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SOME NEW CHARACTERIZATIONS OF BLOCH SPACES

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Abstract. In this paper we obtain some new characterizations for Bloch spaces on the unit ball of $\mathbb{C}^n$. These characterizations are new even in the unit disk.

1. Introduction

Let B be the open unit ball of $\mathbb{C}^n$ and $H(B)$ the class of all holomorphic functions on B. When $n = 1$, B is the open unit disk in the complex plane and we will denote it by D. Let $Aut(B)$ be the group of all biholomorphic self maps of B. It is well known that $Aut(B)$ is generated by the unitary operators on $\mathbb{C}^n$ and the involutions φ_a of the form

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle},$$

where $s_a = (1 - |a|^2)^{1/2}$, P_a is the orthogonal projection into the space spanned by $a \in B$, i.e., $P_a z = \frac{\langle z,a\rangle a}{|a|^2}$, $|a|^2 = \langle a, a\rangle$, $P_0 z = 0$ and $Q_a = I - P_a$ (see [11, 17]).

For $f \in C^1(B)$, the invariant gradient $\widetilde{\nabla} f$ is defined by $(\widetilde{\nabla} f)(z) = \nabla(f \circ \varphi_z)(0)$, where $\nabla f(z) = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ is the complex gradient of f. For $f \in H(B)$, let $\mathcal{R}f$ denote the radial derivative of f, that is, $\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$.

Let dv be the normalized Lebesgue measure of B and $d\lambda(z) = (1-|z|^2)^{-n-1} dv(z)$. Then $d\lambda(z)$ is a Möbius invariant measure, which means that for any $\psi \in Aut(B)$ and $f \in L^1(B)$,

$$\int_B f(z) d\lambda(z) = \int_B f \circ \psi(z) d\lambda(z).$$

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When $n = 1$, dv is the normalized Lebesgue measure of D and we will denote it by dA.

Suppose $0 < p < \infty$, recall that the Bergman space A^p consists of those functions $f \in H(B)$ for which

$$\|f\|_{A^p}^p = \int_B |f(z)|^p dv(z) < \infty.$$

The Bloch space $\mathcal{B}$, introduced by Timoney (see [13, 14]), is the space of all $f \in H(B)$ such that $\|f\|_{\mathcal{B}} = \sup_{z\in B} Q_f(z) < \infty$, where

$$Q_f(z) = \sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{|\langle \nabla f(z), \bar{w}\rangle|}{\sqrt{\frac{n+1}{2}\frac{(1-|z|^2)|w|^2+|\langle w,z\rangle|^2}{(1-|z|^2)^2}}},\ f \in H(B),\ z \in B.$$

It is well known that $f \in \mathcal{B}$ if and only if (see, e.g. [17])

$$\sup_{z\in B}(1-|z|^2)\,|\nabla f(z)| < \infty$$

if and only if $\sup_{z\in B}(1-|z|^2)\,|\mathcal{R}f(z)| < \infty$. We denote by $\mathcal{B}(D)$ the Bloch space in the unit disk.

For $f \in H(B)$, Nowak proved that $f \in \mathcal{B}$ if and only if (see [6])

$$\sup_{\substack{z,w\in B\\ z\neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \cdot \frac{|f(z)-f(w)|}{|w-P_w z - s_w Q_w z|} < \infty. \tag{1}$$

Recently, Ren and Tu proved that $f \in \mathcal{B}$ if and only if (see [10])

$$\sup_{\substack{z,w\in B\\ z\neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \cdot \frac{|f(z)-f(w)|}{|z-w|} < \infty. \tag{2}$$

In [4], we proved that $f \in \mathcal{B}$ if and only if

$$\sup_{\substack{z,w\in B\\ z\neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \cdot \frac{|f(z)-f(w)|}{|1-\langle z,w\rangle|} < \infty. \tag{3}$$

These characterizations can be seen as derivative-free characterizations of Bloch spaces on the unit ball. See [1-8, 10, 13-15, 17] for more characterizations of the Bloch space in the unit ball.

In this paper, we add some other derivative-free characterizations for Bloch spaces in the unit ball of $\mathbb{C}^n$, which can be seen as continuation work of [3, 4].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2. Preliminaries

In this section, we collect some known results and technical results that will be needed in the proof of our main result. We begin with the following estimate (see [12]).

Lemma 1. *Let* $-1 < t < \infty$ *and* $s \geq 0$. *If* $c > 0$, *then there is a finite constant* C *such that*

$$\int_B \frac{(1-|z|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}}\Big(\log\frac{1}{1-|\varphi_w(z)|^2}\Big)^s dv(z) \leq \frac{C}{(1-|w|^2)^c}, \text{ for all } w \in B. \tag{4}$$

Lemma 2. ([9]). *Suppose* $p > 0$, $0 \leq \alpha < p+2$ *and* $f \in H(B)$. *Then* $f \in A^p$ *if and only if*

$$I(f) = \int_B |f(z)|^{p-\alpha}|\widetilde{\nabla} f(z)|^\alpha dv(z) < \infty. \tag{5}$$

Moreover, the quantities $\|f\|^p_{A^p}$ *and* $|f(0)|^p + I(f)$ *are comparable for* $f \in H(B)$.

Lemma 3. ([7]). *Let* $0 < p < \infty$. *A holomorphic function* f *is in the Bloch space* $\mathcal{B}$ *if and only if*

$$\sup_{a\in B}\|f\circ\varphi_a - f(a)\|_{A^p} < \infty. \tag{6}$$

Lemma 4. *Assume that* $f \in H(B)$, $0 < p < \infty$, $-1 < q < \infty$, $0 \leq s < \infty$ *such that* $p+s > n$. *Then for all* $a \in B$, *the following inequality holds.*

$$\int_B |f(z)-f(0)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dv(z) \leq C\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z). \tag{7}$$

Proof. The case $s=0$ is well known (see [17]). Now we assume that $s>0$. If the right side in (7) is infinite, then the result is obvious.

Now we assume that the right side in (7) is finite. For a fixed $r \in (0,1)$, let $E(a,r) = \{z \in B : |\varphi_a(z)| < r\}$. From [6] or [17] we see that

$$(1-|z|^2)^{n+1} \asymp (1-|a|^2)^{n+1} \asymp |1-\langle a,z\rangle|^{n+1} \asymp |E(a,r)| \tag{8}$$

when $z \in E(a,r)$. By the subharmonicity and (8) we have (see [17])

$$\begin{aligned}
&|\mathcal{R}f(a)|^p(1-|a|^2)^{p+q} \\
\le\ & \frac{C}{|E(a,r)|}\int_{E(a,r)} |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}dv(z) \\
\le\ & \frac{C(1-r^2)^{-s}}{|E(a,r)|}\int_{E(a,r)} |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z) \\
\le\ & \frac{C}{(1-|a|^2)^{n+1}}\int_{B} |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{a\in B}|\mathcal{R}f(a)|^p(1-|a|^2)^{p+q+n+1} \\
&\le \sup_{a\in B}\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z) < \infty,
\end{aligned}$$

from which and exercise 7.7 of [17] we see that $K(f) = \sup_{a\in B}|f(a)|^p(1-|a|^2)^{q+n+1} < \infty$. Therefore, for $a \in B$, from Theorem 2.16 of [17] and Lemma 1 we get

$$\begin{aligned}
&\int_B |f(z)-f(0)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dv(z) \\
=\ & (1-|a|^2)^s\int_B \left|\frac{f(z)-f(0)}{(1-\langle a,z\rangle)^{2s/p}}\right|^p(1-|z|^2)^{q+s}dv(z) \\
\le\ & C(1-|a|^2)^s\int_B \left|\mathcal{R}\left(\frac{f(z)-f(0)}{(1-\langle a,z\rangle)^{2s/p}}\right)\right|^p(1-|z|^2)^{p+q+s}dv(z) \\
\le\ & C(1-|a|^2)^s\int_B \frac{|\mathcal{R}f(z)|^p}{|1-\langle a,z\rangle|^{2s}}(1-|z|^2)^{p+q+s}dv(z) \\
& +C(1-|a|^2)^s\int_B \frac{|f(z)-f(0)|^p}{|1-\langle a,z\rangle|^{2s+p}}(1-|z|^2)^{p+q+s}dv(z) \\
\le\ & C\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z) \\
& +CK(f)(1-|a|^2)^s\int_B \frac{(1-|z|^2)^{p+q+s-q-n-1}}{|1-\langle a,z\rangle|^{2s+p}}dv(z) \\
\le\ & C\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z),
\end{aligned}$$

as desired. ∎

Lemma 5. *Assume that* $f \in H(B)$, $0 < p < \infty$, $-1 < q < \infty$, $0 \leq t < p+2n$, $0 \leq s < \infty$ *such that* $p + s > n$. *Then for* $a \in B$,

$$\begin{aligned}&\int_B \frac{|f(z) - f(0)|^p}{|z|^t}(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dv(z)\\ &\leq C\int_B |\mathcal{R}f|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z)\\ &\leq C\int_B |\widetilde{\nabla} f|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dv(z).\end{aligned}$$

Proof. It is elementary to shown that there exists a constant C (independent of f) such that

$$\int_B \frac{|f(z) - f(0)|^p}{|z|^t}(1-|z|^2)^q dv(z) \leq C\int_B |f(z)-f(0)|^p(1-|z|^2)^q dv(z).$$

This together with Theorem 2.16 of [17] show that

$$\int_B \frac{|f(z) - f(0)|^p}{|z|^t}(1-|z|^2)^q dv(z) \leq C\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q} dv(z). \tag{9}$$

Taking $g(z) = \frac{f(z)-f(0)}{(1-\langle a,z\rangle)^{2s/p}}$, then

$$|\mathcal{R}g(z)| = \frac{|\mathcal{R}f(z)(1-\langle a,z\rangle)^{2s/p} + 2s/p(f(z)-f(0))(1-\langle a,z\rangle)^{2s/p-1}\langle a,z\rangle|}{|1-\langle a,z\rangle|^{4s/p}}.$$

Applying g to the inequality (9) and from Lemma 4 we obtain

$$\begin{aligned}&\int_B \frac{|f(z) - f(0)|^p}{|z|^t}(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dv(z)\\ &= (1-|a|^2)^s\int_B \frac{\left|\frac{f(z)-f(0)}{(1-\langle a,z\rangle)^{2s/p}} - 0\right|^p}{|z|^t}(1-|z|^2)^{q+s} dv(z)\\ &= (1-|a|^2)^s\int_B \frac{|g(z)-g(0)|^p}{|z|^t}(1-|z|^2)^{q+s} dv(z)\\ &\leq C(1-|a|^2)^s\int_B |\mathcal{R}g(z)|^p(1-|z|^2)^{p+q+s} dv(z)\\ &\leq C\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z)\\ &\quad +C\int_B \frac{|f(z)-f(0)|^p}{|1-\langle a,z\rangle|^p}(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z)\end{aligned}$$

$$\leq C\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z)$$
$$+C\int_B |f(z)-f(0)|^p(1-|z|^2)^{q}(1-|\varphi_a(z)|^2)^s dv(z)$$
$$\leq C\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p+q}(1-|\varphi_a(z)|^2)^s dv(z),$$

as desired. The second inequality follows from the following well-known inequality (see [17])

$$(1-|z|^2)|\mathcal{R}f(z)| \leq (1-|z|^2)|\nabla f(z)| \leq |\widetilde{\nabla} f(z)|.$$

This completes the proof of the lemma. ∎

3. Main Results and Proofs

In this section, we give our main results and proofs.

Theorem 1. *Assume that $f \in H(B)$, $0 < p, c < \infty$ and $0 \leq t < \infty$. Then $f \in \mathcal{B}$ if and only if*

$$\sup_{a\in B}\int_B |f(z)-f(a)|^p \frac{(1-|z|^2)^t(1-|a|^2)^c}{|1-\langle z,a\rangle|^{n+1+t+c}} dv(z) < \infty. \tag{10}$$

Proof. Assume that (10) holds. It follows from [17] and (8) that there exists a constant C such that

$$\begin{aligned}
&(1-|a|^2)^p|\nabla f(a)|^p\\
&\leq \frac{C}{(1-|a|^2)^{n+1}}\int_{E(a,r)} |f(z)-f(a)|^p dv(z)\\
&\leq C\int_{E(a,r)} |f(z)-f(a)|^p \frac{(1-|z|^2)^t(1-|a|^2)^c}{|1-\langle z,a\rangle|^{n+1+t+c}} dv(z)\\
&\leq C\int_B |f(z)-f(a)|^p \frac{(1-|z|^2)^t(1-|a|^2)^c}{|1-\langle z,a\rangle|^{n+1+t+c}} dv(z),
\end{aligned} \tag{11}$$

from which we see that $f \in \mathcal{B}$.

Conversely, assume that $f \in \mathcal{B}$. It follows from the Cauchy-Schwarz inequality and the inequality $|\frac{\partial f}{\partial z_k}| \leq |\nabla f|$ that there exists a constant C such that

$$\begin{aligned}
|f(z)-f(0)| &= \left|\int_0^1 \frac{df}{dt}(tz)dt\right| \leq \left|\int_0^1 \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(tz)dt\right|\\
&\leq C\|f\|_{\mathcal{B}}|z|\int_0^1 \frac{1}{1-|zt|}dt \leq C\|f\|_{\mathcal{B}} \log\frac{1}{1-|z|}.
\end{aligned}$$

From the Mӧbius invariant property of the Bloch space, we have

$$|f\circ\varphi_a(z)-f(a)|\le C\|f\circ\varphi_a\|_{\mathcal{B}}\log\frac{1}{1-|z|}\le C\|f\|_{\mathcal{B}}\log\frac{1}{1-|z|}.$$

Making the change of variables $z\mapsto\varphi_a(z)$ we obtain

$$|f(a)-f(z)|\le C\|f\|_{\mathcal{B}}\log\frac{1}{1-|\varphi_a(z)|^2}. \tag{12}$$

By the last inequality and Lemma 1, it gives

$$\begin{aligned}&\sup_{a\in B}\int_B|f(z)-f(a)|^p\frac{(1-|z|^2)^t(1-|a|^2)^c}{|1-\langle z,a\rangle|^{n+1+t+c}}dv(z)\\ &\le C\|f\|_{\mathcal{B}}^p\sup_{a\in B}(1-|a|^2)^c\int_B\frac{(1-|z|^2)^t}{|1-\langle z,a\rangle|^{n+1+t+c}}\Big(\log\frac{1}{1-|\varphi_a(z)|^2}\Big)^p dv(z)\\ &\le C\|f\|_{\mathcal{B}}^p<\infty.\end{aligned}$$

This completes the proof of the theorem. ■

Remark 1. Set $t=0, c=n+1$ in Theorem 1. We get that $f\in\mathcal{B}$ if and only if

$$\sup_{a\in B}\int_B|f(z)-f(a)|^p(1-|\varphi_a(z)|^2)^{n+1}d\lambda(z)<\infty, \tag{13}$$

which is equivalent to Lemma 3. Hence Theorem 1 can be seen as a generalization of Lemma 3.

Theorem 2. *Assume that* $f\in H(B)$, $0<p,c<\infty$, $0\le t<\infty$ *such that* $t+c<p+n-1$. *Then* $f\in\mathcal{B}$ *if and only if*

$$\sup_{a\in B}\int_B|f(z)-f(a)|^p\frac{(1-|z|^2)^t(1-|a|^2)^c}{|a-P_az-s_aQ_az|^{n+1+t+c}}dv(z)<\infty. \tag{14}$$

Proof. Suppose that (14) holds. Since

$$\frac{1}{|1-\langle z,a\rangle|}\le\frac{1}{|a-P_az-s_aQ_az|},\ z,a\in B \tag{15}$$

it follows from Theorem 1 that $f\in\mathcal{B}$.

Conversely, suppose that $f\in\mathcal{B}$. Making the change of variables $z\mapsto\varphi_a(z)$ and using the following equalities (see [17])

$$1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2},\ |1-\langle\varphi_a(z),a\rangle|=\frac{1-|a|^2}{|1-\langle z,a\rangle|},$$

we obtain

$$\begin{aligned} K &= \int_B |f(z)-f(a)|^p \frac{(1-|a|^2)^c(1-|z|^2)^t}{|\varphi_a(z)|^{n+1+t+c}|1-\langle z,a\rangle|^{n+1+t+c}}dv(z) \\ &= C\int_B |f(\varphi_a(z))-f(a)|^p \frac{(1-|a|^2)^c(1-|\varphi_a(z)|^2)^{t+n+1}}{|z|^{n+1+t+c}|1-\langle \varphi_a(z),a\rangle|^{n+1+t+c}}d\lambda(z) \\ &= C\int_B \frac{|f\circ\varphi_a(z)-f\circ\varphi_a(0)|^p}{|z|^{n+1+t+c}} \frac{(1-|z|^2)^t dv(z)}{|1-\langle z,a\rangle|^{2(t+n+1)-(n+1+t+c)}}. \end{aligned}$$

It is elementary to check that there exists a positive constant C(independent of f) such that

$$\begin{aligned} K &\le C\int_B \frac{|f\circ\varphi_a(z)-f\circ\varphi_a(0)|^p(1-|z|^2)^t}{|1-\langle z,a\rangle|^{2(t+n+1)-(n+1+t+c)}}dv(z) \\ &= C\int_B |f(\varphi_a(z))-f(a)|^p \frac{(1-|a|^2)^c(1-|\varphi_a(z)|^2)^{t+n+1}}{|1-\langle \varphi_a(z),a\rangle|^{n+1+t+c}}d\lambda(z). \end{aligned}$$

Making the change of variables $z\mapsto\varphi_a(z)$ again, we get

$$K \le C\int_B |f(z)-f(a)|^p \frac{(1-|z|^2)^t(1-|a|^2)^c}{|1-\langle a,z\rangle|^{n+1+t+c}}dv(z).$$

Then the result follows from Theorem 1. ■

Theorem 3. *Assume that* $f\in H(B)$, $0\le q<\infty$, $0<p<\infty$ *and* $p-q>-2$. *Then* $f\in\mathcal{B}$ *if and only if*

$$\sup_{a\in B}\int_B |\widetilde{\nabla}f(z)|^q|f(z)-f(a)|^{p-q}(1-|\varphi_a(z)|^2)^{n+1}d\lambda(z)<\infty. \tag{16}$$

Proof. From Lemmas 2 and 3, we see that $f\in\mathcal{B}$ if and only if

$$\begin{aligned} \infty &> \sup_{a\in B}\int_B |f(\varphi_a(z))-f(a)|^{p-q}|\widetilde{\nabla}f\circ\varphi_a(z)|^q dv(z) \\ &= \sup_{a\in B}\int_B |f(\varphi_a(z))-f(a)|^{p-q}|\widetilde{\nabla}f(\varphi_a(z))|^q(1-|z|^2)^{n+1}d\lambda(z). \end{aligned}$$

Making the change of variable $z\mapsto\varphi_a(z)$, we get the desired result. ■

Remark 2. Taking $q=p$ in Theorem 3, we obtain that $f\in\mathcal{B}$ if and only if

$$\sup_{a\in B}\int_B |\widetilde{\nabla}f(z)|^p(1-|\varphi_a(z)|^2)^{n+1}d\lambda(z)<\infty. \tag{17}$$

Taking $q = 0$ in (16), we get (13). Hence Theorem 3 can also be seen as a generalization of Lemma 3.

The following Theorem was first proved in [3], here we give a different proof for the completeness.

Theorem 4. *Assume that* $f \in H(B)$ *and* $0 < p < \infty$. *Then* $f \in \mathcal{B}$ *if and only if*

$$\sup_{a\in B}\int_B\int_B \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|\varphi_a(w)|^2)^{n+1}dv(z)dv(w)<\infty. \tag{18}$$

Proof. Suppose that $f \in \mathcal{B}$. Making the change of variables, from (12) and Lemma 1 we have

$$\begin{aligned}
&\sup_{a\in B}\int_B\int_B \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|\varphi_a(w)|^2)^{n+1}dv(z)dv(w)\\
\le\ &\sup_{a\in B}\int_B (1-|\varphi_a(w)|^2)^{n+1}dv(w)\int_B \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}dv(z)\\
\le\ &C\|f\|_{\mathcal{B}}^p \sup_{a\in B}\int_B (1-|\varphi_a(w)|^2)^{n+1}dv(w)\\
&\int_B \frac{1}{|1-\langle z,w\rangle|^{2(n+1)}}\Big(\log\frac{1}{1-|\varphi_w(z)|^2}\Big)^p dv(z)\\
\le\ &C\|f\|_{\mathcal{B}}^p \sup_{a\in B}\int_B (1-|\varphi_a(w)|^2)^{n+1}d\lambda(w)\\
\le\ &C\|f\|_{\mathcal{B}}^p \int_B (1-|w|^2)^{n+1}d\lambda(w)<\infty.
\end{aligned}$$

Conversely, suppose that (18) holds. From [16] we see that $f \in \mathcal{B}$ if and only if

$$\sup_{a\in B}\int_B |\nabla f(z)|^p(1-|z|^2)^p(1-|\varphi_a(z)|^2)^{n+1}d\lambda(z)<\infty. \tag{19}$$

From the proof of Theorem 1 we see that there exists a constant C such that

$$(1-|w|^2)^p|\nabla f(w)|^p \le \frac{C}{(1-|w|^2)^{n+1}}\int_{E(w,r)}|f(z)-f(w)|^p dv(z),$$

i.e.

$$\begin{aligned}
&(1-|w|^2)^p|\nabla f(w)|^p(1-|\varphi_a(w)|^2)^{n+1}\\
\le\ &\frac{C}{(1-|w|^2)^{n+1}}\int_{E(w,r)}|f(z)-f(w)|^p(1-|\varphi_a(w)|^2)^{n+1}dv(z)\\
\le\ &C\int_{E(w,r)}\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{n+1}}(1-|\varphi_a(w)|^2)^{n+1}dv(z).
\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_B (1-|w|^2)^p|\nabla f(w)|^p(1-|\varphi_a(w)|^2)^{n+1}d\lambda(w)\\ &\le C\int_B\int_{E(w,r)}|f(z)-f(w)|^p\frac{1}{|1-\langle z,w\rangle|^{n+1}}(1-|\varphi_a(w)|^2)^{n+1}dv(z)d\lambda(w)\\ &\le C\int_B\int_{E(w,r)}|f(z)-f(w)|^p\frac{1}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|\varphi_a(w)|^2)^{n+1}dv(z)dv(w)\\ &\le C\int_B\int_B\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|\varphi_a(w)|^2)^{n+1}dv(z)dv(w).\end{aligned}$$

From the last inequality and (19), we see that $f\in\mathcal{B}$, as desired. ∎

Theorem 5. *Assume that* $f\in H(B)$ *and* $2<p<\infty$. *Then* $f\in\mathcal{B}$ *if and only if*

$$\sup_{a\in B}\int_B\int_B\frac{|f(z)-f(w)|^p}{|w-P_wz-s_wQ_wz|^{2(n+1)}}(1-|\varphi_a(w)|^2)^{n+1}dv(z)dv(w)<\infty. \tag{20}$$

Proof. Suppose that (20) holds. Then the result follows from Theorem 4 and (15).

Conversely, suppose that $f\in\mathcal{B}$. Making the change of variables of $z\mapsto\varphi_w(z)$ and similarly to the proof of Theorem 2 we obtain

$$\begin{aligned}M &=\int_B\int_B\frac{|f(z)-f(w)|^p}{|w-P_wz-s_wQ_wz|^{2(n+1)}}(1-|\varphi_a(w)|^2)^{n+1}dv(z)dv(w)\\ &=\int_B(1-|\varphi_a(w)|^2)^{n+1}dv(w)\int_B\frac{|f(z)-f(w)|^p}{|\varphi_w(z)|^{2(n+1)}|1-\langle z,w\rangle|^{2(n+1)}}dv(z)\\ &=\int_B(1-|\varphi_a(w)|^2)^{n+1}dv(w)\int_B\frac{|f(\varphi_w(z))-f(w)|^p(1-|\varphi_w(z)|^2)^{n+1}}{|z|^{2(n+1)}|1-\langle\varphi_w(z),w\rangle|^{2(n+1)}}d\lambda(z)\\ &=\int_B(1-|\varphi_a(w)|^2)^{n+1}d\lambda(w)\int_B\frac{|f(\varphi_w(z))-f(w)|^p}{|z|^{2(n+1)}}dv(z).\end{aligned}$$

It is elementary to show that there exists a positive constant C(independent of f) such that

$$M\le C\int_B(1-|\varphi_a(w)|^2)^{n+1}d\lambda(w)\int_B|f(\varphi_w(z))-f(w)|^pdv(z).$$

From Lemma 3 we get

$$\begin{aligned}M &\le C\int_B(1-|\varphi_a(w)|^2)^{n+1}d\lambda(w)\sup_{w\in B}\|f\circ\varphi_w-f(w)\|_{A^p}\\ &\le C\int_B(1-|w|^2)^{n+1}d\lambda(w)\sup_{w\in B}\|f\circ\varphi_w-f(w)\|_{A^p}<\infty,\end{aligned}$$

as desired. This completes the proof of the theorem. ■

Remark 3. When $n = 1$ and $2 < p < \infty$, then $f \in \mathcal{B}(D)$ if and only if

$$\sup_{a\in D}\int_D\int_D \frac{|f(z)-f(w)|^p}{|z-w|^4}(1-|\varphi_a(w)|^2)^2 dA(z)dA(w) < \infty. \tag{21}$$

Theorem 6. *Assume that $f \in H(B)$ and $0 < p < \infty$. Then $f \in \mathcal{B}$ if and only if*

$$\begin{aligned}\sup_{a\in B}\int_B\int_B &\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|\varphi_a(z)|^2)^{\frac{n+1}{2}}\\ &(1-|\varphi_a(w)|^2)^{\frac{n+1}{2}} dv(z)dv(w) < \infty.\end{aligned} \tag{22}$$

Proof. Assume that (22) holds. For a fixed $r \in (0,1)$, when $z \in E(w,r)$, it holds

$$|1-\langle z,a\rangle| \asymp |1-\langle w,a\rangle|, \tag{23}$$

for any $a \in B$ (see [17]). Hence, from the proof of Theorem 4 we have

$$\begin{aligned}&(1-|w|^2)^p|\nabla f(w)|^p(1-|\varphi_a(w)|^2)^{n+1}\\ \le\ & C\int_{E(w,r)}\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|w|^2)^{n+1}(1-|\varphi_a(w)|^2)^{\frac{n+1}{2}}(1-|\varphi_a(z)|^2)^{\frac{n+1}{2}}dv(z).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_B(1-|w|^2)^p|\nabla f(w)|^p(1-|\varphi_a(w)|^2)^{n+1}d\lambda(w)\\ \le\ & C\int_B\int_{E(w,r)}\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|\varphi_a(z)|^2)^{\frac{n+1}{2}}\times(1-|\varphi_a(w)|^2)^{\frac{n+1}{2}}dv(z)dv(w)\\ \le\ & C\int_B\int_B\frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|\varphi_a(z)|^2)^{\frac{n+1}{2}}(1-|\varphi_a(w)|^2)^{\frac{n+1}{2}}dv(z)dv(w).\end{aligned}$$

It follows from (19) and (22) that $f \in \mathcal{B}$.

Conversely, suppose that $f \in \mathcal{B}$. From Theorem 3 we see that $f \in \mathcal{B}$ if and only if

$$\sup_{a\in B}\int_B|\widetilde{\nabla} f(z)|^p(1-|\varphi_a(z)|^2)^{n+1}d\lambda(z) < \infty.$$

Making the change of variables $z \mapsto \varphi_a(z)$, we see that $f \in \mathcal{B}$ if and only if

$$\sup_{a\in B}\int_B |\widetilde{\nabla}(f\circ\varphi_a)(z)|^p dv(z) = \sup_{a\in B}\int_B |\widetilde{\nabla} f(\varphi_a(z))|^p dv(z) < \infty. \tag{24}$$

We claim that for any $g \in A^p$,

$$\begin{aligned} Y &= \int_B\int_B \frac{|g(z)-g(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|z|^2)^{\frac{n+1}{2}}(1-|w|^2)^{\frac{n+1}{2}}dv(z)dv(w) \\ &\le C\int_B |g(z)|^p dv(z) \asymp \int_B |\widetilde{\nabla} g(z)|^p dv(z) < \infty. \end{aligned} \tag{25}$$

In fact, using Lemma 1 we obtain

$$\begin{aligned} Y \le& \int_B\int_B \frac{|g(z)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|z|^2)^{\frac{n+1}{2}}(1-|w|^2)^{\frac{n+1}{2}}dv(z)dv(w) \\ &+\int_B\int_B \frac{|g(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|z|^2)^{\frac{n+1}{2}}(1-|w|^2)^{\frac{n+1}{2}}dv(z)dv(w) \\ \le& \ C\int_B |g(z)|^p(1-|z|^2)^{\frac{n+1}{2}}dv(z)\int_B \frac{(1-|w|^2)^{\frac{n+1}{2}}}{|1-\langle z,w\rangle|^{2(n+1)}}dv(w) \\ &+C\int_B |g(w)|^p(1-|w|^2)^{\frac{n+1}{2}}dv(w)\int_B \frac{(1-|z|^2)^{\frac{n+1}{2}}}{|1-\langle z,w\rangle|^{2(n+1)}}dv(z) \\ \le& \ C\int_B |g(z)|^p dv(z) + C\int_B |g(w)|^p dv(w) \le C\int_B |g(z)|^p dv(z). \end{aligned}$$

For $f \in \mathcal{B}$ and $a \in B$, we have that $f\circ\varphi_a - f(a) \in A^p$. It follows from (24) and (25) that

$$\begin{aligned} \sup_{a\in B}\int_B\int_B &\frac{|f\circ\varphi_a(z)-f\circ\varphi_a(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}} \\ &(1-|z|^2)^{\frac{n+1}{2}}(1-|w|^2)^{\frac{n+1}{2}}dv(z)dv(w) < \infty. \end{aligned} \tag{26}$$

Making the change of variables $z \mapsto \varphi_a(z)$, $w \mapsto \varphi_a(w)$ and using the following equality

$$\frac{(1-|\varphi_a(z)|^2)(1-|\varphi_a(w)|^2)}{|1-\langle\varphi_a(z),\varphi_a(w)\rangle|^2} = 1-|\varphi_w(z)|^2,$$

we see that (26) is equivalent to (22). The proof is completed. ■

Theorem 7. *Assume that* $f \in H(B)$ *and* $\max\{2, \frac{n-1}{2}\} < p < \infty$. *Then* $f \in \mathcal{B}$ *if and only if*

$$\begin{aligned} \sup_{a\in B}\int_B\int_B &\frac{|f(z)-f(w)|^p}{|w-P_w z - s_w Q_w z|^{2(n+1)}} \\ &(1-|\varphi_a(z)|^2)^{\frac{n+1}{2}}(1-|\varphi_a(w)|^2)^{\frac{n+1}{2}}dv(z)dv(w) < \infty. \end{aligned} \tag{27}$$

Proof. Suppose that (27) holds. Then by Theorem 6 and (15) we see that $f \in \mathcal{B}$.

Conversely, suppose that $f \in \mathcal{B}$. By Lemma 5 and the condition $\max\{2, \frac{n-1}{2}\} < p < \infty$, we obtain

$$\begin{aligned}
&\int_B \int_B \frac{|f(z) - f(w)|^p}{|w - P_w z - s_w Q_w z|^{2(n+1)}} \\
&\qquad (1 - |\varphi_a(z)|^2)^{\frac{n+1}{2}} (1 - |\varphi_a(w)|^2)^{\frac{n+1}{2}} dv(z) dv(w) \\
&= \int_B \int_B \frac{|f(z) - f(w)|^p}{|\varphi_w(z)|^{2(n+1)} |1 - \langle w, z\rangle|^{2(n+1)}} \\
&\qquad (1 - |\varphi_a(z)|^2)^{\frac{n+1}{2}} (1 - |\varphi_a(w)|^2)^{\frac{n+1}{2}} dv(z) dv(w) \\
&= \int_B \int_B \frac{|f \circ \varphi_w(u) - f \circ \varphi_w(0)|^p}{|u|^{2(n+1)}} \\
&\qquad (1 - |\varphi_a(\varphi_w(u))|^2)^{\frac{n+1}{2}} dv(u) (1 - |\varphi_a(w)|^2)^{\frac{n+1}{2}} d\lambda(w) \\
&\le C \int_B \int_B |\widetilde{\nabla} f \circ \varphi_w(u)|^p \\
&\qquad (1 - |\varphi_a(\varphi_w(u))|^2)^{\frac{n+1}{2}} dv(u) (1 - |\varphi_a(w)|^2)^{\frac{n+1}{2}} d\lambda(w) \\
&\le C \int_B \int_B |\widetilde{\nabla} f(z)|^p (1 - |\varphi_w(z)|^2)^{n+1} \\
&\qquad (1 - |\varphi_a(z)|^2)^{\frac{n+1}{2}} d\lambda(z) (1 - |\varphi_a(w)|^2)^{\frac{n+1}{2}} d\lambda(w) \\
&\le C \int_B |\widetilde{\nabla} f(z)|^p (1 - |\varphi_a(z)|^2)^{n+1} d\lambda(z) \times I.
\end{aligned} \tag{28}$$

Here

$$I = \sup_{a,z \in B} \int_B \frac{1}{(1 - |\varphi_a(z)|^2)^{\frac{n+1}{2}}} (1 - |\varphi_w(z)|^2)^{n+1} (1 - |\varphi_a(w)|^2)^{\frac{n+1}{2}} d\lambda(w).$$

Making the change of variables $w \mapsto \varphi_z(u)$ and using the fact that $|\varphi_z(w)| = |\varphi_w(z)|$ we have

$$I = \sup_{a,z \in B} \int_B \frac{1}{(1 - |\varphi_z(a)|^2)^{\frac{n+1}{2}}} (1 - |u|^2)^{n+1} (1 - |\varphi_a(\varphi_z(u))|^2)^{\frac{n+1}{2}} d\lambda(u).$$

From the exercises 1.24 of [17] we see that $|\varphi_a(\varphi_z(u))| = |\varphi_{\varphi_z(a)}(u)|$. It follows from Theorem 1.12 of [17] that

$$
\begin{aligned}
I &= \sup_{a,z\in B}\int_B \frac{1}{(1-|\varphi_z(a)|^2)^{\frac{n+1}{2}}}(1-|\varphi_{\varphi_z(a)}(u)|^2)^{\frac{n+1}{2}}dv(u)\\
&= \sup_{a,z\in B}\int_B \frac{(1-|u|^2)^{\frac{n+1}{2}}}{|1-\langle u,\varphi_z(a)\rangle|^{n+1}}dv(u) \qquad (29)\\
&= \sup_{w\in B}\int_B \frac{(1-|u|^2)^{\frac{n+1}{2}}}{|1-\langle u,w\rangle|^{n+1}}dv(u)<\infty.
\end{aligned}
$$

Combining (28) with (29), the result follows from Theorem 3. ■

Remark 4. When $n=1$ and $2<p<\infty$, from Theorem 7 we see that $f\in\mathcal{B}(D)$ if and only if

$$
\sup_{a\in D}\int_D\int_D \frac{|f(z)-f(w)|^p}{|z-w|^4}(1-|\varphi_a(z)|^2)(1-|\varphi_a(w)|^2)dA(z)dA(w)<\infty. \qquad (30)
$$

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