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THE L_p UNIQUE SOLVABILITY OF THE FIRST INITIAL BOUNDARY-VALUE PROBLEM FOR HYPERBOLIC SYSTEMS

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Abstract. The main purpose of the paper is to prove the existence, uniqueness and smoothness with respect to time variable of the generalized L_p-solution of the first initial boundary value problem for higher hyperbolic systems in cylinders with non-smooth base. We also show that the smoothness with respect to time variable of the generalized L_p-solution is independent of the smoothness of base of cylinders.

1. INTRODUCTION

We are concerned with initial boundary value problems for higher hyperbolic systems in non-smooth domains. These problems have been studied by many authors [3-8], whose main results are on the unique existence of the solution and asymptotic expansions of the solution. However, they are based on L_2-theories.

In the present paper, we will establish the well-posedness and the regularity with respect to time variable of L_p-solutions of the first initial boundary value problem for higher hyperbolic systems, which bases on a generalization of Garding's inequality and the approximating boundary method [8]. First, we prove the lemma which is denoted as *"approximating boundary lemma"*, then we use it to establish the existence and the uniqueness of the generalized L_p-solution with $1 < p < +\infty$. After that, by modifying the arguments used in the section above, we can prove the smoothness of the generalized L_p-solution with respect to time variable.

Our paper is organized as follows. In Section 2, we introduce some notation and formulation of the problem, and we also state and prove the approximating boundary theorem, the essential tool in solving the problem. The main results, Theorem 3.1, 3.2 and 3.3, are stated in Section 3, and the proofs of the main results are given in Section 4.

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2. Notation and Formulation of Problem

Let Ω be bounded domain in $\mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$. Set $Q_T = \Omega \times (0,T), 0 < T < +\infty, S_T = \partial\Omega \times (0,T)$. Let $u = (u_1, \ldots, u_s)$ be a complex - valued vector function, we denote:

$$u_{jt^k} = \frac{\partial u_j}{\partial t^k}, u_{t^k} = (u_{1t^k}, \ldots, u_{st^k}), D^\alpha u_j = \frac{\partial^{|\alpha|} u_j}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}, D^\alpha u = (D^\alpha u_1, \ldots, D^\alpha u_s),$$

for each multi index $\alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = \alpha_1 + \cdots + \alpha_n$. Let p be a real number with $1 < p < +\infty$.

We denote by $W_p^m(\Omega)$ the space of all functions $u = u(x), x \in \Omega$ that have generalized derivatives $D^\alpha u \in L_p(\Omega), |\alpha| \leq m$. The norm in this space is defined as follows:

$$\|u\|_{m;p} = \Big(\int_\Omega \sum_{|\alpha|=0}^m |D^\alpha u|^p \, dx \Big)^{1/p}.$$

In particular, $W_p^0(\Omega) \equiv L_p(\Omega)$. $\overset{\circ}{W}{}_p^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in norm of the space $W_p^m(\Omega)$. $W_p^{m,1}(Q_T)$ is the space consisting of all functions $u = u(x,t), (x,t) \in Q_T$ having generalized derivatives $D^\alpha u \in L_p(Q_T), |\alpha| \leq m$, and $u_t \in L_p(Q_T)$, with norm

$$\|u\|_{m,1;p} = \Big(\int_{Q_T} \Big(\sum_{|\alpha|=0}^m |D^\alpha u|^p + |u_t|^p \Big) \, dx\, dt \Big)^{1/p}.$$

$\overset{\circ}{W}{}_p^{m,1}(Q_T)$ is the closure in $W_p^{m,1}(Q_T)$ of the set consisting of all functions in $C^\infty(Q_T)$, vanish near S_T denoted by $C_0^\infty(Q_T)$.

We introduce the partial differential operator of order $2m$

$$L = L(x,t;D) = \sum_{|\alpha|,|\beta|=0}^m D^\alpha \Big(a_{\alpha\beta}(x,t) D^\beta \Big), \tag{2.1}$$

where $a_{\alpha\beta}$ are $s \times s-$ matrices of functions with complex values, and $a_{\alpha\beta}$ are infinity differentiable in $\overline{Q}_T$ and $a_{\alpha\beta} = a^*{}_{\alpha\beta}$, where $a^*{}_{\alpha\beta}$ denotes the transposed conjugate matrix of $a_{\alpha\beta}$. We have the following Green's formula

$$\int_\Omega Lu\,\overline{v}\, dx = B(u, v\,; t)$$

which is valid for all $u, v \in C_0^\infty(\Omega)$ and a.e. $t \in [0,T)$, where

$$B(u, v\,; t) = \sum_{|\alpha|,|\beta|=0}^m \int_\Omega a_{\alpha\beta}(.,t) D^\beta u \, \overline{D^\alpha v}\, dx.$$

We also suppose that the form $B(.,.;t)$ is $W_2^m(\Omega)$- elliptic uniformly with respect to $t \in [0,T)$, i.e., the inequality

$$B(u,u,;t) \geq \gamma_0 \|u\|^2_{W_2^m(\Omega)} \tag{2.2}$$

is valid for all $u \in \overset{\circ}{W}{}_2^m(\Omega)$ and all $t \in [0,T)$, where γ_0 is a positive constant independent of u and t.

From condition (2.2), we get the generalized Garding inequality

$$\sup_{v \in S_q} \Big| B(u,v\,;t) \Big| \geq \gamma_1 \|u\|_{m,p} \tag{2.3}$$

with a.e. $t \in [0,T)$, for every $u \in \overset{\circ}{W}{}_p^m(\Omega)$, where the constant $\gamma_1 = \gamma_1(n,m,p,\Omega) > 0$, $1 < p,q < \infty$ are real numbers with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and S_q is the unit ball in $\overset{\circ}{W}{}_q^m(\Omega)$, (see [9, 10, 11]).

Set

$$B_1(u,\eta) = \sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T} a_{\alpha\beta} D^\beta u \, \overline{D^\alpha \eta}\, dxdt + \int_{Q_T} u_t \overline{\eta_t} dxdt.$$

for all $u \in \overset{\circ}{W}{}_p^m(Q_T), \eta \in \overset{\circ}{W}{}_q^m(Q_T)$.

If $1 < p < 2$, then we have the following lemma:

Lemma 2.1. *There exists a constant $\gamma_2 = \gamma_2(p,n,m,|\Omega|,T) > 0$, such that*

$$\sup\{\big|B_1(u,\eta)\big| : \eta \in \overset{\circ}{W}{}_q^{m,1}(Q_T), \|\eta\|_{m,1;q} \leq 1\} \geq \gamma_2 \|u\|_{m,1;p}, \tag{2.4}$$

for every $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$.

Proof. We prove this result with $u \in C_0^\infty(Q_T)$. Suppose that there is no $\gamma_2 > 0$ such that (2.4) holds true. Then there is a sequence $\{u_k\} \subset C_0^\infty(Q_T)$ with $\|u_k\|_{m,1;p} = 1$ and

$$\sup\{\big|B_1(u_k,\eta)\big| : \eta \in \overset{\circ}{W}{}_q^{m,1}(Q_T), \|\eta\|_{m,1;q} \leq 1\} \leq \frac{1}{k}, \text{ for every } k \geq 1. \tag{2.5}$$

Using condition (2.2), we obtain

$$\big|B_1(u_k,u_k)\big| \geq \gamma_0 \|u_k\|^2_{m,0;2} + \int_{Q_T} |u_{kt}|^2 dxdt \geq C_1 \|u\|^2_{m,1;2}. \tag{2.6}$$

On the other hand, by using Hölder's inequality with $1 < p < 2, p^* = \frac{2}{p}, q^* = \frac{2}{2-p}$, we have

$$\|u_k\|^p_{m,1;p} = \sum_{|\alpha|=0}^{m} \int_{Q_T} |D^\alpha u|^p dxdt + \int_{Q_T} |u_t|^p \, dx\, dt \le C_2 \|u_k\|^p_{m,1;2} \tag{2.7}$$

$C_2 = C_2(p, |\Omega|, T) > 0$. Combining (2.6) and (2.7), we get

$$\big|B_1(u_k, u_k)\big| \ge C\|u_k\|^2_{m,1;p},$$

where C is a constant independent of k.

We get from the inequality above and (2.5) that

$$\|u_k\|^2_{m,1;p} \le \frac{1}{kC}, \text{ for every } k = 1, 2, \ldots.$$

which contradicts $\|u_k\|_{m,1;p} = 1$. Therefore, there is a constant $\gamma_2 > 0$ such that (2.4) holds true. Since $u \in C_0^\infty(Q_T)$ which is dense in $\overset{\circ}{W}{}^{m,1}_p(Q_T)$, this completes the proof. ■

In this paper, we consider the following problem in the cylinder Q_T:

$$Lu - u_{tt} = f,\ f \in L_p(Q_T) \tag{2.8}$$

with the initial conditions

$$u\big|_{t=0} = 0, u_t\big|_{t=0} = 0 \tag{2.9}$$

and the boundary conditions

$$\frac{\partial^j u}{\partial \nu^j}\Big|_{S_T} = 0; j = 0, 1, \ldots, m-1, \tag{2.10}$$

where $\dfrac{\partial^j u}{\partial \nu^j}$ are derivatives with respect to the outer unit normal of S_T.

Definition 2.1. A function u is called a generalized L_p-solution of problem (2.8) -(2.10) if and only if u belongs to $\overset{\circ}{W}{}^{m,1}_p(Q_T), u(x,0) = 0,$ and the equality

$$\sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T} a_{\alpha\beta} D^\beta u \overline{D^\alpha \eta}\, dx\, dt + \int_{Q_T} u_t\, \overline{\eta}_t\, dx\, dt = \int_{Q_T} f\overline{\eta}\, dx\, dt \tag{2.11}$$

holds for all $\eta \in \overset{\circ}{W}{}^{m,1}_q(Q_T)$.

In the case $p = 2$, u is called a generalized L_2-solution and its unique existence is established in [7, 8]. To consider problem (2.8) -(2.10), we need to prove the following approximating boundary result.

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists a sequence of smooth domains $\{\Omega^\varepsilon\}$ such that $\Omega^\varepsilon \subset \Omega$ and $\lim_{\varepsilon\to 0}\Omega^\varepsilon = \Omega$.*

Proof. For $\varepsilon > 0$ arbitrary, set $S^\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) \le \varepsilon\}, \Omega^\varepsilon = \Omega \setminus S^\varepsilon$ and $\partial\Omega^\varepsilon$ is the boundary of Ω^ε. Denote by $J(x)$ the characteristic function of Ω^ε and by $J_h(x)$ the mollification of $J(x)$, i.e.

$$J_h(x) = \int_{\mathbb{R}^n} \theta_h(x-y)J(y)dy,$$

where θ_h is a mollifier. If $h < \frac{\varepsilon}{2}$, then $J_h(x)$ has following properties:

(1) $J_h(x) = 0$ if $x \notin \Omega^{\frac{\varepsilon}{2}}$;

(2) $0 \le J_h(x) \le 1, \forall x \in \Omega$;

(3) $J_h(x) = 1$ in $\Omega^{2\varepsilon}$;

(4) $J_h \in C_0^\infty(\mathbb{R}^n)$.

We now fix a constant $c \in (0,1)$, set $\Omega_c^\varepsilon = \{x \in \Omega : J_h(x) > c\}$. It is obvious that $\Omega^{\frac{\varepsilon}{2}} \supset \Omega_c^\varepsilon \supset \Omega^{2\varepsilon}$. Therefore, $\Omega_c^\varepsilon \subset \Omega$ and $\lim\limits_{\varepsilon\to 0}\Omega_c^\varepsilon = \Omega$.

Assume that K is the critical set of J_h, i.e. K consisting of all point x, such that the gradient of J_h at x vanishes. A number $c \in \mathbb{R}$ such that $J_h^{-1}(c)$ contains at least one $x \in K$ is called a critical value. By Sard's theorem then the set of critical values of J_h is of measure zero (see[12, Theorem 1.30]), it implies that there exists a constant $c_0 \in (0,1)$ such that c_0 is not a critical value of J_h. Denote $\Omega_{c_0}^\varepsilon = \{x \in \Omega : J_h(x) > c_0\}$ and $F(x) = J_h(x) - c_0$. For all $x^0 \in \partial\Omega_{c_0}^\varepsilon$, then $F(x^0) = J_h(x^0) - c_0 = 0$ and $gradJ_h(x^0) \neq 0$. This implies that there exists a $\dfrac{\partial J_h}{\partial x_i}(x^0) \neq 0$, without loss of generality we can suppose that $\dfrac{\partial J_h}{\partial x_n}(x^0) \neq 0$. Using the implicit function theorem, we obtain that there exists a neighbourhood W of $(x_1^0, ..., x_{n-1}^0)$ in $\mathbb{R}^{n-1}$ a neighbourhood V of x_n^0 in $\mathbb{R}$ and an infinitely differentiable function $z : W \longrightarrow \mathbb{R}$ such that $x \in U_{x^0} \cap \partial\Omega_{c_0}^\varepsilon$, where $\partial\Omega_c^\varepsilon = \{x \in \Omega : J_h(x) = c\}$, $U_{x^0} = W \times V$, if and only if $x = (x_1, ..., x_n) \in U_{x^0}, x_n = z(x_1, ..., x_{n-1})$. Hence, $\Omega_{c_0}^\varepsilon$ is smooth and $\lim\limits_{\varepsilon\to 0}\Omega_{c_0}^\varepsilon = \Omega$. The lemma proved. ∎

Suppose that $\{\Omega^\epsilon\}$ is a smooth domain subsequence and $\lim\limits_{\varepsilon\to 0}\Omega^\varepsilon = \Omega$. Set $Q_T^\epsilon = \Omega^\epsilon \times (0,T), S_T^\epsilon = \partial\Omega^\epsilon \times (0,T)$.

We consider the following problem in the cylinder Q_T^ϵ:

$$\begin{cases} Lu^\epsilon - u^\epsilon_{tt} = f, \ f \in C^\infty(\overline{Q_T^\epsilon}) \\ u^\epsilon\big|_{t=0} = 0, u^\epsilon_t\big|_{t=0} = 0 \\ \dfrac{\partial^j u^\epsilon}{\partial \nu^j}\Big|_{S_T^\epsilon} = 0, \text{for } j = 0, 1, \ldots, m-1. \end{cases}$$

If $f_{t^k}\big|_{t=0} = 0$, for $k = 0, 1, \ldots$, then this problem has a unique function $u^\epsilon(x,t) \in C^\infty(\overline{Q_T^\epsilon})$. Moreover, $u^\epsilon(.,t) \in \overset{\circ}{W}{}_2^m(\Omega^\epsilon)$, for all $t \in [0,T]$, (see[3]). Set

$$\widetilde{u^\epsilon}(x,t) = \begin{cases} u^\epsilon(x,t) & \text{if } (x,t) \in Q_T^\epsilon \\ 0 & \text{if } (x,t) \notin Q_T^\epsilon \end{cases}$$

Then $\widetilde{u^\epsilon}(t) = \widetilde{u^\epsilon}(.,t) \in \overset{\circ}{W}{}_p^m(\Omega), \forall t \in [0,T]$ and $\widetilde{u^\epsilon} \in \overset{\circ}{W}{}_p^{m,1}(Q_T), p > 1$.

3. Formulation of the Main Results

In this section, we give the main results of the present paper:

Theorem 3.1. *Assume that there is a positive constant μ, such that coefficients of operator 2.1 satisfy*

$$\sup\left\{|a_{\alpha\beta}| : (x,t) \in \overline{Q_T}, 0 \le |\alpha|, |\beta| \le m\right\} \le \mu$$

and $f \in L_p(Q_T)$. Then problem (2.8) -(2.10) has at most one generalized L_p-solution.

Theorem 3.2. *Let the assumptions of Theorem 3.1 be satisfied.*

(i) *If $1 < p < 2$ and $f \in L_(Q_T)$, then problem (2.8)- (2.10) has a generalized L_p-solution $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$, and the following estimate holds*

$$\|u\|^p_{m,1;p} \le C\|f\|^p_{0,p}, \tag{3.1}$$

where C is a constant independent of u and f.

(ii) *If $2 < p, \dfrac{2n}{n+2m} \le q$ and $f, f_t, f_{tt} \in L_p(Q_T)$, then problem (2.8)- (2.10) in the cylinder Q_T has a unique generalized L_p-solution $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$ and the following estimate holds*

$$\|u\|^p_{m,1;p} \leq C \sum_{k=0}^{2} \|f_{t^k}\|^p_{0,p}, \tag{3.2}$$

where C is a constant independent of u and f.

The following theorem shows that generalized L_p-solution $u \in \overset{\circ}{W}{}^{m,1}_p(Q_T)$ of problem (2.8)-(2.10) is smooth with respect to time variable t if right hand-side f and coefficients of operator (2.1) is smooth enough with respect to t.

Theorem 3.3. *Let h be the nonnegative integer, and we assume that*

(1) $f_{t^k} \in L_p(Q_T), k \leq h+2,$

(2) $f_{t^k}\big|_{t=0} = 0, x \in \Omega, k \leq h,$

(3) $\sup\Big\{\Big|\dfrac{\partial^k a_{\alpha\beta}}{\partial t^k}\Big|, k < h+1 : (x,t) \in Q_T, 0 \leq |\alpha|, |\beta| \leq m\Big\} \leq \mu,$

(4) $1 < p < 2$ *or* $2 < p, \dfrac{2n}{n+2m} \leq q.$

Then the generalized solution $u \in \overset{\circ}{W}{}^{m,1}_p(Q_T)$ of problem (2.8)-(2.10) has generalized derivatives with respect to t up to oder h in $\overset{\circ}{W}{}^{m,1}_p(Q_T)$ and satisfies the estimate

$$\|u_{t^h}\|^p_{m,1;p} \leq C \sum_{k=0}^{h+2} \|f_{t^k}\|^p_{L_p(Q_T)}, \tag{3.3}$$

where C is a constant independent of u and f.

4. Proofs of the Main Results

4.1. Proof of Theorem 3.1

Firstly, we will prove the theorem in the cases $p > 2$. Since $p > 2$, $\overset{\circ}{W}{}^{m,1}_p(Q_T) \subset \overset{\circ}{W}{}^{m,1}_2(Q_T)$, implying that if u is a generalized L_p-solution, and then u is a generalized L_2-solution. Hence, we obtain the uniqueness of a generalized L_p-solution from the uniqueness of a generalized L_2-solution.

Further, we will prove the theorem in the case $1 < p < 2$. Suppose that problem (2.8) -(2.10) has two generalized L_p-solutions u_1, u_2. Put $u = u_1 - u_2$, then (2.11) implies that

$$B_1(u,\eta) = \sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T} a_{\alpha\beta}(x,t) D^\beta u \overline{D^\alpha \eta}\, dx\, dt + \int_{Q_T} u_t\, \overline{\eta}_t\, dx\, dt = 0 \tag{4.1}$$

holds for all $\eta \in \overset{\circ}{W}{}_q^{m,1}(Q_T)$.

Combining inequality (2.4) with equality (4.1), we obtain

$$\gamma_2 \|u\|_{m,1;p} \leq \sup\{\left|B_1(u,\eta)\right| : \eta \in \overset{\circ}{W}{}_q^{m,1}(Q_T), \|\eta\|_{m,1;q} \leq 1\} = 0.$$

Hence, $u \equiv 0$ in Q_T. This completes the proof of theorem.

4.2. Proof of Theorem 3.2

4.2.1. In the case $1 < p < 2$

We need the following assertion:

Proposition 4.1. *Suppose that* $f \in C^\infty(\overline{Q}_T), f_{t^k}\big|_{t=0} = 0$, *for* $k = 0, 1, \ldots$ *and*

$$\sup\left\{|a_{\alpha\beta}| : (x,t) \in Q_T, 0 \leq |\alpha|, |\beta| \leq m\right\} \leq \mu.$$

Then $\widetilde{u^\epsilon}$ *is a generalized* L_p*-solution of problem (2.8) -(2.9) in* Q_T^ϵ *satisfying*

$$\|\widetilde{u^\epsilon}\|_{m,1;p}^p \leq C\|f\|_{0,p}^p$$

where the constant C *is independent of* ϵ, u *and* f.

Proof. From $\widetilde{u^\epsilon}$ satisfying system (2.8) in Q_T replacing f by $\widetilde{f}$, where

$$\widetilde{f} = \begin{cases} f(x,t), & (x,t) \in \overline{Q_T^\epsilon} \\ 0, & (x,t) \notin \overline{Q_T^\epsilon} \end{cases}$$

after multiplying (2.8) by $\overline{\eta}, \eta \in \overset{\circ}{W}{}_q^{m,1}(Q_T)$, integrating on Q_T, we get

$$\int_{Q_T} L\widetilde{u^\epsilon}\,\overline{\eta}\,dxdt - \int_{Q_T} \widetilde{u^\epsilon}_{tt}\,\overline{\eta}\,dxdt = \int_{Q_T} \widetilde{f}\overline{\eta}dxdt$$

By using Green's formula and integrating by parts with respect to t, we obtain from the equality above that

$$B_1(\widetilde{u^\epsilon}, \eta) = \int_{Q_T} \widetilde{f}\overline{\eta}dxdt \tag{4.2}$$

This clearly shows that $\widetilde{u^\epsilon}$ is a generalized L_p-solutions of problem (2.8) -(2.9) in Q_T^ϵ; otherwise, using Hölder's inequality and inequality (2.4), we conclude from (4.2) that

$$\|\widetilde{u^\epsilon}\|_{m,1;p}^p \leq C\|f\|_{0,p}^p. \qquad \blacksquare$$

Now we prove the existence of the generalized L_p-solution of problem (2.8) – (2.10) in Q_T, when the assumptions of Proposition 4.1 are satisfied.

Proposition 4.2. *Let the assumptions of Proposition 4.1 be satisfied. Then problem* (2.8)-(2.10) *in cylinder* Q_T *has a generalized* L_p*-solution* $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$ *which satisfies*

$$\|u\|_{m,1;p}^p \leq C\|f\|_{0,p}^p \tag{4.3}$$

where C *is a constant independent of* u *and* f.

Proof. By Proposition 4.1 we have

$$\|\widetilde{u}^\epsilon\|_{m,1;p}^p \leq C\|f\|_{0,p}^p \tag{4.4}$$

where the constant C does not depend on ϵ. It means that the set $\{\widetilde{u}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in the space $\overset{\circ}{W}{}_p^{m,1}(Q_T)$. So we can take a subsequence, denoted also by $\widetilde{u}^\varepsilon$ for convenience, which converges weakly to a function $u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$. We will show that u is a generalized L_p-solution of problem (2.8)- (2.10) in cylinder Q_T. In fact for all $\eta \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$, there exists $\eta_\delta \in C^\infty(\overline{Q_T})$ such that $\eta_\delta \equiv 0$ in $Q_T \setminus Q_T^\varepsilon$, and $\|\eta_\delta - \eta\|_{m,1;p} \longrightarrow 0$ when $\delta \to 0$. Since $\widetilde{u}^\varepsilon$ is a generalized solution of problem (2.8)- (2.10) in the smooth cylinder Q_T^ε, we have

$$\sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T^\epsilon} a_{\alpha\beta} D^\beta \widetilde{u}^\epsilon \overline{D^\alpha \eta_\delta}\, dx\, dt + \int_{Q_T^\epsilon} \widetilde{u}_t^\epsilon \overline{\eta_\delta}_t\, dx\, dt = \int_{Q_T^\epsilon} f\overline{\eta_\delta}\, dx\, dt.$$

Reforming this equality in to

$$\sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T} a_{\alpha\beta} D^\beta \widetilde{u}^\epsilon \overline{D^\alpha \eta_\delta}\, dx\, dt + \int_{Q_T} \widetilde{u}^\epsilon{}_t \overline{\eta_\delta}_t\, dx\, dt = \int_{Q_T^\epsilon} f\overline{\eta_\delta}\, dx\, dt.$$

Passing to the limit when $\varepsilon \to 0, \delta \to 0$ for the weakly convergent sequence, we get

$$\sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T} a_{\alpha\beta} D^\beta u \overline{D^\alpha \eta}\, dx\, dt + \int_{Q_T} u_t \overline{\eta}_t\, dx\, dt = \int_{Q_T} f\overline{\eta}\, dx\, dt.$$

Since $\overset{\circ}{W}{}_p^{m,1}(Q_T)$ is imbedded continuously into $L_p(\Omega)$, the trace sequence $\{\widetilde{u}^\varepsilon(x,0)\}$ of $\{\widetilde{u}^\varepsilon(x,t)\}$ converges weakly to the trace $u(x,0)$ of $u(x,t)$ in $L_p(\Omega)$. On the other hand, $\widetilde{u}^\varepsilon(x,0) = 0$, so that $u(x,0) = 0$. Hence, $u(x,t)$ is a generalized L_p-solution of problem (2.8)- (2.10). Moreover, from (4.7) we have

$$\|u\|_{m,1;p}^p \leq \varliminf_{\varepsilon\to 0} \|\widetilde{u}^\varepsilon\|_{m,1;p}^p \leq C\|f\|_{0,p}^p. \qquad \blacksquare$$

Proposition 4.2 stated the existence of generalized L_p-solutions of problem (2.8)-(2.10) in $\overset{\circ}{W}{}_p^{m,1}(Q_T)$ when $f \in C^\infty(\overline{Q}_T)$ and $f_{t^k}\big|_{t=0} = 0$, for $k = 0, 1, \ldots$. We now establish the problem when $f \in L_p(Q_T)$.

Proof of Theorem 3.2.
Denote

$$f_h(x,t) = \begin{cases} 0 & \text{if } (x,t) \neq Q_T \\ f(x,t) & \text{if } (x,t) \in Q_T, t > h \\ 0 & \text{if } (x,t) \in Q_T, t \leq h \end{cases}$$

for all $h > 0$. We will denote by $g_{\frac{h}{2}}$ the mollification of f_h, then $g_{\frac{h}{2}} \in C^\infty(\overline{Q_T}), g_{\frac{h}{2}} \equiv 0, t < \frac{h}{2}$ and $g_{\frac{h}{2}} \to f$ in $L_p(Q_T)$. By Proposition 4.2, problem (2.8)-(2.10) has a generalized L_p-solution $u_h \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$ with replacing f by $g_{\frac{h}{2}}$, and the following estimates holds

$$\|u_h\|^p_{m,1;p} \leq C\|g_{\frac{h}{2}}\|^p_{0,p} \tag{4.5}$$

where C is a constant independent of h, u and f. Since $\{g_{\frac{h}{2}}\}$ is a Cauchy sequence in $L_p(Q_T)$ and inequality (4.5), it follows that $\{u_h\}$ is a Cauchy sequence in $\overset{\circ}{W}{}_p^{m,1}(Q_T)$. Hence, $u_h \to u \in \overset{\circ}{W}{}_p^{m,1}(Q_T)$, then u is a generalized L_p-solutions of problem (2.8)- (2.10) and satisfies

$$\|u\|^p_{m,1;p} \leq C\|f\|^p_{0,p}\,.$$

Thus, the theorem is proved in the case $1 < p < 2$.

4.2.2. In the case $2 < p < \infty$

Analogous to cases above $1 < p < 2$, we need prove the following assertion:

Proposition 4.3. *Suppose that* $f \in C^\infty(\overline{Q}_T), f_{t^k}\big|_{t=0} = 0$, *for* $k = 0, 1, \ldots$ *and*

$$\sup\Big\{\Big|\frac{\partial a_{\alpha\beta}}{\partial t}\Big|, |a_{\alpha\beta}| : (x,t) \in Q_T, 0 \leq |\alpha|, |\beta| \leq m\Big\} \leq \mu.$$

Then $\widetilde{u}^\epsilon$ *satisfies*

$$\|\widetilde{u^\epsilon}\|^p_{m,1;p} \leq C\sum_{k=0}^{2}\|f_{t^k}\|^p_{0,p}.$$

where the constant C *is independent of* ϵ, u *and* f.

Proof. We have that $\widetilde{u^\epsilon}$ satisfies system (2.8) in Q_T with replacing f by $\widetilde{f}$; therefore, after multiplying (2.8) by $\overline{v}, v \in \overset{\circ}{C}{}^{\infty}(\Omega)$, integrating with respect to x on Ω, we get

$$\int_\Omega L\widetilde{u^\epsilon}(t)\,\overline{v}\,dx = \int_\Omega \big(\widetilde{f}(t) + \widetilde{u^\epsilon}_{tt}(t)\big)\,\overline{v}\,dx$$

where we set $f(t) = f(.,t)$. By using Green's formula, we obtain

$$B(\widetilde{u^\epsilon}(t), v; t) = \Big(\int_\Omega \widetilde{f}(t)\overline{v}\,dx + \int_\Omega \widetilde{u^\epsilon}_{tt}(t)\overline{v}\,dx\Big). \tag{4.6}$$

By the equality above, it is easy to see that

$$\Big|B(\widetilde{u^\epsilon}(t), v; t)\Big| \le \Big|\int_\Omega \widetilde{f}(t)\overline{v}\,dx\Big| + \Big|\int_\Omega \widetilde{u^\epsilon}_{tt}(t)\overline{v}\,dx\Big|. \tag{4.7}$$

Set

$$F(v) = \int_\Omega \widetilde{u^\epsilon}_{tt}(t)\overline{v}\,dx, \forall v \in C_0^\infty(\Omega), \text{for a.e. } t \in (0,T)$$

and using Hӧl der's inequality, we have

$$\Big|F(v)\Big| \le \|\widetilde{u^\epsilon}_{tt}(t)\|_{0,2}\|v\|_{0,2}.$$

Since $\dfrac{2n}{n+2m} \le q < 2$, it is easy to check validity of the following inequalities:

$$n \le mq \text{ or } \begin{cases} mq & < n \\ 2 & \le \dfrac{nq}{n-mq} \end{cases}.$$

Therefore, $\overset{\circ}{W}{}^m_q(\Omega)$ is imbedded continuously into $L_2(\Omega)$ (see[1]). Hence, we have

$$\|v\|_{0,2} \le C_0\|v\|_{0,q}.$$

It implies

$$|F(v)| \le \|\widetilde{u^\epsilon}_{tt}(t)\|_{0,2}\|v\|_{0,q} \le C_0\|u^\epsilon_{tt}(t)\|_{0,2}, \forall v \in S_q.$$

Using the estimates for the generalized L_2-solution (see[7, 8]), we get

$$\|u^\epsilon_{tt}(t)\|_{0,2} \le C_1\Big(\|f(t)\|_{0,2} + \|f_t(t)\|_{0,2}\Big), \text{ for a.e. } t \in (0,T).$$

As $p > 2$ and Hӧlder's inequality, we obtain

$$\|f(t)\|^2_{0,2} = \int_\Omega |f(t)|^2\,dx \le \Big(\int_\Omega (|f(t)|^2)^{\frac{p}{2}}\,dx\Big)^{\frac{2}{p}} |\Omega|^{\frac{p}{p-2}}.$$

Therefore, $\|f(t)\|_{0,2} \leq |\Omega|^{\frac{p}{2(p-2)}} \|f(t)\|_{0,p}$. It implies

$$\|u^{\epsilon}_{tt}(t)\|_{0,2} \leq C_2\Big(\|f(t)\|_{0,p} + \|f_t(t)\|_{0,p}\Big).$$

It means that

$$|F(v)| \leq C_2\Big(\|f(t)\|_{0,p} + \|f_t(t)\|_{0,p}\Big), \forall v \in S_q. \tag{4.8}$$

Otherwise, by using Hölder's inequality, we have the following inequality:

$$\Big|\int_\Omega \widetilde{f}(t)\,\overline{v}\,dx\Big| \leq \|f(t)\|_{0,p}\|v\|_{0,q} \leq \|f(t)\|_{0,p}, \forall v \in S_q. \tag{4.9}$$

Because $\overset{\circ}{C}{}^{\infty}(\Omega)$ is dense in $\overset{\circ}{W}{}^m_q(\Omega)$, (4.8) and (4.9) holds for all $v \in \overset{\circ}{W}{}^m_q(\Omega)$. Substituting (4.8) and (4.9) into (4.7), we get

$$\sup_{v\in S_q}\Big|B(\widetilde{u^{\epsilon}}(t), v; t)\Big| \leq C_3\Big(\|f(t)\|_{0,p} + \|f_t(t)\|_{0,p}\Big).$$

From the inequality above and Garding's inequality (2.3), we obtain

$$\gamma_1\|\widetilde{u^{\epsilon}}(t)\|_{m,p} \leq C_3\Big(\|f(t)\|_{0,p} + \|f_t(t)\|_{0,p}\Big).$$

Therefore,

$$\|\widetilde{u^{\epsilon}}(t)\|_{m,p} \leq C_4\Big(\|f(t)\|_{0,p} + \|f_t(t)\|_{0,p}\Big) \tag{4.10}$$

where $C_4 = \dfrac{C_3}{\gamma_1}$ is a constant independent of ϵ. Due to the use of Hölder's inequality again

$$\Big|\sum_{k=0}^{r} a_k b_k\Big| \leq \Big(\sum_{k=0}^{r}|a_k|^p\Big)^{\frac{1}{p}}\Big(\sum_{k=0}^{r}|b_k|^q\Big)^{\frac{1}{q}}$$

and putting $b_k = 1$ in this inequality, we get

$$\Big(\|f(t)\|_{0,p} + \|f_t(t)\|_{0,p}\Big)^p \leq 2^{\frac{p}{q}}\Big(\|f(t)\|^p_{0,p} + \|f_t(t)\|^p_{0,p}\Big). \tag{4.11}$$

Combining (4.10), (4.11), we conclude that

$$\|\widetilde{u^{\epsilon}}(t)\|^p_{m,p} \leq C_5\Big(\|f(t)\|^p_{0,p} + \|f_t(t)\|^p_{0,p}\Big) \tag{4.12}$$

where the constant C_5 does not depend on ϵ.

By the derivation of the equality (4.6) with respect to t and the arguments analogous to the estimates above, we obtain

$$\|\widetilde{u}_t^\epsilon(t)\|_{0,p}^p \leq C_6\bigg(\|f_t(t)\|_{0,p}^p + \|f_{tt}(t)\|_{0,p}^p\bigg) \tag{4.13}$$

From (4.12), (4.13), it implies

$$\|\widetilde{u^\epsilon}(t)\|_{m,p}^p + \|\widetilde{u}_t^\epsilon(t)\|_{0,p}^p \leq C\sum_{k=0}^{2}\|f_{t^k}(.,t)\|_{0,p}^p, \text{ for a.e. } t \in (0,T) \tag{4.14}$$

where constant C does not depend on ε. So by integrating with respect to t from 0 to T we get

$$\|\widetilde{u}^\epsilon\|_{m,1;p}^p \leq C\sum_{k=0}^{2}\|f_{t^k}\|_{0,p}^p.$$

This completes the proof. ■

By the arguments analogous to the proof of the case $1 < p < 2$, we get proof of Theorem 3.2 in the case $p > 2$.

4.3. Proof of Theorem 3.3

We only need to prove the theorem in the condition $f \in C^\infty(\overline{Q}_T), f_{t^k}|_{t=0} = 0$, for $k = 0, 1, \ldots$; in other conditions, the theorem is proved by arguments analogous to the proof of Proposition 4.2 and Theorem 3.2.

4.3.1. In the case $1 < p < 2$

The theorem is proved by the induction on h. According to Theorem 3.2, the theorem is valid for $h = 0$. Now let the theorem be true for $h - 1$; we will prove that this also holds for h.

From $\widetilde{u^\epsilon}$ satisfies system (2.8) in Q_T with replacing f by $\widetilde{f}$, we have

$$Lu^\epsilon - u_{tt}^\epsilon = \widetilde{f}. \tag{4.15}$$

After differentiating equality (4.15) h times with respect to t and multiplying that retrieved equality by $\overline{\eta}, \eta \in C_0^\infty(Q_T)$, we integrate this equality on Q_T, and the obtained equality will be

$$\sum_{k=0}^{h}\binom{h}{k}\int_{Q_T}\sum_{|\alpha|,|\beta|=0}^{m} D^\alpha(a_{\alpha\beta t^{h-k}}D^\beta\widetilde{u_{t^k}^\epsilon})\overline{\eta}\,dx\,dt - \int_{Q_T}\widetilde{u}^\epsilon{}_{t^{h+1}}\overline{\eta}\,dx\,dt = \int_{Q_T}\widetilde{f}_{t^h}\overline{\eta}\,dx\,dt.$$

By using Green's formula and integrating by parts, we get

$$B_1(\widetilde{u^\epsilon_{t^h}},\eta)=\int_{Q_T}\widetilde{f}_{t^h}\overline{\eta}\,dx\,dt-\sum_{k=0}^{h-1}\binom{h-1}{k}\int_{Q_T}\sum_{|\alpha|,|\beta|=0}^{m}a_{\alpha\beta t^{h-k}}D^\beta\widetilde{u^\epsilon_{t^k}}\overline{D^\alpha\eta}\,dx\,dt.$$

Therefore,

$$|B_1(\widetilde{u^\epsilon_{t^h}},\eta)|\le\Big|\int_{Q_T}\widetilde{f}_{t^h}\overline{\eta}\,dx\,dt\Big|+\Big|\sum_{k=0}^{h-1}\binom{h-1}{k}\int_{Q_T}\sum_{|\alpha|,|\beta|=0}^{m}a_{\alpha\beta t^{h-k}}D^\beta\widetilde{u^\epsilon_{t^k}}\overline{D^\alpha\eta}\,dx\,dt\Big|.$$

From the inequality above and Hölder's inequality, we have

$$|B_1(\widetilde{u^\epsilon_{t^h}},\eta)|\le C(\|f_{t^h}\|_{0,p}+\sum_{k=0}^{h-1}\|\widetilde{u^\epsilon_{t^k}}\|_{m,1;p})\|\eta\|_{m,1;q} \tag{4.16}$$

Since $C_0^\infty(Q_T)$ is dense in $\overset{\circ}{W}{}^{m,1}_q(Q_T)$, (4.16) holds for all $\eta\in\overset{\circ}{W}{}^{m,1}_q(Q_T)$. By using equality (2.4) and the induction assumption, we obtain

$$\|\widetilde{u^\epsilon_{t^h}}\|^p_{m,1;p}\le C'\sum_{k=0}^{h}\|f_{t^k}\|^p_{0,p}$$

where C is a constant independent of ϵ,u.

Since $\widetilde{u}^\varepsilon$ converges weakly to the generalized solution $u\in\overset{\circ}{W}{}^{m,1}_p(Q_T)$, the u has generalized derivatives with respect to t up to oder h in $\overset{\circ}{W}{}^{m,1}_p(Q_T)$ and

$$\|u_{t^h}\|^p_{m,1;p}\le\underline{\lim}_{\varepsilon\to0}\|\widetilde{u}^\varepsilon_{t^h}\|^p_{m,1;p}\le C\sum_{k=0}^{h}\|f_{t^k}\|^p_{L_p(Q_T)}.$$

4.3.2. In the case $2<p<\infty$

The theorem is proved by the induction on h, a similar method used to prove the theorem in the cases $1<p<2$. According to Theorem 3.2, the theorem is valid for $h=0$. Now let the theorem be true for $h-1$; we will prove that this also holds for h.

Differentiating equality (4.6) h times with respect to t, we get

$$\begin{aligned}&\int_\Omega\sum_{|\alpha|,|\beta|=0}^{m}a_{\alpha\beta}(x,t)D^\beta\widetilde{u^\epsilon_{t^h}}\overline{D^\alpha v}\,dx\\&=\sum_{k=0}^{h-1}\binom{h-1}{k}\int_\Omega\sum_{|\alpha|,|\beta|=0}^{m}(a_{\alpha\beta})_{t^{h-k}}D^\beta\widetilde{u^\epsilon_{t^k}}\overline{D^\alpha v}\,dx\\&\quad+\Big(\int_\Omega\widetilde{f}_{t^h}\overline{v}\,dx+\int_\Omega\widetilde{u}^\epsilon{}_{t^{h+2}}\overline{v}\,dx\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\Big|B(\widetilde{u^\epsilon}_{t^h}, v; t)\Big| &\le \Big|\int_\Omega \widetilde{f}_{t^h}\overline{v}\,dx\Big| + \Big|\int_\Omega \widetilde{u^\epsilon}_{t^{h+2}}\overline{v}\,dx\Big| \\ &\quad + \Big|\sum_{k=0}^{h-1}\binom{h}{k}\int_\Omega \sum_{|\alpha|,|\beta|=0}^{m}(a_{\alpha\beta})_{t^{h-k}}D^\beta\widetilde{u^\epsilon_{t^k}}\,\overline{D^\alpha v}\,dx\Big|.\end{aligned} \tag{4.17}$$

By the arguments analogous to the proof of Proposition 4.3, we have inequalities

$$\Big|\int_\Omega \widetilde{u^\epsilon}_{t^{h+2}}\overline{v}\,dx\Big| \le C_1\sum_{k=0}^{h+1}\|f_{t^k}(t)\|_{0,p}, \forall v\in S_q, \tag{4.18}$$

and

$$\Big|\int_\Omega \widetilde{f}_{t^h}\overline{v}\,dx\Big| \le \|f_{t^h}(t)\|_{0,p}, \forall v\in S_q. \tag{4.19}$$

By using Hölder's inequality, it is easy to see that

$$\begin{aligned}&\Big|\sum_{k=0}^{h-1}\binom{h}{k}\int_\Omega \sum_{|\alpha|,|\beta|=0}^{m}(a_{\alpha\beta})_{t^{h-k}}D^\beta\widetilde{u^\epsilon_{t^k}}\,\overline{D^\alpha v}\,dx\Big| \\ &\le \sum_{k=0}^{h-1}\mu\binom{h}{k}\sum_{|\alpha|,|\beta|=0}^{m}\Big|\int_\Omega D^\beta\widetilde{u^\epsilon_{t^k}}\,\overline{D^\alpha v}\,dx\Big| \\ &\le \sum_{k=0}^{h-1}\mu\binom{h}{k}\sum_{|\alpha|,|\beta|=0}^{m}\|D^\beta\widetilde{u^\epsilon_{t^k}}(t)\|_{0,p}\|D^\alpha v\|_{0,q} \\ &\le C_2\sum_{k=0}^{h-1}\|\widetilde{u^\epsilon_{t^k}}(t)\|_{m,p}\|v\|_{m,q} \le C_2\sum_{k=0}^{h-1}\|\widetilde{u^\epsilon_{t^k}}(t)\|_{m,p}, \forall v\in S_q.\end{aligned} \tag{4.20}$$

Substituting (4.18), (4.19) and (4.20) into (4.17), we obtain

$$\Big|B(\widetilde{u^\epsilon}_{t^h}(t), v; t)\Big| \le C_3\Big(\sum_{k=0}^{h-1}\|\widetilde{u^\epsilon_{t^k}}(t)\|_{m,p} + \sum_{k=0}^{h+1}\|f_{t^k}(t)\|_{0,p}\Big), \forall v\in S_q. \tag{4.21}$$

Because $\overset{\circ}{C}{}^\infty(\Omega)$ is dense in $\overset{\circ}{W}{}^m_q(\Omega)$, (4.17) holds for all $v\in \overset{\circ}{W}{}^m_q(\Omega)$. By using Garding's inequality (2.3), we obtain

$$\gamma_1\|\widetilde{u^\epsilon}_{t^h}(t)\|_{m,p} \le \sup_{v\in S_q}\Big|B(\widetilde{u^\epsilon}_{t^h}, v; t)\Big| \le C_3\Big(\sum_{k=0}^{h-1}\|\widetilde{u^\epsilon_{t^k}}(t)\|_{m,p} + \sum_{k=0}^{h+1}\|f_{t^k}(t)\|_{0,p}\Big).$$

Therefore,

$$\|\widetilde{u^{\epsilon}}_{t^h}(t)\|_{m,p} \le C_4\Big(\sum_{k=0}^{h-1}\|\widetilde{u^{\epsilon}_{t^k}}(t)\|_{m,p} + \sum_{k=0}^{h+1}\|f_{t^k}(t)\|_{0,p}\Big)$$

for all most everywhere $t \in (0,T)$, where the constant C_4 does not depend on ϵ. From the inequality above and Hölder's inequality, we get

$$\|\widetilde{u^{\epsilon}}_{t^h}(t)\|^p_{m,p} \le C_5\Big(\sum_{k=0}^{h-1}\|\widetilde{u^{\epsilon}_{t^k}}(t)\|^p_{m,p} + \sum_{k=0}^{h+1}\|f_{t^k}(t)\|^p_{0,p}\Big). \tag{4.22}$$

where the constant C_5 does not depend on ϵ. So by integrating with respect to t from 0 to T and using the induction assumption, we have the inequality

$$\int_0^T \|\widetilde{u^{\epsilon}}_{t^h}(t)\|^p_{m,p}dt \le C_6\sum_{k=0}^{h+1}\|f_{t^k}\|^p_{L_p(Q_T)}. \tag{4.23}$$

By differentiating equality (4.6) repeatedly $h+1$ times with respect to t and the arguments analogous to estimates above, we have the inequality

$$\int_0^T \|\widetilde{u^{\epsilon}}_{t^{h+1}}(t)\|^p_{m,p}dt \le C_7\sum_{k=0}^{h+2}\|f_{t^k}\|^p_{L_p(Q_T)}. \tag{4.24}$$

Combining (4.23) with(4.24), we obtain

$$\|\widetilde{u^{\epsilon}}_{t^h}\|^p_{m,1;p} \le C\sum_{k=0}^{h+2}\|f_{t^k}\|^p_{L_p(Q_T)},$$

where C is a constant independent of ϵ, u and f. Since $\widetilde{u}_{\varepsilon}$ converges weakly to generalized solution $u \in \overset{\circ}{W}{}^{m,1}_p(Q_T)$, the u has generalized derivatives with respect to t up to oder h in $\overset{\circ}{W}{}^{m,1}_p(Q_T)$ and

$$\|u_{t^h}\|^p_{m,1;p} \le \varliminf_{\varepsilon\to 0}\|\widetilde{u}^{\varepsilon}_{t^h}\|^p_{m,1;p} \le C\sum_{k=0}^{h+2}\|f_{t^k}\|^p_{L_p(Q_T)}.$$

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