

Reasoning as above, we will obtain the analogue of Theorem 3 and Corollary 1, respectively.

Theorem 4. *Assume that* (*H1*)-(*H3*) *hold. If* $u = \left\{u^{(t)}\right\}_{t\geq k}$ *is the solution of* (*3*), (*6*) *and there is a unique, uniformly positive and bounded on* $\mathbf{N}$ *solution* $\{L_t\}_{t\in\mathbf{N}}$ *of*

$$L_t = \mathcal{M}_t L_{t+1} + \Phi,$$

then there exist $\beta > 1$ *and* $\alpha \in (0,1)$ *such that for all* $l \in \mathbf{Z}$, $x \in \mathbf{R}^6$ *and* $t, k \in \mathbf{N}$, $t \geq k$,

$$E\left[\sum_{\{i,j\} \in \{\{1,2\},\{6,3\},\{5,4\}\}} \left[u_i^{(t)} - u_j^{(t)}\right]^2 |r(k) = l\right] \leq \beta\alpha^{t-k} \|v\|^2 .$$

Corollary 2. *If the hypotheses of the above theorem hold, then , for any* $x \in \mathbf{R}^6$*, the system* (3)*,* (6) *is* $\{u_1, u_2\}$*-,* $\{u_6, u_3\}$*- and* $\{u_5, u_4\}$*-synchronized both in conditional mean and mean error.*

If the hypotheses of Theorems 3, 4 hold, then the system (3),(6) is synchronized both in conditional mean and mean error for any $x \in \mathbf{R}^6$. Indeed, if we consider the graph with vertices $1, 2, 3, 4, 5, 6$ and the edges $\{1, 5\}$, $\{2, 4\}$, $\{2, 6\}$, $\{3, 5\}$, $\{3, 1\}$, $\{4, 6\}$, $\{1, 2\}$, $\{6, 3\}$ and $\{5, 4\}$ we see that it is connected. The conclusion follows. Therefore we now have a one colored pattern formed in our stochastic network.

5. General Results

Now, we turn our attention to the cases $n = 2m$ or $n = 2m + 1$ where $m \geq 1$. For all $t \in \mathbf{N}$ and $i \in \mathbf{Z}$, we denote $\pi(t,i) = \begin{pmatrix} |1 - 3a(t,i)| & |a(t,i)| \\ |a(t,i)| & |1 - 3a(t,i)| \end{pmatrix}$, and we introduce the functions $\Lambda_t, \Psi_t : l^\infty_{\mathbf{R}} \to l^\infty_{\mathbf{R}}$ and $\Pi_t : l^\infty_{\mathbf{R}^2} \to l^\infty_{\mathbf{R}^2}$ defined by

$$\begin{aligned}
(\Lambda_t h)(i) &= [1 - 4a(t,i)]^2 \Gamma(t)^2 \sum_{j=-\infty}^{\infty} q_{i,j} h(j), \\
(\Psi_t h)(i) &= [1 - 3a(t,i)]^2 \Gamma(t)^2 \sum_{j=-\infty}^{\infty} q_{i,j} h(j), \\
(\Pi_t h)(i) &= \Gamma(t)^2 \sum_{j=-\infty}^{\infty} q_{i,j} \pi^*(t,i) h(j) \pi(t,i), i \in \mathbf{Z}.
\end{aligned}$$

Obviously $\Lambda_t, \Psi_t \in L(l^\infty_{\mathbf{R}}), \Pi_t \in L\left(l^\infty_{\mathbf{R}^2}\right)$ and Λ_t, Ψ_t, Π_t satisfy the requirements of Lemma 1.

Let us consider system (3), (6) in the cases $n = 2, 3, 4$.

Suppose $n = 2$. As in [1] it follows

$$\left|u_1^{(t+1)} - u_2^{(t+1)}\right| \leq \Gamma(t) |1 - 4a(t, r(t))| \left|u_1^{(t)} - u_2^{(t)}\right| .$$

If $n=3$ then

$$\begin{aligned}\left|u_1^{(t+1)}-u_2^{(t+1)}\right| &\leq \Gamma(t)\left|1-3a(t,r(t))\right|\left|u_1^{(t)}-u_2^{(t)}\right|,\\ \left|u_1^{(t+1)}-u_3^{(t+1)}\right| &\leq \Gamma(t)\left|1-3a(t,r(t))\right|\left|u_1^{(t)}-u_3^{(t)}\right|\end{aligned}$$

and for $n=4$ we have

$$\begin{pmatrix}\left|u_1^{(t+1)}-u_5^{(t+1)}\right|\\ \left|u_2^{(t+1)}-u_4^{(t+1)}\right|\end{pmatrix}\leq\Gamma(t)\,\pi(t,r(t))\begin{pmatrix}\left|u_1^{(t)}-u_5^{(t)}\right|\\ \left|u_2^{(t)}-u_4^{(t)}\right|\end{pmatrix}$$

Let $l\in\mathbf{Z}$, $t\in\mathbf{N}$, $t>k$. Arguing as in the proof of Theorem 2, we see that for $n=2$ we get

$$E\left[\left|u_1^{(t)}-u_2^{(t)}\right|^2|r(k)=l\right]\leq\left\langle\Lambda_k\cdots\Lambda_{t-1}(\phi)(l)\left|x_1-x_2\right|,\left|x_1-x_2\right|\right\rangle,$$

and for $n=3$ we obtain

$$\begin{aligned}E\left[\left|u_1^{(t)}-u_2^{(t)}\right|^2|r(k)=l\right] &\leq \left\langle\Psi_k\cdots\Psi_{t-1}(\phi)(l)\left|x_1-x_2\right|,\left|x_1-x_2\right|\right\rangle,\\ E\left[\left|u_1^{(t)}-u_3^{(t)}\right|^2|r(k)=l\right] &\leq \left\langle\Psi_k\cdots\Psi_{t-1}(\phi)(l)\left|x_1-x_3\right|,\left|x_1-x_3\right|\right\rangle,\end{aligned}$$

where $\phi=(...,1,1,1,...)\in l_{\mathbf{R}}^{\infty}$. If $n=4$ we have

$$\begin{aligned}&E\left[\left\|\begin{pmatrix}\left|u_1^{(t)}-u_2^{(t)}\right|\\ \left|u_4^{(t)}-u_3^{(t)}\right|\end{pmatrix}\right\|^2|r(k)=l\right]\\ &\leq\left\langle\Pi_k\cdots\Pi_{t-1}(\Phi)(l)\begin{pmatrix}\left|x_1-x_2\right|\\ \left|x_4-x_3\right|\end{pmatrix},\begin{pmatrix}\left|x_1-x_2\right|\\ \left|x_4-x_3\right|\end{pmatrix}\right\rangle.\end{aligned}$$

Let us introduce the following equations

$$z_t=\Lambda_t z_{t+1}+\phi, \tag{28}$$

$$z_t=\Psi_t z_{t+1}+\phi, \tag{29}$$

$$\mathcal{Z}_t=\Pi_t\mathcal{Z}_{t+1}+\Phi, \tag{30}$$

The next result is a version of Theorem 3 and Corollary 1 for the cases $n=2,3,4$. (The proof is very similar and will be omitted).

Theorem 5. *Assume that* (*H1*)-(*H3*) *hold.*

(a) *If* $n = 2$ (*respectively* $n = 3$) *and there exists a unique, uniformly positive and bounded on* $\mathbf{N}$ *solution* $\{z_t\}_{t\in\mathbf{N}}$ *of* (28) (*respectively* (29)), *then system* (3), (6) *is synchronized both in conditional mean and mean error for any* $x \in \mathbf{R}^2$ (*respectively* $x \in \mathbf{R}^3$).

(b) *If* $n = 4$ *and there is a unique, uniformly positive and bounded on* $\mathbf{N}$ *solution* $\{\mathcal{Z}_t\}_{t\in\mathbf{N}}$ *of* (30), *then system* (3), (6) *is synchronized both in conditional mean and mean error for any* $x \in \mathbf{R}^4$.

Now we consider the system (3) for $n = 2m, m \geq 3$. For any $t \in \mathbf{N}, i \in \mathbf{Z}$, we set

$$A_1(t,i) = \begin{pmatrix} 1-2a(t,i) & a(t,i) & 0 & 0 \\ a(t,i) & 1-2a(t,i) & a(t,i) & 0 \\ 0 & a(t,i) & .... & a(t,i) \\ 0 & 0 & a(t,i) & 1-2a(t,i) \end{pmatrix}_{m-1\times m-1}$$

and

$$B_1(t,i) = \begin{pmatrix} 1-3a(t,i) & a(t,i) & 0 & 0 \\ a(t,i) & 1-2a(t,i) & a(t,i) & 0 \\ 0 & a(t,i) & .... & a(t,i) \\ 0 & 0 & a(t,i) & 1-3a(t,i) \end{pmatrix}_{m\times m}.$$

It is easy to verify that

$$(31)\quad \begin{pmatrix} u_1^{(t+1)} - u_3^{(t+1)} \\ u_{2m}^{(t+1)} - u_4^{(t+1)} \\ u_{2m-1}^{(t+1)} - u_5^{(t+1)} \\ .... \\ u_{m+3}^{(t+1)} - u_{m+1}^{(t+1)} \end{pmatrix} = A_1(t, r(t)) \begin{pmatrix} f\left(t, u_1^{(t)}\right) - f\left(t, u_3^{(t)}\right) \\ f\left(t, u_{2m}^{(t)}\right) - f\left(t, u_4^{(t)}\right) \\ f\left(t, u_{2m-1}^{(t)}\right) - f\left(t, u_5^{(t)}\right) \\ .... \\ f\left(t, u_{m+3}^{(t)}\right) - f\left(t, u_{m+1}^{(t)}\right) \end{pmatrix},$$

respectively

$$(32)\quad \begin{pmatrix} u_1^{(t+1)} - u_2^{(t+1)} \\ u_{2m}^{(t+1)} - u_3^{(t+1)} \\ u_{2m-1}^{(t+1)} - u_4^{(t+1)} \\ .... \\ u_{m+2}^{(t+1)} - u_{m+1}^{(t+1)} \end{pmatrix} = B_1(t, r(t)) \begin{pmatrix} f\left(t, u_1^{(t)}\right) - f\left(t, u_2^{(t)}\right) \\ f\left(t, u_{2m}^{(t)}\right) - f\left(t, u_3^{(t)}\right) \\ f\left(t, u_{2m-1}^{(t)}\right) - f\left(t, u_4^{(t)}\right) \\ .... \\ f\left(t, u_{m+2}^{(t)}\right) - f\left(t, u_{m+1}^{(t)}\right) \end{pmatrix}.$$

The initial distributions, at the time period $t = k$, of the systems (31) and (32) are $w_{(m-1)} = (x_1 - x_3, x_{2m} - x_4, x_{2m-1} - x_5, ..., x_{m+3} - x_{m+1})^{\dagger}$ and $v_{(m)} = (x_1 - x_2, x_{2m} - x_3, x_{2m-1} - x_4, ..., x_{m+2} - x_{m+1})^{\dagger}$), respectively, where $x = (x_1, x_2, ..., x_{2m})^{\dagger}$ is the initial distribution of the system (3).

In what follows we will denote by $[M]_{p,q}$ the element on line p and column q of a given matrix M. As in the previous section, for any $i \in \mathbf{Z}$ and $t \in \mathbf{N}$, $A(t,i)$ (respectively $B(t,i)$), denotes the $(m-1) \times (m-1)$ (respectively $m \times m$) matrix defined by $[A(t,i)]_{p,q} = \left| [A_1(t,i)]_{p,q} \right|, p, q \in \{1, ..m-1\}$ (respectively $[B(t,i)]_{p,q} = \left| [B_1(t,i)]_{p,q} \right|, p, q \in \{1, ..m\}$).

Now we consider the function $\mathcal{L}_t : l^{\infty}_{SL(\mathbf{R}^{m-1})} \to l^{\infty}_{SL(\mathbf{R}^{m-1})}$ defined by formula (22), where $A(t,i)$ is defined above and $h \in l^{\infty}_{SL(\mathbf{R}^{m-1})}$. It is clear that $\mathcal{L}_t \in L\left(l^{\infty}_{SL(\mathbf{R}^{m-1})}\right)$, $\mathcal{L}_t(\mathcal{K}^{\infty}) \subset \mathcal{K}^{\infty}$ and $\mathcal{L}_t(h)(i), i \in \mathbf{Z}$, $t \in \mathbf{N}$ has the property **(P1)** if $h(i)$ has the property **(P1)** for all $i \in \mathbf{Z}$.

Analogously, we define the function $\mathcal{M}_t : l^{\infty}_{SL(\mathbf{R}^m)} \to l^{\infty}_{SL(\mathbf{R}^m)}$ by formula (27), where $B(t,i)$ is the matrix introduced above and $h \in l^{\infty}_{SL(\mathbf{R}^m)}$.

All the properties of $\mathcal{M}_t$ mentioned in the last section for $n = 6$ remain true (in a corresponding form) for $n = 2m$.

Reasoning exactly as in the case $n = 6$, we obtain the following result, which extends Theorems 3, 4 to the general case $n = 2m, m \geq 3$.

Theorem 6. *Assume that* (*H1*)-(*H3*) *hold and* $n = 2m$ *where* $m \geq 3$.

(i) *If there is a unique, uniformly positive and bounded on* $\mathbf{N}$ *solution* $\{Z_t\}_{t\in\mathbf{N}} \subset l^{\infty}_{SL(\mathbf{R}^{m-1})}$ *to*

$$Z_t = \mathcal{L}_t Z_{t+1} + \Phi, \tag{33}$$

then there are $\beta > 1$ *and* $\alpha \in (0,1)$ *such that for all* $t, k \in \mathbf{N}$,$t \geq k$ *and all* $l \in \mathbf{Z}$,

$$E\left[\sum_{\{i,j\}\in\{\{1,3\},\{2m,4\},\{2m-1,5\},....\{m+3,m+1\}\}} \left[u_i^{(t)} - u_j^{(t)}\right]^2 | r(k) = l\right] \leq \beta\alpha^{t-k} \left\|w_{(m-1)}\right\|^2.$$

(ii) *If there is a unique, uniformly positive and bounded on* $\mathbf{N}$ *solution* $\{L_t\}_{t\in\mathbf{N}} \subset l^{\infty}_{SL(\mathbf{R}^m)}$ *to*

$$L_t = \mathcal{M}_t L_{t+1} + \Phi, \tag{34}$$

then there are $\beta > 1$ *and* $\alpha \in (0,1)$ *such that for all* $t, k \in \mathbf{N}$ *that satisfy* $t \geq k$ *and all* $l \in \mathbf{Z}$,

$$E\left[\sum_{\{i,j\}\in\{\{1,2\},\{2m,3\},\{2m-1,4\},\ldots,\{m+2,m+1\}\}}\left[u_i^{(t)}-u_j^{(t)}\right]^2|r\left(k\right)=l\right]\leq\beta\alpha^{t-k}\left\|v_{(m)}\right\|^2.$$

A direct consequence of the above theorem is the following result.

Theorem 7. *Assume that* $(H1)$*-*$(H3)$ *hold and* $n=2m$ *where* $m\geq 3$*. If there exists a unique uniformly positive and bounded (on* $\mathbf{N}$*) solution* $\{Z_t\}_{t\in\mathbf{N}}\subset l^{\infty}_{SL(\mathbf{R}^{m-1})}$ *and* $\{L_t\}_{t\in\mathbf{N}}\subset l^{\infty}_{SL(\mathbf{R}^m)}$ *to* (33) *and* (34)*, respectively, then the solution of* (3)*,* (6) *is synchronized both in conditional mean and mean error for any* $x\in\mathbf{R}^{2m}$.

Proof. It is easy to see that Theorem 6 (i) implies that the solution of (3), (6) is $\{u_1,u_3\}$-, $\{u_{2m},u_4\}$-, $\{u_{2m-1},u_5\}$-, ..., $\{u_{m+3},u_{m+1}\}$-synchronized both in conditional mean and mean error. Analogously, applying Theorem 6 (ii), it follows that the solution of (3), (6) is $\{u_1,u_2\}$-, $\{u_{2m},u_3\}$-, $\{u_{2m-1},u_4\}$-, ..., $\{u_{m+2},u_{m+1}\}$-synchronized both in conditional mean and mean error. Now, we consider the graph with the vertices $1,2,\ldots,2m$ and the edges $\{1,3\}$, $\{2m,4\}$, $\{2m-1,5\}$, ..., $\{m+3,m+1\}$, $\{1,2\}$, $\{2m,3\}$, $\{2m-1,4\}$, ..., $\{m+2,m+1\}$. Obviously, this is a connected graph and it is easy to deduce that any solution of (3), (6) is synchronized both in conditional mean and mean error.

Proposition 2. *Assume that* $(H1)$*-*$(H3)$ *hold and* $n=2m$*, where* $m\geq 3$*. If there is a unique, uniformly positive and bounded on* $\mathbf{N}$ *solution* $\{Z_t\}_{t\in\mathbf{N}}\subset l^{\infty}_{SL(\mathbf{R}^{m-1})}$ *to (33), then the solution of* (3)*,* (6) *is* $\{u_1,u_3,u_5,\ldots,u_{2m-1}\}$*- and* $\{u_2,u_4,u_6,\ldots,u_{2m}\}$*-synchronized both in conditional mean and mean error.*

Proof. It is clear that system (3) is 'invariant' with respect to the numeration of its units. More precisely, as in [1], we define the rotation σ of a vector $(z_1,z_2,\ldots,z_n)$ by $\sigma(z_1)=z_n,\sigma(z_2)=z_1,\ldots,\sigma(z_n)=z_{n-1}$. In view of Theorem 6(i), the solution of (3), (6) is $\{u_1,u_3\}$-, $\{u_{2m},u_4\}$-, $\{u_{2m-1},u_5\}$-, ..., $\{u_{m+3},u_{m+1}\}$-synchronized both in conditional mean and mean error. After one rotation, we apply Theorem 6(i) again and we deduce that (3), (6) is $\{u_2,u_4\}$-, $\{u_1,u_5\}$-, $\{u_{2m},u_6\}$-, ..., $\{u_{m+4},u_{m+2}\}$-synchronized both in conditional mean and mean error. If we consider the graph with the vertices $1,2,\ldots,2m$ and the edges $\{u_1,u_3\}$, $\{u_{2m},u_4\}$, $\{u_{2m-1},u_5\}$, ..., $\{u_2,u_4\}$, $\{u_1,u_5\}$, $\{u_{2m},u_6\}$, ..., $\{u_{m+4},u_{m+2}\}$, then we may identify two connected components $\{1,3,5,\ldots,2m-1\}$ and $\{2,4,6,\ldots,2m\}$. The conclusion follows.

Now let us consider the system (3), (6) for $n=2m+1,m\geq 2$. For all $i\in\mathbf{Z}$ and $t\in\mathbf{N}$, we set

$$C_1(t,i)=\begin{pmatrix} 1-2a(t,i) & a(t,i) & 0 & 0 \\ a(t,i) & 1-2a(t,i) & a(t,i) & 0 \\ 0 & a(t,i) & .... & a(t,i) \\ 0 & 0 & a(t,i) & 1-3a(t,i) \end{pmatrix}_{m\times m}$$

and we define the matrix $C(t,i)$ by $[C(t,i)]_{p,q}=\left|[C_1(t,i)]_{p,q}\right|$, for all $p,q\in\{1,..m\}$.

We obtain the system

$$\begin{pmatrix} u_1^{(t+1)}-u_3^{(t+1)} \\ u_{2m+1}^{(t+1)}-u_4^{(t+1)} \\ u_{2m}^{(t+1)}-u_5^{(t+1)} \\ .... \\ u_{m+3}^{(t+1)}-u_{m+2}^{(t+1)} \end{pmatrix}=C_1(t,r(t))\begin{pmatrix} f\left(t,u_1^{(t)}\right)-f\left(t,u_3^{(t)}\right) \\ f\left(t,u_{2m+1}^{(t)}\right)-f\left(t,u_4^{(t)}\right) \\ f\left(t,u_{2m}^{(t)}\right)-f\left(t,u_5^{(t)}\right) \\ .... \\ f\left(t,u_{m+3}^{(t)}\right)-f\left(t,u_{m+2}^{(t)}\right) \end{pmatrix}.$$

with the initial distribution, at time $t=k$, $\nu_{(m)}=(x_1-x_3,x_{2m+1}-x_4,x_{2m}-x_5,...,x_{m+3}-x_{m+2})^{\dagger}$.

We introduce the function $\mathcal{X}_t: l^{\infty}_{SL(\mathbf{R}^m)}\to l^{\infty}_{SL(\mathbf{R}^m)}$, $t\in\mathbf{N}$ defined by

$$(\mathcal{X}_t h)(i)=\Gamma(t)^2\sum_{j=-\infty}^{\infty}q_{i,j}C^*(t,i)\,h(j)\,C(t,i)\,,\ i\in\mathbf{Z},h\in l^{\infty}_{SL(\mathbf{R}^m)}. \tag{35}$$

Clearly $\mathcal{X}_t\in L\left(l^{\infty}_{SL(\mathbf{R}^m)}\right),\mathcal{X}_t(\mathcal{K}^{\infty})\subset(\mathcal{K}^{\infty})$ and $(\mathcal{X}_t h)(i)$ has the property **(P1)** for any $i\in\mathbf{Z}$ and $t\in\mathbf{N}$ provided $h(i)$ has the property **(P1)** for any $i\in\mathbf{Z}$.

Arguing as in Section 4. we obtain the following results.

Theorem 8. *Assume that (H1)-(H3) hold and* $n=2m+1$ *where* $m\geq 2$*. If there is a unique, uniformly positive and bounded on* $\mathbf{N}$ *solution* $\{P_t\}_{t\in\mathbf{N}}\subset l^{\infty}_{SL(\mathbf{R}^m)}$ *to the equation*

$$P_t=\mathcal{X}_tP_{t+1}+\Phi, \tag{36}$$

then there are $\beta>1$ *and* $\alpha\in(0,1)$ *such that for all* $t,k\in\mathbf{N}$,$t\geq k$ *and all* $l\in\mathbf{Z}$,

$$E\left[\sum_{\{i,j\}\in\{\{1,3\},\{2m+1,4\},\{2m,5\},....\{m+3,m+2\}\}}\left[u_i^{(t)}-u_j^{(t)}\right]^2|r(k)=l\right]\leq\beta\alpha^{t-k}\left\|\nu_{(m)}\right\|^2.$$

Theorem 9. *Assume that the hypotheses of the above theorem hold. Then the solution of system* $(3),(6)$ *is synchronized both in conditional mean and mean error for any* $x \in \mathbf{R}^{2m+1}$.

Proof. We deduce by Theorem 8 that (3), (6) is $\{u_1, u_3\}$-, $\{u_{2m+1}, u_4\}$-, $\{u_{2m}, u_5\}$-, ..., $\{u_{m+3}, u_{m+2}\}$-synchronized both in conditional mean and mean error. After one rotation, we use again Theorem 8 to show that (3),(6) is $\{u_2, u_4\}$-, $\{u_1, u_5\}$-, $\{u_{2m+1}, u_6\}$-, ..., $\{u_{m+4}, u_{m+3}\}$-synchronized both in conditional mean and mean error. The graph with the vertices $1, 2, ..., 2m$ and the edges $\{1, 3\}$, $\{2m + 1, 4\}$, $\{2m, 5\}$,....$\{m + 3, m + 2\}$, $\{2, 4\}$, $\{1, 5\}$, $\{2m + 1, 6\}$, ..., $\{m + 4, m + 3\}$ is connected and the conclusion follows.

Example 2. Let us consider system (3), (6) for $n = 6$. Assume that (H1) holds and the transition matrix $Q = (q_{i,j})$ is defined by $q_{i,i-1} = 1/2, q_{i,i} = 1/2$, for all $i \in \mathbf{Z}$ and $q_{i,j} = 0$ otherwise. We also assume that $a\,(t, 2i + 1) = 1/2$, $a\,(t, 2i) = 0, t \in \mathbf{N}, i \in \mathbf{Z}$ and $f\,(t, u) = \left(1/\sqrt{2}\right) u, u \in \mathbf{R}, t \in \mathbf{N}$. Obviously, for all $t \in \mathbf{N}$,$i \in \mathbf{Z}$ we have $\Gamma\,(t) = 1/\sqrt{2}$, $A\,(t, 2i) = I_2$, $B\,(t, 2i) = I_3, A\,(t, 2i + 1) = A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ and $B\,(t, 2i + 1) = B = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$. Note that (H1)-(H3) hold and we are in the time invariant case. The problem is whether (3), (6) is synchronized or not. In view of Proposition 1, Remark 1 and Theorem 7, it suffices to study the existence of positive solutions to equations (33) and (34). Since (33) and (34) can be equivalently rewritten as

$$\left\{ \begin{array}{c} Z\,(2i) = \frac{1}{4}\left[Z\,(2i-1) + Z\,(2i)\right] + I_2 \\ \\ Z\,(2i+1) = \frac{1}{4}A^*\left[Z\,(2i) + Z\,(2i+1)\right]A + I_2 \end{array} \right. \tag{37}$$

and

$$\left\{ \begin{array}{c} L\,(2i) = \frac{1}{4}\left[L\,(2i-1) + L\,(2i)\right] + I_3 \\ \\ L\,(2i+1) = \frac{1}{4}B^*\left[L\,(2i) + L\,(2i+1)\right]B + I_2 \end{array} \right. , \tag{38}$$

respectively, we only have to discuss the solvability of (37) and (38). First, we notice that $Z\,(2i+1) = \begin{pmatrix} 13/11 & 0 \\ 0 & 13/11 \end{pmatrix}, Z\,(2i) = \begin{pmatrix} 19/11 & 0 \\ 0 & 19/11 \end{pmatrix}, i \in \mathbf{Z}$, is a positive solution to (37). Obviously, such a solution is unique according to Remark 1. It remains to study (38). Since $L\,(2i) = \frac{1}{3}L\,(2i-1) + \frac{4}{3}I_3$, we see that

$$L\,(2i+1) = \frac{1}{4}B^*\left[\frac{1}{3}L\,(2i-1) + L\,(2i+1)\right]B + \frac{1}{3}B^*B + I_3. \tag{39}$$

Assume that $L(2i+1) = L$ for all $i \in \mathbf{Z}$. Then we obtain the equation

$$L = \frac{1}{3}B^* L B + \frac{1}{3}B^* B + I_3. \tag{40}$$

Since the eigenvalues of $1/\sqrt{3}B$ are $1/\sqrt{3}, -1/\left(2\sqrt{3}\right)$ and $1/\left(2\sqrt{3}\right)$, we deduce that $\rho\left(1/\sqrt{3}B\right) < 1$ and $\left\|1/\sqrt{3}B\right\| < 1$. Therefore, the deterministic system $x_{n+1} = 1/\sqrt{3}Bx_n$ is uniformly exponentially stable (see Definition 4 in the next section). It is known (see Theorem 11, for example) that (40) has a unique positive solution. By a direct computation we get

$$L = \begin{pmatrix} \frac{16}{11} & \frac{3}{11} & \frac{3}{11} \\ \frac{3}{11} & \frac{16}{11} & \frac{3}{11} \\ \frac{3}{11} & \frac{3}{11} & \frac{16}{11} \end{pmatrix}.$$

Then $L(2i+1) = L$ and $L(2i) = \frac{1}{3}L + \frac{4}{3}I_3, i \in \mathbf{Z}$, is a positive solution of (38).

Applying Theorem 7 it follows that the solution of system (3), (6) is synchronized both in conditional mean and mean error for any $x \in \mathbf{R}^6$.

Using (3), (6), it is not difficult to deduce that $\lim_{t\to\infty} E\left\|u^{(t)}|r(k) = i\right\|^2 = 0$. Consequently $u^{(t)} \to 0\ P.a.s.$

Now, we observe that the hypotheses of Theorem 7 remain true if in the above example (Example 2) we replace the function $f(t,u) = \left(1/\sqrt{2}\right)u$ by any of the following functions $\widetilde{f}(t,u) = \left(1/\sqrt{2}\right)u + 2$, $\widehat{f}(t,u) = \left(1/\sqrt{2}\right)u + 2 + (-1)^t$ and $\overline{f}(t,u) = \left(1/\sqrt{2}\right)u + t(-1)^t$, $u \in \mathbf{R}, t \in \mathbf{N}$. Let us denote by $\widetilde{u}^{(t)}$, $\widehat{u}^{(t)}$ and $\overline{u}^{(t)}$ the solutions of (3),(6) corresponding to $\widetilde{f}(t,u)$, $\widehat{f}(t,u)$ and $\overline{f}(t,u)$, respectively. Reasoning as in Example 2 we see that $\lim_{t\to\infty} E\left\|\left(\widetilde{u}^{(t+1)} - (2\sqrt{2}+4)\right)|r(k) = i\right\|^2 = 0$, i.e. $\widetilde{u}^{(t+1)} \to 2\sqrt{2}+4\ P.a.s.$ Since

$$\widehat{u}^{(t)} = \widetilde{u}^{(t)} + \frac{\sqrt{2}}{\sqrt{2}+1}\left((-1)^{t-1} + (-1)^k(\sqrt{2})^{k-t}\right)(1,1,1,1,1,1)^T$$

it follows that (3), (6) (where $f(t,u)$ is replaced by $\widehat{f}(t,u)$) is synchronized both in conditional mean and mean error and generates a dynamic pattern which looks as a blinking star. Also, if we consider (3), (6) with $\overline{f}(t,u)$ replacing $f(t,u)$ we observe that it is synchronized both in conditional mean and mean error and the vector

$$\widehat{u}^{(t)} = u^{(t)} + (-1)^{t-1}\frac{t\left(2+\sqrt{2}\right) - 2 - \left(-\sqrt{2}\right)^{k-t}\left[k\left(2+\sqrt{2}\right) - 2\right]}{\left(\sqrt{2}+1\right)^2}(1,1,1,1,1,1)^T$$

mimics a glowing and blinking star.

Example 3. The mathematical setting remains as in Example 2, excepting that $n = 5$. We have to study the synchronization property of the system (3), (6), via

Theorem 9. We note that $C\left(t, 2i\right) = I_2$ and $C\left(t, 2i+1\right) = C = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ for all $i \in \mathbf{Z}$. Equation (36) can be rewritten as

(41) $$\left\{ \begin{array}{c} P\left(2i\right) = \frac{1}{4}\left[P\left(2i-1\right) + P\left(2i\right)\right] + I_2 \\ P\left(2i+1\right) = \frac{1}{4}C^*\left[P\left(2i\right) + P\left(2i+1\right)\right]C + I_2 \end{array} \right. .$$

Reasoning as above, we deduce that if there exists a solution $P\left(2i+1\right) = P$, $P\left(2i\right) = 1/3P + 4/3I_2$ for all $i \in \mathbf{Z}$ of equation (41) then

(42) $$P = \frac{1}{3}C^*PC + \frac{1}{3}C^*C + I_2.$$

By computation, we see that $P = \begin{pmatrix} \frac{131}{109} & \frac{24}{109} \\ \frac{24}{109} & \frac{155}{109} \end{pmatrix}$ is a positive solution of (42) and consequently $P\left(2i+1\right) = P$, $P\left(2i\right) = 1/3P + 4/3I_2, i \in \mathbf{Z}$ is positive solution of (41). This solution is unique from Remark 1.

Applying Theorem 9, it follows that the solution of system (3),(6) is synchronized both in conditional mean and mean error for any $x \in \mathbf{R}^5$.

6. Connections with the Deterministic Case

In this section we assume that $n \in \mathbf{N}, n > 5$ and $a\left(t, i\right) = a\left(t\right), i \in \mathbf{Z}, t \in \mathbf{N}$ in system (3). Evidently (3) is deterministic and the matrices $A\left(t, i\right), B\left(t, i\right), C\left(t, i\right)$ (or $A_1\left(t, i\right), B_1\left(t, i\right), C_1\left(t, i\right)$), defined in the previous section, do not depend upon $i \in \mathbf{Z}$. Hence we will use the short notation $A\left(t\right), B\left(t\right), C\left(t\right)$ (respectively $A_1\left(t\right), B_1\left(t\right), C_1\left(t\right)$) for these matrices. Actually it has no sense to work on spaces of infinite sequences of linear operators. Therefore, keeping the notation, we replace the functions defined by (22), (27) and (35) with

$$\mathcal{L}_t : SL\left(\mathbf{R}^{m-1}\right) \to SL\left(\mathbf{R}^{m-1}\right), \mathcal{L}_t\left(h\right) = \Gamma\left(t\right)^2 A^*\left(t\right) h A\left(t\right)$$
$$\mathcal{M}_t : SL\left(\mathbf{R}^{m}\right) \to SL\left(\mathbf{R}^{m}\right), \mathcal{M}_t\left(h\right) = \Gamma\left(t\right)^2 B^*\left(t\right) h B\left(t\right),$$
$$\mathcal{X}_t : SL\left(\mathbf{R}^{m}\right) \to SL\left(\mathbf{R}^{m}\right), \mathcal{X}_t\left(h\right) = \Gamma\left(t\right)^2 C^*\left(t\right) h C\left(t\right).$$

Then the equations (33), (34) and (36), may be rewritten as

$$\begin{aligned} Z_t &= \Gamma\left(t\right)^2 A^*\left(t\right) Z_{t+1} A\left(t\right) + I, \\ L_t &= \Gamma\left(t\right)^2 B^*\left(t\right) L_{t+1} B\left(t\right) + I, \\ P_t &= \Gamma\left(t\right)^2 C^*\left(t\right) L_{t+1} C\left(t\right) + I. \end{aligned}$$

Directly from Theorems 7, 9, we obtain the following results.

Corollary 3. *Assume that* $(H3)$ *holds and* $n = 2m$ *where* $m \in \mathbf{N}, m \geq 3$. *If there are unique, uniformly positive and bounded on* $\mathbf{N}$ *solutions* $\{Z_t\}_{t\in\mathbf{N}} \subset SL\left(\mathbf{R}^{m-1}\right)$ *and* $\{L_t\}_{t\in\mathbf{N}} \subset SL\left(\mathbf{R}^{m}\right)$ *to the equations*

$$Z_t = \Gamma(t)^2 A^*(t) Z_{t+1} A(t) + I_{\mathbf{R}^{m-1}} \text{ and} \tag{43}$$

$$L_t = \Gamma(t)^2 B^*(t) Z_{t+1} B(t) + I_{\mathbf{R}^{m}}, \tag{44}$$

respectively, then the solution of system (3), (6) *is synchronized for any* $x \in \mathbf{R}^{2m}$.

Corollary 4. *Assume that* $(H3)$ *holds and* $n = 2m+1$ *where* $m \in \mathbf{N}$,$m \geq 2$. *If there is a unique, uniformly positive and bounded on* $\mathbf{N}$ *solution,* $\{P_t\}_{t\in\mathbf{N}} \subset SL\left(\mathbf{R}^{m}\right)$, *of the equation*

$$P_t = \Gamma(t)^2 C^*(t) P_{t+1} C(t) + I_{\mathbf{R}^{m}},$$

then the solution of system (3), (6) is synchronized for any $x \in \mathbf{R}^{2m+1}$.

In [1] (see Theorems 1, 2) it is proved the following result:

Theorem 10. *Assume that* $\Gamma(t) = \Gamma$ *for all* $t \in \mathbf{N}$.

(i) *If* $n = 2m, m \geq 3, \limsup_{t\to\infty} \rho(A_1(t)) < 1/\Gamma$ *and* $\limsup_{t\to\infty} \rho(B_1(t)) < 1/\Gamma$, *then every solution of (3) is synchronized.*

(ii) *If* $n = 2m+1, m \geq 2$, $\limsup_{t\to\infty} \rho(C_1(t)) < 1/\Gamma$, *then every solution of (3) is synchronized.*

In what follows we will compare the above results. First, we recall some well-known results (see [7] and the references therein).

Let H be a real separable Hilbert space and $A_n \in L(H), n \in \mathbf{N}$. We consider the system

$$x_{n+1} = A_n x_n, x_k = x \in H, n \geq k \tag{45}$$

and we denote by $X(n,k) = A_{n-1} \cdots A_k$ the evolution operator associated with (45).

Definition 4. The system **(45)** is uniformly exponentially stable *iff* there exist $\beta \geq 1$, $a \in (0,1)$ such that

$$\|X(n,k)x\| \leq \beta a^{n-k} \|x\| \tag{46}$$

for all $n \geq k \geq 0$ and $x \in H$.

The following result is known [7].

Theorem 11.

(i) *The system (45) is uniformly exponentially stable iff there exist* $m, M > 0$ *and a unique solution* $P = (P_n)_{n\in\mathbf{N}}$ *to the Lyapunov equation*

$$P_n = A_n^* P_{n+1} A_n + I. \tag{47}$$

satisfying

$$m\|x\|^2 \leq \langle P_n x, x\rangle \leq \ M\|x\|^2 \tag{48}$$

for all $n\in \mathbf{N}$ and $x \in H$.

(ii) *In the time invariant case, where* $A_n = A$ *for all* $n \in \mathbf{N}$*, the system (45) is uniformly exponentially stable* iff *the Lyapunov equation*

$$P = A^* P A + I,$$

has a unique positive solution.

Proposition 3. *Assume that* $A_n \in L(H), n \in \mathbf{N}$ *is a sequence of normal operators. If* $\lim_{n\to\infty} \rho(A_n) < 1$, *then* there exist $\beta \geq 1$ and $a \in (0,1)$ such that (46) holds for all $n \geq k \geq 0$ and $x \in H$.

Proof. Clearly $\|X(n,k)x\| = \|A_{n-1}\cdots A_k x\| \leq \|A_{n-1}\|\cdots\|A_k\|\|x\| = \rho(A_{n-1})\cdots\rho(A_k)\|x\|$ for all $x \in H$. From the hypothesis it follows that there exists $n_0 \in \mathbf{N}$ and $l \in (0,1)$ such that $\rho(A_n) \leq l$ for all $n \geq n_0$. Hence for all $n \geq k \geq n_0$ we have $\|A_{n-1}\|\cdots\|A_k\| = \rho(A_{n-1})\cdots\rho(A_k) \leq l^{n-k}$. Now it is clear that for all $n \geq k \geq 0$ and $x \in H$

$$\|X(n,k)x\| \leq l^{n-k}\|x\|.$$

The conclusion follows.

The converse is not true as we will show in the following counter example.

Counter Example. Let $A_{2m+1} = A_1 = 1/2\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = 1/2$ $\begin{pmatrix} 1-2a & a \\ a & 1-2a \end{pmatrix}_{a=1}$ and $A_{2m} = A_0 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1-2a & a \\ a & 1-2a \end{pmatrix}_{a=1/2}$ for all $m \in \mathbf{N}$. Evidently, $A_n \in L(H), n \in \mathbf{N}$ is a sequence of normal operators. The spectrum of A_1, A_0 and $A_1A_0 = 1/2\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is $\{0,-1\}, \{\frac{1}{2}, -\frac{1}{2}\}$ and $\{0, 1/2\}$, respectively. Now, it is clear that $\rho(A_1A_0) < 1, \rho(A_1) = 1$ and $\rho(A_2) < 1$. An easy computation shows that (46) holds, while $\lim_{n\to\infty}\rho(A_n)$ does not exist.

We can easily prove the following result. The details are left to the reader.

Lemma 2. *With the notation of this section, assume that $A_1(t)$ and $A(t)$ (respectively $B_1(t)$ and $B(t)$) are $(m-1)\times(m-1), m\in\mathbf{N}, m\geq 3$ (respectively $m\times m, m\in\mathbf{N}, m\geq 3$) matrices and let $t\in\mathbf{N}$ be fixed. We denote by $P_m^{(t)}(\lambda)$, $Q_m^{(t)}(\lambda)$, $G_m^{(t)}(\lambda)$ and $H_m^{(t)}(\lambda)$ the characteristic polynomials of the matrices $A_1(t)$, $A(t)$, $B_1(t)$ and $B(t)$, respectively.*

- (a) *If $m\geq 3$ and $1-2a(t)<0$, then $Q_m^{(t)}(\lambda)=(-1)^m P_m^{(t)}(-\lambda)$, otherwise the two characteristic polynomials coincide.*
- (b) *If $m\geq 5$ and $1/2<a(t)$, we have $H_m^{(t)}(\lambda)=(-1)^m G_m^{(t)}(-\lambda), \lambda\in\mathbf{R}$; if $m\geq 5$ and $a(t)\leq 1/3$, then $H_n^{(t)}(\lambda)=G_n^{(t)}(\lambda), \lambda\in\mathbf{R}$.*

Remark 3. Assume that $n=2m, m\in\mathbf{N}, m\geq 3$ and $t\in\mathbf{N}$ are fixed. The following statements are easy consequences of the above lemma:

- (a) If $1-2a(t)\geq 0$, the eigenvalues of the matrices $A(t)$ and $A_1(t)$ coincide; if $1-2a(t)<0$, then λ is an eigenvalue of $A(t)$ iff $-\lambda$ is an eigenvalue of $A_1(t)$. (In other words, the spectral radii of $A(t)$ and $A_1(t)$ are equal for all $t\in\mathbf{N}$.)
- (b) If $a(t)\in[0,1/3]\cup(1/2,\infty)$, then the spectral radii of $B(t)$ and $B_1(t)$ are equal for all $t\in\mathbf{N}$.

We already show that the results in this paper stay true in the deterministic case (see Corollaries 3, 4). The next proposition prove that they are stronger then the one in [1], for a large class of values of $a(t)$.

Proposition 4.

- (i) *Assume $a(t)\in[0,1/3]\cup(1/2,\infty)$ for all $t\in\mathbf{N}$. If the hypotheses of Theorem 10 (i) and $(H3)$ hold, then the hypotheses of Corollary 3 are fulfilled.*
- (ii) *Assume $a(t)\in[0,1/3]$ for all $t\in\mathbf{N}$. If the hypotheses of Theorem 10 (ii) and $(H3)$ hold, then the hypotheses of Corollary 4 are fulfilled.*

Proof. (i) Lemma 2 and Remark 3 show that $\limsup_{t\to\infty}\rho(A(t)), \limsup_{t\to\infty}\rho(B(t))<1/\Gamma$. By Proposition 3, it follows that, for all $t\geq k\geq 0$, there exist $\beta\geq 1$, $a\in(0,1)$ such that $\|X_Y(t,k)x\|\leq\beta a^{t-k}\|x\|, Y=A,B$, where $X_Y(t,k), Y=A,B$, are the evolution operators associated to the systems

$$x_{t+1}=\Gamma A(t)x_t, x_k=x\in\mathbf{R}^{m-1},$$

$$x_{t+1}=\Gamma B(t)x_t, x_k=x\in\mathbf{R}^{m},$$

respectively. In view of Theorem 11, we see that (43), (44) admit unique, bounded on $\mathbf{N}$ and uniformly positive solutions. The proof of statement ii) is similar and will be omitted.

The above proposition shows that if $n \geq 5$, then the sufficient conditions given in [1] for the synchronization of the system (3), are stronger then the ones introduced in this paper. It can easily be proved that this conclusion remains true in the cases where $n = 2, 3$ and 4.

7. Concluding Remarks

In our previous discussions, we have emphasized on one or two colored patterns. We can also ask for three or more colored patterns (see the following figure)

$$\begin{array}{cccc} & A & B & \\ D & & & C \\ C & & & D \\ & B & A & \end{array}$$

or we can discuss patterns formation in stochastic networks with different underlying structure such as the star-shaped structure in [14] and the two coupled ring structure in [3] (in which a number of interesting patterns can be found).

We have also emphasized on probabilistic parameters that satisfy Markov chain assumptions. But we can also deal with more general stochastic networks. Therefore we have only established some elementary results in an open territory of research. It is hoped, however, that our investigations will make it easier for the reader to enter this vast territory in nonlinear statistical science.

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