TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 6, pp. 2291-2305, December 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

$l^\infty(X) - l^p(Y)$ SUMMABILITY OF MAPPING MATRICES

Li Ronglu and Zhong Shuhui*

Abstract. For Banach spaces X and Y, $\mathcal{F}_{C,\delta}(X,Y)$ is a large and meaningful extension of the family $L(X,Y)$ of linear operators. For classical Banach sequence spaces $l^\infty(X)$ and $l^p(Y)\,(p \geq 1)$ we find a characterization of the $l^\infty(X) - l^p(Y)$ transformation of matrices of mappings in $\mathcal{F}_{C,\delta}(X,Y)$.

0. Introduction

Let X and Y be topological vector spaces and $f_{ij} \in Y^X,\ i,\ j \in \mathbb{N}$. For sequence families $\lambda(X) \subset X^{\mathbb{N}}$ and $\mu(Y) \subset Y^{\mathbb{N}}$ the matrix $(f_{ij}) \in (\lambda(X), \mu(Y))$ means that $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges when $(x_j) \in \lambda(X),\ i \in \mathbb{N}$ and $(\sum_{j=1}^{\infty} f_{ij}(x_j))_{i=1}^{\infty} \in \mu(Y)$ for each $(x_j) \in \lambda(X)$.

We are interested in the sequence families such as

$$\begin{aligned} c_0(X) &= \{(x_j) \in X^{\mathbb{N}} : x_j \to 0\}, \\ c(X) &= \{(x_j) \in X^{\mathbb{N}} : \lim x_j \text{ exists}\}, \\ l^\infty(X) &= \{(x_j) \in X^{\mathbb{N}} : (x_j) \text{ is bounded}\} \text{ and} \\ l^p(X) &= \{(x_j) \in X^{\mathbb{N}} : \sum_{j=1}^{\infty} \|x_j\|^p < \infty\} \text{ when } X \text{ is normed and } p \geq 1. \end{aligned}$$

There is a nice result for the family $(l^\infty(X), c(Y))$ as follows.

Theorem A. ([1; 2]). *Let $X,\ Y$ be topological vector spaces and $f_{ij} : X \to Y$ a mapping such that $f_{ij}(0) = 0,\ \forall\, i,\ j \in \mathbb{N}$. If $(f_{ij}) \in (l^\infty(X), c(Y))$, then for every bounded $B \subset X$ the series $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$, and $\lim_i f_{ij}(x)$ exists for every $x \in X,\ j \in \mathbb{N}$.*

Received January 8, 2009, accepted April 17, 2009.
Communicated by Bor-Luh Lin.
2000 *Mathematics Subject Classification*: 47H99, 46A45.
Key words and phrases: Dissecting mapping, Equicontinuity, Summability.
*Corresponding author.

If, in addition, Y *is sequentially complete, then the converse holds. Especially, if* Y *is a Banach space, then* $(f_{ij}) \in (l^\infty(X), c(Y))$ *if and only if*

(1) $\lim_i f_{ij}(x)$ *exists,* $\forall\, x \in X,\ j \in \mathbb{N}$*, and*

(2) *for every bounded* $B \subset X$*,* $\lim_n \sup_{i\in\mathbb{N},\, m\geq n,\, \{x_j\}\subset B} \| \sum_{j=n}^{m} f_{ij}(x_j)\| = 0.$

Note that for the case of Banach spaces $X,\ Y$ and continuous linear operators $T_{ij} : X \to Y$, I.J. Maddox gave a characterization of $(T_{ij}) \in (l^\infty(X), c(Y))$ [3; 4, p.46]. It is easy to see that the Maddox theorem is a special case of Theorem A.

As an immediate consequence of Theorem A, we have

Theorem B. *Let* $X,\ Y$ *be topological vector spaces and* $f_{ij} : X \to Y$ *a mapping such that* $f_{ij}(0) = 0,\ \forall\, i, j \in \mathbb{N}$*. Then* $(f_{ij}) \in (l^\infty(X), c_0(Y))$ *if and only if for every bounded* $B \subset X$*,* $\sum_{j=1}^{\infty} f_{ij}(x_j)$ *converges uniformly with respect to both* $i \in \mathbb{N}$ *and* $\{x_j\} \subset B$*, and* $\lim_i f_{ij}(x) = 0,\ \forall\, x \in X,\ j \in \mathbb{N}$; $(f_{ij}) \in (l^\infty(X), l^\infty(Y))$ *if and only if for every bounded* $B \subset X$ *and* $(s_i) \in c_0$*,* $\sum_{j=1}^{\infty} s_i f_{ij}(x_j)$ *converges uniformly with respect to both* $i \in \mathbb{N}$ *and* $\{x_j\} \subset B$*, and* $\{f_{ij}(x)\}_{i=1}^{\infty}$ *is bounded,* $\forall\, x \in X,\ j \in \mathbb{N}$.

Since linear operators form a small subfamily of $\{f \in Y^X : f(0) = 0\}$, the most interesting point in Theorem A and B is just that these results were established for matrices of mappings satisfying $f(0) = 0$ only. However, for Banach spaces $X,\ Y$ and matrices of mappings in $\{f \in Y^X : f(0) = 0\}$, we have had no any description of the matrix family $(l^\infty(X), l^p(Y))\ (p \geq 1)$ as yet.

In this paper we would like to consider a very meaningful subfamily $\mathcal{F}_{C,\delta}(X, Y)$ of $\{f \in Y^X : f(0) = 0\}$. This is a large extension of the family of linear operators, and for matrices of mappings in $\mathcal{F}_{C,\delta}(X, Y)$ we will give a clear-cut characterization of the matrix family $(l^\infty(X), l^p(Y))\ (p \geq 1)$.

1. The Family of Dissecting Mappings

Let $X,\ Y$ be vector spaces over the scalar field $\mathbb{K}$. Every linear operator $T : X \to Y$ has the absolutely exact dissecting property:

$$T(x + tu) = T(x) + tT(u), \quad \forall\, x, u \in X,\ t \in \mathbb{K}.$$

A function $T : \mathbb{R} \to \mathbb{R}$ is linear if and only if $T(x) = \alpha x$ for all $x \in \mathbb{R}$, where $\alpha = T(1)$ is a constant. Therefore, linear operators in $\mathbb{R}^{\mathbb{R}}$ can be used only to describe Newton's First Law of Motion, i.e., we have

Proposition 1. *The physical meaning of a linear operator* $T : \mathbb{R} \to \mathbb{R}$ *is just a motion at constant speed along a straight-line path.*

Definition 1.1. ([5; 6; 7]). Let $X,\ Y$ be normed spaces over $\mathbb{K}$. A mapping $f : X \to Y$ is said to be dissecting if $f(0) = 0$ and there exist $C \geq 1$ and $\delta > 0$ such that every $x,\ u \in X$ and $t \in \mathbb{K}$ with $\|u\| \leq \delta$ and $|t| \leq 1$ determine $r,\ s \in \mathbb{K}$ for which $|r - 1| \leq C|t|$, $|s| \leq C|t|$ and $f(x + tu) = rf(x) + sf(u)$.

Let $\mathcal{F}_{C,\delta}(X, Y)$ be the family of dissecting maps related to $C \geq 1$ and $\delta > 0$, and let

$\mathcal{E}_{C,\delta}(X, Y) = \big\{f \in \mathcal{F}_{C,\delta}(X, Y) :$ if $x,\ u \in X$ and $t \in \mathbb{K}$ for which $\|u\| \leq \delta$ and $|t| \leq 1,$ then $f(x + tu) = f(x) + sf(u)$ with $|s| \leq C|t|\big\}$.

The family of dissecting maps is a large extension of the family of linear operators. Especially, each nonzero linear operator produces uncountably many of nonlinear dissecting maps (see [5; 6; 7]).

Dissecting maps in $\mathbb{R}^{\mathbb{R}}$ contrast sharply with linear operators in $\mathbb{R}^{\mathbb{R}}$. In fact, since no material object can travel at or beyond the speed of light, it follows from Theorem 1.1 of [7] that every motion in the classical mechanics can be described by a dissecting map in $\mathbb{R}^{\mathbb{R}}$.

Proposition 2. *Let $(x(t),\ y(t),\ z(t))$ be the position in $\mathbb{R}^3$ of a material object at the moment t. Let $a \geq 0$ for which $\dot{x}(a) \neq 0$ and $s = t - a$ for $t \geq a$, $f(s) = x(t) - x(a)$ and $f(-s) = -f(s)$. Then $f \in \mathcal{E}_{C,\delta}(\mathbb{R}, \mathbb{R})$ for some $C \geq 1$ and $\delta > 0$, and similar conclusions hold for $y(t)$ and $z(t)$.*

The family of dissecting maps is a very important extension of the family of linear operators. Indeed, we have the following very important result which is an essential improvement of a basic principle of functional analysis.

Recall that for topological vector spaces X and Y a mapping $f : X \to Y$ is bounded if $f(B)$ is bounded when $B \subset X$ is bounded.

Theorem 1.1. ([5; 6; 7]). *Let $X,\ Y$ be normed spaces and X is of second category. Let $C \geq 1$ and $\delta > 0$. If $\Gamma \subset \mathcal{F}_{C,\delta}(X, Y)$ is a pointwise bounded family of bounded dissecting maps, then Γ is equicontinuous on X, i.e., for every $x \in X$ and $\varepsilon > 0$ there is an $\alpha > 0$ such that $\|f(z) - f(x)\| < \varepsilon$ for all $f \in \Gamma$ whenever $\|z - x\| < \alpha$, and Γ is uniformly bounded on each bounded subset of X, i.e., $\sup_{f\in\Gamma,\, x\in B} \|f(x)\| < +\infty$ whenever $\sup_{x\in B} \|x\| < +\infty$.*

2. Basic Propositions

First, we improve the simple but useful Theorem 1 of [8] which implies that if G is an abelian topological group and $\{x_j\} \subset G$ such that $\sum x_j$ is subseries convergent, then $\{\sum_{j\in\Delta} x_j : \Delta \subseteq \mathbb{N}\}$ is both compact and sequentially compact [8, Corollary 2].

Lemma 2.1. *Let G be an abelian topological group. Then for every $\Omega \neq \emptyset$ and $\{f_j\} \subset G^\Omega$, the following (α) and (β) are equivalent.*

(α) $\sum_{j=1}^{\infty} f_j(\omega_j)$ *converges for each* $\{\omega_j\} \subset \Omega$.

(β) $\sum_{j=1}^{\infty} f_j(\omega_j)$ *converges uniformly with respect to* $\{\omega_j\} \subset \Omega$.

Proof. If (α) holds but (β) fails, then there exist a neighborhood U of $0 \in G$ and integers $m_1 < n_1 < m_2 < n_2 \cdots$ and $\{\omega_{ij} : m_i \leq j \leq n_i, i \in \mathbb{N}\} \subset \Omega$ such that $\sum_{j=m_i}^{n_i} f_j(\omega_{ij}) \notin U,\ i = 1, 2, 3, \cdots$. Pick an $\omega_0 \in \Omega$ and let

$$\omega_j = \begin{cases} \omega_{ij}, & m_i \leq j \leq n_i,\ i = 1, 2, 3, \cdots, \\ \omega_0, & \text{otherwise,} \end{cases}$$

then $\sum_{j=1}^{\infty} f_j(\omega_j)$ diverges. This contradicts (α). ∎

Henceforth, X and Y are Banach spaces and $C \geq 1$. Since $\mathcal{F}_{C,\gamma}(X, Y) \subset \mathcal{F}_{C,\delta}(X, Y)$ for $0 < \delta \leq \gamma$, we always assume that $0 < \delta \leq 1$ in the notion $\mathcal{F}_{C,\delta}(X, Y)$. For $(x_j) \in l^\infty(X)$, let $\|(x_j)\|_\infty = \sup_j \|x_j\|$.

Lemma 2.2. *For $f \in \mathcal{F}_{C,\delta}(X, Y)$ the following* (1), (2), (3) *and* (4) *are equivalent.*

(1) f *is continuous.*

(2) f *is bounded, i.e., for every* $a > 0$, $\|f\|_{(a)} = \sup_{\|x\| \leq a} \|f(x)\| < +\infty$.

(3) *If* $(u_j) \in c_0(X)$, *i.e.,* $u_j \to 0$ *in* X, *then* $\{f(u_j) : j \in \mathbb{N}\}$ *is bounded:*

$$\sup_j \|f(u_j)\| < +\infty.$$

(4) f *is continuous at* $0 \in X$.

Proof. (1)$\Longrightarrow$(2). Let $\theta \in (0, \delta)$ for which $\|f(x)\| < 1$ whenever $\|x\| < \theta$, and $a > 0$. Pick an integer $n > 1$ for which $\frac{a}{n} < \theta$. Then $\|f(\frac{x}{n})\| < 1,\ \forall \|x\| \leq a$.

Let $\|x\| \leq a$. Since $\|\frac{x}{n}\| < \theta < \delta$, it follows from $f \in \mathcal{F}_{C,\delta}(X, Y)$ that

$$\begin{aligned} f(x) &= f\big(\frac{n-1}{n}x + \frac{x}{n}\big) \\ &= r_1 f\big(\frac{n-1}{n}x\big) + s_1 f\big(\frac{x}{n}\big) \\ &= r_1 r_2 f\big(\frac{n-2}{n}x\big) + r_1 s_2 f\big(\frac{x}{n}\big) + s_1 f\big(\frac{x}{n}\big) \\ &\ \ \vdots \\ &= (r_1 r_2 \cdots r_{n-1} + r_1 r_2 \cdots r_{n-2} s_{n-1} + \cdots + r_1 s_2 + s_1) f\big(\frac{x}{n}\big), \end{aligned}$$

where $|r_i| \le 1 + C$, $|s_i| \le C$. Thus, $\sup_{\|x\| \le a} \|f(x)\| \le n(1+C)^{n-1} < +\infty$.

(2) $\Longrightarrow$ (3). Obvious.

(3) $\Longrightarrow$ (4). Let $x_j \to 0$ in X. May assume that all $x_j \ne 0$ and $\sqrt{\|x_j\|} < \delta$, $\forall\, j \in \mathbb{N}$. Then $x_j = \sqrt{\|x_j\|}\frac{x_j}{\sqrt{\|x_j\|}}$, $\sqrt{\|x_j\|} \to 0$ and $\frac{x_j}{\sqrt{\|x_j\|}} \to 0 : \|\frac{x_j}{\sqrt{\|x_j\|}}\| = \sqrt{\|x_j\|} \to 0$. By (3), $\sup_j \|f(\frac{x_j}{\sqrt{\|x_j\|}})\| = M < +\infty$ and so

$$\|f(x_j)\| = \Big\|f\big(\sqrt{\|x_j\|}\frac{x_j}{\sqrt{\|x_j\|}}\big)\Big\| = \Big\|s_j f\big(\frac{x_j}{\sqrt{\|x_j\|}}\big)\Big\| \le C\sqrt{\|x_j\|}M \to 0,$$

i.e., $f(x_j) \to 0 = f(0)$.

(4) $\Longrightarrow$ (1). Let $x_j \to x$ in X. May assume that all $x_j \ne x$ and $\sqrt{\|x_j - x\|} < \delta$, $\forall\, j \in \mathbb{N}$. Then $f(x_j) = f(x + x_j - x) = f(x + \sqrt{\|x_j - x\|}\frac{x_j - x}{\sqrt{\|x_j - x\|}}) = r_j f(x) + s_j f(\frac{x_j - x}{\sqrt{\|x_j - x\|}})$ where $|r_j - 1| \le C\sqrt{\|x_j - x\|} \to 0$ and $|s_j| \le C\sqrt{\|x_j - x\|} \to 0$. Since $\|\frac{x_j - x}{\sqrt{\|x_j - x\|}}\| = \sqrt{\|x_j - x\|} \to 0$, $f(\frac{x_j - x}{\sqrt{\|x_j - x\|}}) \to 0$ by (4), $f(x_j) = r_j f(x) + s_j f(\frac{x_j - x}{\sqrt{\|x_j - x\|}}) \to f(x)$. ■

Lemma 2.3. *Let $k \in \mathbb{N}$ and $f_{ij} \in \mathcal{F}_{C,\delta}(X, Y)$ be bounded, $\forall\, i, j \in \mathbb{N}$. If $(f_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), l^1(Y))$ and*

$$\Phi_k((x_j)) = \Big(\sum_{j=1}^{\infty} f_{1j}(x_j), \sum_{j=1}^{\infty} f_{2j}(x_j), \cdots, \sum_{j=1}^{\infty} f_{kj}(x_j), 0, 0, \cdots\Big),\ (x_j) \in l^\infty(X),$$

then $\Phi_k : l^\infty(X) \to l^1(Y)$ is continuous at $0 \in l^\infty(X)$, and for every $a > 0$,

$$\|\Phi_k\|_{(a)} = \sup_{\|(x_j)\|_\infty \le a} \sum_{i=1}^{k} \Big\|\sum_{j=1}^{\infty} f_{ij}(x_j)\Big\| < +\infty.$$

Proof. Let $a > 0$, $\varepsilon > 0$. Since $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges for each $\{x_j\} \subset \{x \in X : \|x\| \le a\}$ and $i \in \mathbb{N}$, Lemma 2.1 shows that there is an $n_0 \in \mathbb{N}$ for which

$$\Big\|\sum_{j=n_0+1}^{\infty} f_{ij}(x_j)\Big\| < \varepsilon/2k,\ \ \forall\, \|(x_j)\|_\infty \le a,\ 1 \le i \le k.$$

By Lemma 2.2, for $i, j \in \mathbb{N}$ there is $\theta_{ij} > 0$ such that $\|f_{ij}(x)\| < \varepsilon/2kn_0$ whenever $\|x\| \le \theta_{ij}$, and let $\theta = \min\{\theta_{ij} : 1 \le i \le k,\ 1 \le j \le n_0\}$. Then $\|f_{ij}(x)\| < \varepsilon/2kn_0$, $\forall\, \|x\| \le \theta$, $1 \le i \le k$, $1 \le j \le n_0$. Hence for $\|(x_j)\|_\infty \le \min(\theta, a)$ we have that $\|\Phi_k((x_j))\|_1 = \sum_{i=1}^{k} \|\sum_{j=1}^{\infty} f_{ij}(x_j)\| \le \sum_{i=1}^{k}\sum_{j=1}^{n_0} \|f_{ij}(x_j)\| + \sum_{i=1}^{k} \|\sum_{j=n_0+1}^{\infty} f_{ij}(x_j)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, Φ_k is continuous at $0 \in l^\infty(X)$.

Let $\|(x_j)\|_\infty \le a$. It follows from Lemma 2.2 that

$$\begin{aligned}\sum_{i=1}^{k}\Big\|\sum_{j=1}^{\infty} f_{ij}(x_j)\Big\| &\le \sum_{i=1}^{k}\sum_{j=1}^{n_0}\|f_{ij}(x_j)\| + \sum_{i=1}^{k}\Big\|\sum_{j=n_0+1}^{\infty} f_{ij}(x_j)\Big\| \\ &\le \sum_{i=1}^{k}\sum_{j=1}^{n_0}\sup_{\|x\|\le a}\|f_{ij}(x)\| + \frac{\varepsilon}{2} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{n_0}\|f_{ij}\|_{(a)} + \frac{\varepsilon}{2} < +\infty. \qquad \blacksquare\end{aligned}$$

Lemma 2.4. *Let* $f_i \in \mathcal{F}_{C,\delta}(X,Y)$ *be bounded,* $\forall\, i \in \mathbb{N}$. *If* $\sum_{i=1}^\infty \|f_i(x)\| < +\infty$ *for each* $x \in X$*, then the function sequence* $\{\sum_{i=1}^n \|f_i(\cdot)\| \in \mathbb{R}^X : n \in \mathbb{N}\}$ *is equicontinuous at* $0 \in X$*, i.e.,* $\forall\, \varepsilon > 0\ \exists\, \gamma > 0$ *such that* $\sum_{i=1}^n \|f_i(x)\| \le \varepsilon$ *for all* $\|x\| < \gamma$ *and all* $n \in \mathbb{N}$. *Thus,* $\sum_{i=1}^\infty \|f_i(\cdot)\| : X \to \mathbb{R}$ *is continuous at* $0 \in X$.

Proof. Let $0<\varepsilon<1$, $I=[0, \frac{\varepsilon}{2(1+C)}]$ and $M=\{x\in X : \|x\|\le\delta,\ \sum_{i=1}^n \|f_i(x)\|\in I$ for all $n \in \mathbb{N}\}$. Then $0 \in M$.

By Lemma 2.2, $\sum_{i=1}^n \|f_i(\cdot)\| : X \to \mathbb{R}$ is continuous for all $n \in \mathbb{N}$ and so M is a nonempty closed set in X.

Let $x \in X$ and $\alpha \in (0,1)$ for which $\|\alpha x\| \le \delta$, and $\theta = \frac{\varepsilon}{2(1+C)(1+\sum_{i=1}^\infty \|f_i(\alpha x)\|)}$. If $0 \le t \le \frac{\theta}{C}$, then

$$\begin{aligned}\sum_{i=1}^{n}\|f_i(t\alpha x)\| &= \sum_{i=1}^{n}\|s_i f_i(\alpha x)\| \qquad\qquad (|s_i| \le Ct \le \theta) \\ &\le \theta\sum_{i=1}^{n}\|f_i(\alpha x)\| < \frac{\varepsilon}{2(1+C)}, \quad \forall\, n \in \mathbb{N},\end{aligned}$$

i.e., $t\alpha x \in M,\ \forall\, 0 \le t \le \frac{\theta}{C}$. Hence, $\frac{1}{n}x = \frac{1}{n\alpha}\alpha x \in M$ whenever $n \ge \frac{C}{\alpha\theta}$.

This shows that $X = \bigcup_{n=1}^\infty nM$. Since X is a Banach space, it follows from Baire category theorem that $\{x \in X : \|x\| < \gamma\} \subset M - M = \{u - v : u, v \in M\}$ for some $\gamma > 0$.

Let $x,\ u \in M$. Then $\|u\| \le \delta$. Since $\|y + tz\| = \|y\| + s\|z\|$ with $|s| \le |t|$ for $y,\ z \in Y$ and $t \in \mathbb{K}$, it follows that $\sum_{i=1}^n \|f_i(x-u)\| = \sum_{i=1}^n \|r_i f_i(x)+s_i f_i(u)\| = \sum_{i=1}^n (\|r_i f_i(x)\| + s_i'\|f_i(u)\|) = \sum_{i=1}^n |r_i|\|f_i(x)\| + \sum_{i=1}^n s_i'\|f_i(u)\|$, $\forall\, n \in \mathbb{N}$, where $|r_i - 1| \le C|-1| = C,\ |s_i'| \le |s_i| \le C|-1| = C$. Hence,

$$\sum_{i=1}^{n}\|f_i(x-u)\| \le (1+C)\Big(\sum_{i=1}^{n}\|f_i(x)\| + \sum_{i=1}^{n}\|f_i(u)\|\Big)$$

$$\leq (1+C)\Big(\frac{\varepsilon}{2(1+C)} + \frac{\varepsilon}{2(1+C)}\Big) = \varepsilon, \quad \forall\, n \in \mathbb{N}.$$

Since $\{x \in X : \|x\| < \gamma\} \subset M - M$, $\sum_{i=1}^{n} \|f_i(x)\| \leq \varepsilon$ when $n \in \mathbb{N}$ and $\|x\| < \gamma$. ■

Lemma 2.5. *Let $n \in \mathbb{N}$ and $f_{ij} \in \mathcal{F}_{C,\delta}(X,Y)$ be bounded, $\forall\, i,\, j \in \mathbb{N}$. If $(f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^1(Y))$ and*

$$F_n((x_j)) = \Big(\sum_{j=1}^{n} f_{ij}(x_j)\Big)_{i=1}^{\infty}, \quad (x_j) \in l^\infty(X),$$

then $F_n : l^\infty(X) \to l^1(Y)$ is continuous at $0 \in l^\infty(X)$ and for every $a > 0$,

$$\|F_n\|_{(a)} = \sup_{\|(x_j)\|_\infty \leq a} \sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{n} f_{ij}(x_j) \Big\| < +\infty.$$

Proof. Let $\lim_k (x_{kj})_{j=1}^{\infty} = 0$ in $l^\infty(X)$, i.e., $\lim_k \sup_j \|x_{kj}\| = 0$.

Since $(f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^1(Y))$ and $(0, \cdots, 0, \overset{(j)}{x}, 0, 0, \cdots) \in l^\infty(X)$ when $x \in X$ and $j \in \mathbb{N}$, $\sum_{i=1}^{\infty} \|f_{ij}(x)\| \leq +\infty$ for every $x \in X$ and $j \in \mathbb{N}$. By Lemma 2.4, $\lim_k \sum_{i=1}^{\infty} \|f_{ij}(x_{kj})\| = 0$ for each $j \in \mathbb{N}$ and

$$\lim_k \|F_n((x_{kj})_{j=1}^{\infty})\|_1 = \lim_k \sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{n} f_{ij}(x_{kj}) \Big\| \leq \lim_k \sum_{j=1}^{n} \sum_{i=1}^{\infty} \|f_{ij}(x_{kj})\| = 0.$$

Thus, $F_n : l^\infty(X) \to l^1(Y)$ is continuous at $0 \in l^\infty(X)$.

Let $a > 0$. By Lemma 2.4, for each $j \in \mathbb{N}$ there is a $\theta_j \in (0, \delta)$ such that $\sum_{i=1}^{\infty} \|f_{ij}(x)\| < 1$ whenever $\|x\| < \theta_j$. Letting $\theta = \min(\theta_1, \theta_2, \cdots, \theta_n)$, $\sum_{i=1}^{\infty} \|f_{ij}(x)\| < 1,\ \forall\, \|x\| < \theta,\ 1 \leq j \leq n$.

Pick an integer $n_0 > \frac{a}{\theta}$ and let $j \in \{1, 2, \cdots, n\}$, $\|x\| \leq a$. Then $\|\frac{x}{n_0}\| = \frac{\|x\|}{n_0} < \theta < \delta$ and

$$\begin{aligned}
\sum_{i=1}^{\infty} \|f_{ij}(x)\| &= \sum_{i=1}^{\infty} \Big\| f_{ij}\big(\frac{n_0 - 1}{n_0} x + \frac{x}{n_0}\big) \Big\| \\
&= \sum_{i=1}^{\infty} \Big\| r_{ij1} f_{ij}\big(\frac{n_0 - 1}{n_0} x\big) + s_{ij1} f_{ij}\big(\frac{x}{n_0}\big) \Big\| \\
&= \sum_{i=1}^{\infty} \Big\| r_{ij1} r_{ij2} f\big(\frac{n_0 - 2}{n_0} x\big) + r_{ij1} s_{ij2} f_{ij}\big(\frac{x}{n_0}\big) + s_{ij1} f_{ij}\big(\frac{x}{n_0}\big) \Big\| \\
&\ \ \vdots
\end{aligned}$$

$$= \sum_{i=1}^{\infty} \Big\| (r_{ij1} r_{ij2} \cdots r_{ij\ n_0-1} + r_{ij1} r_{ij2} \cdots r_{ij\ n_0-2} s_{ij\ n_0-1} + \cdots + r_{ij1} s_{ij2} + s_{ij1}) f_{ij}\Big(\frac{x}{n_0}\Big) \Big\|,$$

where $|r_{ijk} - 1| \leq C|1| = C$, $|s_{ijk}| \leq C|1| = C$ and so

$$\begin{aligned} \sum_{i=1}^{\infty} \|f_{ij}(x)\| &\leq n_0(1+C)^{n_0-1} \sum_{i=1}^{\infty} \Big\| f_{ij}\Big(\frac{x}{n_0}\Big) \Big\| \\ &\leq n_0(1+C)^{n_0-1}, \quad \forall \|x\| \leq a,\ 1 \leq j \leq n, \\ \|F_n\|_{(a)} &= \sup_{\|(x_j)\|_\infty \leq a} \sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{n} f_{ij}(x_j) \Big\| \\ &\leq \sup_{\|(x_j)\|_\infty \leq a} \sum_{j=1}^{n} \sum_{i=1}^{\infty} \|f_{ij}(x_j)\| \\ &\leq n n_0 (1+C)^{n_0-1} < +\infty. \end{aligned}$$

■

We also need a useful fact in linear algebra.

Lemma 2.6. ([9, Lemma 3.2]). *Let E be a vector space and V a convex set in E, $0 \in V$. If $x_1, x_2, \cdots, x_n \in E$ and $M > 0$ such that*

$$M \sum_{j \in \Delta} x_j \in V, \quad \forall \Delta \subseteq \{1, 2, \cdots, n\},\ \Delta \neq \emptyset,$$

then $\sum_{j=1}^{n} s_j x_j \in V$, $\forall\, 0 \leq s_j \leq M,\ j = 1, 2, \cdots, n$.

3. A Characterization of the Matrix Family $(l^\infty(X), l^p(Y))$

Let $X,\ Y$ be real Banach spaces and $C \geq 1$, $\delta > 0$. Let Y' be the dual space of Y.

Theorem 3.1. *Let $f_{ij} \in \mathcal{F}_{C,\delta}(X, Y)$ be bounded, $\forall\, i, j \in \mathbb{N}$. If $(f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^1(Y))$ and*

$$[f_{ij}]((x_j)) = \Big(\sum_{j=1}^{\infty} f_{ij}(x_j) \Big)_{i=1}^{\infty}, \quad (x_j) \in l^\infty(X),$$

then for every $a \geq 0$,

$$\|[f_{ij}]\|_{(a)} = \sup_{\|(x_j)\|_\infty \leq a} \sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{\infty} f_{ij}(x_j) \Big\| < +\infty.$$

Proof. Suppose that $a > 0$ and $\|[f_{ij}]\|_{(a)} = +\infty$. Pick $(x_{1j}) \in l^\infty(X)$ for which $\|(x_{1j})\|_\infty \le a$ and $\sum_{i=1}^\infty \| \sum_{j=1}^\infty f_{ij}(x_{1j})\| > 1$. Then $\sum_{i=1}^{i_1} \| \sum_{j=1}^\infty f_{ij}(x_{1j})\| > 1$ for some $i_1 > 1$ and so $\sum_{i=1}^{i_1} \| \sum_{j=1}^{n_1} f_{ij}(x_{1j})\| > 1$ for some $n_1 > 1$.

For each $i \in \{1, 2, \cdots, i_1\}$ pick $y_i' \in Y'$ with $\|y_i'\| = 1$ and $\| \sum_{j=1}^{n_1} f_{ij}(x_{1j})\| = y_i'(\sum_{j=1}^{n_1} f_{ij}(x_{1j}))$. Then $\sum_{i=1}^{i_1} y_i'(\sum_{j=1}^{n_1} f_{ij}(x_{1j})) > 1$.

Let $\Phi_{i_1}((x_j)) = (\sum_{j=1}^\infty f_{1j}(x_j), \cdots, \sum_{j=1}^\infty f_{i_1 j}(x_j), 0, 0, \cdots)$ and $F_{n_1}((x_j)) = (\sum_{j=1}^{n_1} f_{ij}(x_j))_{i=1}^\infty, \ \forall (x_j) \in l^\infty(X)$. By Lemma 2.3 and 2.5,

$$\|\Phi_{i_1}\|_{(a)} = \sup_{\|(x_j)\|_\infty \le a} \sum_{i=1}^{i_1} \Big\| \sum_{j=1}^\infty f_{ij}(x_j) \Big\| < +\infty,$$
$$\|F_{n_1}\|_{(a)} = \sup_{\|(x_j)\|_\infty \le a} \sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{n_1} f_{ij}(x_j) \Big\| < +\infty.$$

Since $\|[f_{ij}]\|_{(a)} = +\infty$, there is $(x_{2j}) \in l^\infty(X)$ with $\|(x_{2j})\|_\infty \le a$ such that $\sum_{i=1}^\infty \| \sum_{j=1}^\infty f_{ij}(x_{2j})\| > 1 + \|\Phi_{i_1}\|_{(a)} + \|F_{n_1}\|_{(a)}$. Then $\sum_{i=1}^{i_2} \| \sum_{j=1}^{n_2} f_{ij}(x_{2j})\| > 1 + \|\Phi_{i_1}\|_{(a)} + \|F_{n_1}\|_{(a)}$ for some $i_2 > i_1$ and $n_2 > n_1$, and it follows from $\|\Phi_{i_1}\|_{(a)} \ge \sum_{i=1}^{i_1} \| \sum_{j=1}^{n_2} f_{ij}(x_{2j}) + \sum_{j=n_2+1}^\infty f_{ij}(0)\| = \sum_{i=1}^{i_1} \| \sum_{j=1}^{n_2} f_{ij}(x_{2j})\|$ and $\|F_{n_1}\|_{(a)} \ge \sum_{i=1}^\infty \| \sum_{j=1}^{n_1} f_{ij}(x_{2j})\|$ that

$$\begin{aligned}\sum_{i=1}^{i_2} \Big\| \sum_{j=1}^{n_2} f_{ij}(x_{2j}) \Big\| &> 1 + \sum_{i=1}^{i_1} \Big\| \sum_{j=1}^{n_2} f_{ij}(x_{2j}) \Big\| + \sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{n_1} f_{ij}(x_{2j}) \Big\| \\ &\ge 1 + \sum_{i=1}^{i_1} \Big\| \sum_{j=1}^{n_2} f_{ij}(x_{2j}) \Big\| + \sum_{i=i_1+1}^{i_2} \Big\| \sum_{j=1}^{n_1} f_{ij}(x_{2j}) \Big\|,\end{aligned}$$

i.e., $\sum_{i=i_1+1}^{i_2} \| \sum_{j=1}^{n_2} f_{ij}(x_{2j})\| > 1 + \sum_{i=i_1+1}^{i_2} \| \sum_{j=1}^{n_1} f_{ij}(x_{2j})\|$ and

$$\begin{aligned}\sum_{i=i_1+1}^{i_2} \Big\| \sum_{j=n_1+1}^{n_2} f_{ij}(x_{2j}) \Big\| &= \sum_{i=i_1+1}^{i_2} \Big\| \sum_{j=1}^{n_2} f_{ij}(x_{2j}) - \sum_{j=1}^{n_1} f_{ij}(x_{2j}) \Big\| \\ &\ge \sum_{i=i_1+1}^{i_2} \left(\Big\| \sum_{j=1}^{n_2} f_{ij}(x_{2j}) \Big\| - \Big\| \sum_{j=1}^{n_1} f_{ij}(x_{2j}) \Big\| \right) \\ &= \sum_{i=i_1+1}^{i_2} \Big\| \sum_{j=1}^{n_2} f_{ij}(x_{2j}) \Big\| - \sum_{i=i_1+1}^{i_2} \Big\| \sum_{j=1}^{n_1} f_{ij}(x_{2j}) \Big\| > 1.\end{aligned}$$

Now for each $i \in \{i_1 + 1, i_1 + 2, \cdots, i_2\}$ pick $y_i' \in Y'$ with $\|y_i'\| = 1$ and $\| \sum_{j=n_1+1}^{n_2} f_{ij}(x_{2j})\| = y_i'(\sum_{j=n_1+1}^{n_2} f_{ij}(x_{2j}))$. Then

$$\sum_{i=i_1+1}^{i_2} y_i'(\sum_{j=n_1+1}^{n_2} f_{ij}(x_{2j})) > 1$$

.

Continuing this construction inductively gives integer sequences $0 = i_0 < i_1 < i_2 < i_3 < \cdots$, $0 = n_0 < n_1 < n_2 < n_3 < \cdots$ and $\{y'_i\} \subset Y'$ with $\|y'_i\| = 1$ and $\{x_{kj} : n_{k-1} + 1 \leq j \leq n_k, k \in \mathbb{N}\} \subset \{x \in X : \|x\| \leq a\}$ such that

$$y'_i\Big(\sum_{j=n_{k-1}+1}^{n_k} f_{ij}(x_{kj})\Big) = \Big\|\sum_{j=n_{k-1}+1}^{n_k} f_{ij}(x_{kj})\Big\|,\ i_{k-1}+1 \leq i \leq i_k,\ k = 1, 2, 3, \cdots,$$

$$(*) \qquad \sum_{i=i_{k-1}+1}^{i_k} y'_i\Big(\sum_{j=n_{k-1}+1}^{n_k} f_{ij}(x_{kj})\Big) > 1, \quad k = 1, 2, 3, \cdots.$$

Consider the matrix $[\sum_{i=i_{k-1}+1}^{i_k} y'_i(\sum_{j=n_{p-1}+1}^{n_p} f_{ij}(x_{pj}))]_{k,p\in\mathbb{N}}$. Fix $p \in \mathbb{N}$ and let

$$z_j = \begin{cases} x_{pj}, & n_{p-1} + 1 \leq j \leq n_p, \\ 0, & \text{otherwise}. \end{cases}$$

Then $(z_j) \in l^\infty(X)$ and $\sum_{i=1}^\infty \|\sum_{j=n_{p-1}+1}^{n_p} f_{ij}(x_{pj})\| = \sum_{i=1}^\infty \|\sum_{j=1}^\infty f_{ij}(z_j)\| < +\infty$ and so

$$\begin{aligned} 0 &\leq \lim_k \Big|\sum_{i=i_{k-1}+1}^{i_k} y'_i\Big(\sum_{j=n_{p-1}+1}^{n_p} f_{ij}(x_{pj})\Big)\Big| \\ &\leq \lim_k \sum_{i=i_{k-1}+1}^{i_k} \Big|y'_i\Big(\sum_{j=n_{p-1}+1}^{n_p} f_{ij}(x_{pj})\Big)\Big| \\ &\leq \lim_k \sum_{i=i_{k-1}+1}^{i_k} \|y'_i\| \Big\|\sum_{j=n_{p-1}+1}^{n_p} f_{ij}(x_{pj})\Big\| \\ &= \lim_k \sum_{i=i_{k-1}+1}^{i_k} \Big\|\sum_{j=n_{p-1}+1}^{n_p} f_{ij}(x_{pj})\Big\| = 0, \end{aligned}$$

i.e., $\lim_k \sum_{i=i_{k-1}+1}^{i_k} y'_i(\sum_{j=n_{p-1}+1}^{n_p} f_{ij}(x_{pj})) = 0,\ \forall\, p \in \mathbb{N}$.

Let $p_1 < p_2 < \cdots$ in $\mathbb{N}$ and $u_j = \begin{cases} x_{p_\nu j}, & n_{p_\nu - 1} + 1 \leq j \leq n_{p_\nu},\ \nu = 1, 2, 3, \cdots, \\ 0, & \text{otherwise}. \end{cases}$

Then $(u_j) \in l^\infty(X)$ and $\sum_{i=1}^\infty \|\sum_{j=1}^\infty f_{ij}(u_j)\| < +\infty$. Hence,

$$0 = \lim_k \sum_{i=i_{k-1}+1}^{i_k} \Big\|\sum_{j=1}^\infty f_{ij}(u_j)\Big\|$$

$$\geq \lim_k \sum_{i=i_{k-1}+1}^{i_k} \Big| y_i'\Big(\sum_{j=1}^{\infty} f_{ij}(u_j)\Big)\Big|$$

$$\geq \lim_k \Big| \sum_{i=i_{k-1}+1}^{i_k} y_i'\Big(\sum_{j=1}^{\infty} f_{ij}(u_j)\Big)\Big|$$

$$= \lim_k \Big| \sum_{i=i_{k-1}+1}^{i_k} \sum_{j=1}^{\infty} y_i'(f_{ij}(u_j))\Big|$$

$$= \lim_k \Big| \sum_{i=i_{k-1}+1}^{i_k} \sum_{\nu=1}^{\infty} \sum_{j=n_{p_\nu -1}+1}^{n_{p_\nu}} y_i'(f_{ij}(x_{p_\nu j}))\Big|$$

$$= \lim_k \Big| \sum_{\nu=1}^{\infty} \sum_{i=i_{k-1}+1}^{i_k} y_i'\Big(\sum_{j=n_{p_\nu -1}+1}^{n_{p_\nu}} f_{ij}(x_{p_\nu j})\Big)\Big| \geq 0,$$

i.e., $\lim_k \sum_{\nu=1}^{\infty} \sum_{i=i_{k-1}+1}^{i_k} y_i'(\sum_{j=n_{p_v -1}+1}^{n_{p_\nu}} f_{ij}(x_{p_v j})) = 0$.

Then $\lim_k \sum_{i=i_{k-1}+1}^{i_k} y_i'(\sum_{j=n_{k-1}+1}^{n_k} f_{ij}(x_{kj})) = 0$ by Antosik-Mikusinski matrix theorem [10; 11; 2]. This contradicts $(*)$ and so $\|[f_{ij}]\|_{(a)} < +\infty$. ■

Lemma 3.1. $p, p' > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $f_{ij} \in Y^X$ *for* $i, j \in \mathbb{N}$. *Then* $(f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^p(Y))$ *if and only if* $(s_i f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^1(Y))$ *for each* $(s_i) \in l^{p'}$.

Proof. If $\sum_{i=1}^{\infty} \| \sum_{j=1}^{\infty} f_{ij}(x_j)\|^p < +\infty$ and $(s_i) \in l^{p'}$, then

$$\sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{\infty} s_i f_{ij}(x_j)\Big\| = \sum_{i=1}^{\infty} |s_i| \Big\| \sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|$$
$$\leq \Big(\sum_{i=1}^{\infty} |s_i|^{p'}\Big)^{1/p'} \Big(\sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|^p\Big)^{1/p} < +\infty.$$

Conversely, if $\sum_{i=1}^{\infty} |s_i| \, \| \sum_{j=1}^{\infty} f_{ij}(x_j)\| = \sum_{i=1}^{\infty} \| \sum_{j=1}^{\infty} s_i f_{ij}(x_j)\| < +\infty$, $\forall\, (s_i) \in l^{p'}$, then $\sum_{i=1}^{\infty} \| \sum_{j=1}^{\infty} f_{ij}(x_j)\|^p < +\infty$. ■

Theorem 3.2. *Let* $p \geq 1$ *and* $f_{ij} \in \mathcal{F}_{C,\delta}(X, Y)$ *be bounded,* $\forall\, i, j \in \mathbb{N}$. *If* $(f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^p(Y))$ *and*

$$[f_{ij}]((x_j)) = \Big(\sum_{j=1}^{\infty} f_{ij}(x_j)\Big)_{i=1}^{\infty}, \quad (x_j) \in l^\infty(X),$$

then for every $a > 0$,

$$\sup_{\|(x_j)\|_\infty \le a} \Big(\sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|^p\Big)^{1/p} < +\infty.$$

Proof. By Theorem 3.1, the conclusion holds when $p = 1$. Let $p > 1$, $\frac{1}{p'} + \frac{1}{p} = 1$ and $a > 0$. By Lemma 3.1 and Theorem 3.1,

$$\sup_{\|(x_j)\|_\infty \le a} \Big|\sum_{i=1}^{\infty} s_i \Big\| \sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|\Big| \le \sup_{\|(x_j)\|_\infty \le a} \sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{\infty} s_i f_{ij}(x_j)\Big\| < +\infty,\ \forall\, (s_i) \in l^{p'}.$$

Hence $\{(\| \sum_{j=1}^{\infty} f_{ij}(x_j)\|)_{i=1}^{\infty} : \|(x_j)\|_\infty \le a\}$ is a pointwise bounded subfamily of $l^p = (l^{p'})'$. Then the resonance theorem shows that

$$\sup_{\|(x_j)\|_\infty \le a} \Big(\sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|^p\Big)^{1/p} < +\infty. \qquad \blacksquare$$

Now for $f_{ij} \in \mathcal{F}_{C,\delta}(X, Y)$ ($i,\ j \in \mathbb{N}$) and $i \in \mathbb{N}$ let

$$\|(f_{i\,n}, f_{i\,n+1}, f_{i\,n+2}, \cdots)\| = \sup_{k \ge n,\, \|(x_j)\|_\infty \le 1} \Big\| \sum_{j=n}^{k} f_{ij}(x_j)\Big\|,$$

the group norm of $(f_{ij})_{j \ge n}$ [4, p.5] which is the key object in the Maddox results of [3; 4]. Moreover, for $p \ge 1$ and $a > 0$ let

$$\|[f_{ij}]\|_{p,a} = \sup_{m,n \ge 1,\, \|(x_j)\|_\infty \le a} \Big(\sum_{i=1}^{m} \Big\| \sum_{j=1}^{n} f_{ij}(x_j)\Big\|^p\Big)^{1/p}.$$

Theorem 3.3. *Let* $p \ge 1$ *and* $f_{ij} \in \mathcal{F}_{C,\delta}(X, Y)$ *be bounded,* $\forall\, i,\ j \in \mathbb{N}$. *If* $(f_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), l^p(Y))$, *then for every* $a > 0$,

$$\|[f_{ij}]\|_{p,a} = \sup_{\|(x_j)\|_\infty \le a} \Big(\sum_{i=1}^{\infty} \Big\| \sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|^p\Big)^{1/p} < +\infty.$$

Proof. Let $a > 0$. If $\|(x_j)\|_\infty \le a$, then $\|(x_1, x_2, \cdots, x_n, 0, 0, \cdots)\|_\infty \le a$ for all n and so $(\sum_{i=1}^{\infty} \| \sum_{j=1}^{n} f_{ij}(x_j)\|^p)^{1/p} \le \sup_{\|(z_j)\|_\infty \le a} (\sum_{i=1}^{\infty} \| \sum_{j=1}^{\infty} f_{ij}(z_j)\|^p)^{1/p}$, $\forall\, n \in \mathbb{N}$. By Theorem 3.2,

$$\begin{aligned}\|[f_{ij}]\|_{p,a} &= \sup_{m,n\geq 1,\,\|(x_j)\|_\infty\leq a}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n} f_{ij}(x_j)\Big\|^p\Big)^{1/p}\\ &= \sup_{n\geq 1,\,\|(x_j)\|_\infty\leq a}\Big(\sum_{i=1}^{\infty}\Big\|\sum_{j=1}^{n} f_{ij}(x_j)\Big\|^p\Big)^{1/p}\\ &\leq \sup_{\|(z_j)\|_\infty\leq a}\Big(\sum_{i=1}^{\infty}\Big\|\sum_{j=1}^{\infty} f_{ij}(z_j)\Big\|^p\Big)^{1/p} < +\infty.\end{aligned}$$

Let $a > 0$ and $\|(x_j)\|_\infty \leq a$. Then

$$\begin{aligned}\Big(\sum_{i=1}^{\infty}\Big\|\sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|^p\Big)^{1/p} &= \sup_{m\geq 1}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|^p\Big)^{1/p}\\ &\leq \sup_{m\geq 1,\,\|(z_j)\|_\infty\leq a}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{\infty} f_{ij}(z_j)\Big\|^p\Big)^{1/p}\\ &= \sup_{m\geq 1,\,\|(z_j)\|_\infty\leq a}\lim_n\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n} f_{ij}(z_j)\Big\|^p\Big)^{1/p}\\ &\leq \sup_{m\geq 1,\,\|(z_j)\|_\infty\leq a}\sup_{n\geq 1}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n} f_{ij}(z_j)\Big\|^p\Big)^{1/p}\\ &= \sup_{m,n\geq 1,\,\|(z_j)\|_\infty\leq a}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n} f_{ij}(z_j)\Big\|^p\Big)^{1/p}\\ &= \|[f_{ij}]\|_{p,a}.\end{aligned}$$

∎

Recall that $0 < \delta \leq 1$ and $C \geq 1$ in the notation $\mathcal{F}_{C,\delta}(X,Y)$.

Theorem 3.4. *Let* $p \geq 1$ *and* $f_{ij} \in \mathcal{F}_{C,\delta}(X,Y)$ *be bounded,* $\forall\, i,\, j \in \mathbb{N}$. *Then* $(f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^p(Y))$ *if and only if*

(I) *for each* $i \in \mathbb{N}$, $\lim_n \|(f_{in}, f_{i\,n+1}, f_{i\,n+2}, \cdots)\| = 0$, *and*

(II) $\|[f_{ij}]\|_{p,a} < +\infty,\ \forall\, a > 0$.

Proof. $\Longrightarrow$ By Theorem 3.3, (II) holds. Let $i \in \mathbb{N}$. Then $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges for each $(x_j) \in l^\infty(X)$. By Lemma 2.1, $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges uniformly with respect to $\|(x_j)\|_\infty \leq 1$ and so (I) holds.

$\Longleftarrow$ If $\|(x_j)\|_\infty \leq 1$, then (I) shows that $\{\sum_{j=1}^{k} f_{ij}(x_j)\}_{k=1}^{\infty}$ is Cauchy and so $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges for all $i \in \mathbb{N}$.

Let $(x_j) \in l^\infty(X)$ with $\|(x_j)\|_\infty > 1$ and $\varepsilon > 0$. Pick an $m \in \mathbb{N}$ such that $\frac{\|(x_j)\|_\infty}{m} \leq \delta \leq 1$ and let $i \in \mathbb{N}$, $M = m(1 + C)^{m-1}$. By (I) there is an $n_0 \in \mathbb{N}$

such that $\sup_{k\geq n,\,\|(z_j)\|_\infty\leq 1}\|\sum_{j=n}^{k} f_{ij}(z_j)\| < \frac{\varepsilon}{2M}$ for all $n > n_0$. If $\Delta \subset \mathbb{N}$ is finite and $z_j = \begin{cases} x_j/m, & j \in \Delta, \\ 0, & j \notin \Delta, \end{cases}$ then $\|(z_j)\|_\infty \leq 1$ and, observing $f_{ij}(0) = 0$, we have that

$$\Big\|M\sum_{j\in\Delta} f_{ij}(\frac{x_j}{m})\Big\| < \frac{\varepsilon}{2},\ \ \forall\, k \geq n > n_0,\ \Delta \subseteq \{n, n+1, \cdots, k\},\ \Delta \neq \phi.$$

For convenience, we say that $\sum_{j\in\Delta} y_j = 0$ when Δ is empty. As in the proof of Lemma 2.2, $f_{ij}(x_j) = f_{ij}(m\frac{x_j}{m}) = s_j f_{ij}(\frac{x_j}{m})$ where $s_j \in [-M, M]$. For $k \geq n > n_0$ and $\Delta_1 = \{j \in \mathbb{N} : n \leq j \leq k, s_j \geq 0\}$, $\Delta_2 = \{j \in \mathbb{N} : n \leq j \leq k, s_j < 0\}$, it follows from Lemma 2.6 that

$$\begin{aligned}\Big\|\sum_{j=n}^{k} f_{ij}(x_j)\Big\| &= \Big\|\sum_{j=n}^{k} s_j f_{ij}(\frac{x_j}{m})\Big\| \\ &\leq \Big\|\sum_{j\in\Delta_1} s_j f_{ij}(\frac{x_j}{m})\Big\| + \Big\|\sum_{j\in\Delta_2} s_j f_{ij}(\frac{x_j}{m})\Big\| \\ &= \Big\|\sum_{j\in\Delta_1} s_j f_{ij}(\frac{x_j}{m})\Big\| + \Big\|\sum_{j\in\Delta_2} (-s_j) f_{ij}(\frac{x_j}{m})\Big\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This shows that $\{\sum_{j=1}^{k} f_{ij}(x_j)\}_{k=1}^{\infty}$ is Cauchy and so $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges.

Now let $(x_j) \in l^\infty(X)$ and $a = 1 + \|(x_j)\|_\infty$. Since $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges for all $i \in \mathbb{N}$, it follows from (II) that

$$\begin{aligned}\Big(\sum_{i=1}^{\infty}\Big\|\sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|^p\Big)^{1/p} &= \sup_{m\geq 1}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{\infty} f_{ij}(x_j)\Big\|^p\Big)^{1/p} \\ &\leq \sup_{m\geq 1,\,\|(z_j)\|_\infty\leq a}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{\infty} f_{ij}(z_j)\Big\|^p\Big)^{1/p} \\ &= \sup_{m\geq 1,\,\|(z_j)\|_\infty\leq a}\lim_n\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n} f_{ij}(z_j)\Big\|^p\Big)^{1/p} \\ &\leq \sup_{m,n\geq 1,\,\|(z_j)\|_\infty\leq a}\Big(\sum_{i=1}^{m}\Big\|\sum_{j=1}^{n} f_{ij}(z_j)\Big\|^p\Big)^{1/p} \\ &= \|[f_{ij}]\|_{p,a} < +\infty.\end{aligned}$$

∎

References

1. Li Ronglu, Shin Min Kang and C. Swartz, Operator matrices on topological vector spaces, *J. Math. Anal. Appl.*, **274** (2002), 645-658.
2. Li Ronglu and C. Swartz, A nonlinear Schur theorem, *Acta Sci. Math.*, **58** (1993), 497-508.
3. I. J. Maddox, Schur's theorem for operators, *Bull. Soc. Math. Grèce*, **16** (1975), 18-21.
4. I. J. Maddox, *Infinite Matrices of Operators*, Lecture Notes in Math., Vol. 786, Springer-Verlag, 1980.
5. Li Ronglu, Wang Fubin and Zhong Shuhui, The strongest intrinsic meaning of sequential evaluation convergence, *Topology Appl.*, **154** (2007), 1195-1205.
6. Li Ronglu, Zhong Shuhui and Cui Chengri, New basic principles of functional analysis, *J. Yanbian Univ.*, **30** (2004), 157-160.
7. Li Ronglu, Zhong Shuhui and Wen Songlong, Pan-Linear distributions I, *J. Yanbian Univ.*, **33** (2007), 157-159.
8. Li Ronglu and Bu Qingying, Locally convex spaces containing no copy of c_0, *J. Math. Anal. Appl.*, **172** (1993), 205-211.
9. Li Ronglu and Wang Junming, Invariants in abstract mapping pairs, *J. Aust. Math. Soc.*, **76** (2004), 369-381.
10. P. Antosik and C. Swartz, *Matrix Methods in Analysis*, Lecture Notes in Math., Vol. 1113, Springer-Verlag, 1985.
11. C. Swartz, *Infinite Matrices and the Gliding Hump*, World Scientific, 1996.

Li Ronglu
Department of Mathematics,
Harbin Institute of Technology,
Harbin 150001,
P. R. China
E-mail: rongluli@yahoo.com.cn

Zhong Shuhui
Department of Mathematics,
Tianjin University,
Tianjin 300072,
P. R. China
E-mail: zhshuhui@yahoo.com.cn