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ON THE MAXIMAL ASYMPTOTICS FOR LINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. The work develops the approach proposed in 1982 by the author and V.Ya. Shirman for analysis of asymptotic stability of a linear differential equation in Banach space. It is shown that the method introduced in the mentioned above work allows also to prove the nonexistence of the fastest growing solution for a wide class of linear equations.

1. INTRODUCTION

One of important results of last decades in the asymptotic semigroups theory [3, 9, 12] is the following theorem on asymptotic stability:

Theorem 1. *Consider a linear differential equation in Banach space* X

(1) $$\dot{x} = Ax,$$

where A *is the generator of a* C_0*-semigroup* $\{e^{At}\}, t \geq 0,$ *under assumptions that the set* $\sigma(A) \cap (i\mathbb{R})$ *is at most countable and for some* $C > 0$*:* $\|e^{At}x\| \leq C\|x\|$, $t \geq 0,\ x \in X$. *Then equation* (1) *is asymptotically stable, i.e.* $\|e^{At}x\| \to 0$ *as* $t \to +\infty$ *for any* $x \in X$*, if and only if the adjoint operator* A^* *has no pure imaginary eigenvalues.*

Statement of this theorem and its proof in the case of a bounded operator A were given in 1982 by Sklyar and Shirman [13]. We considered it as a development of the remarkable B. Sz.-Nagy and C. Foias theorem (see [14], p. 102):

Let a complete nonunitary contraction T be given in a Hilbert space H and let

$$\mathrm{mes}\,(\sigma(\mathrm{T}) \cap \mathrm{S}_0(1)) = 0,$$

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where $S_0(1) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\mathrm{mes}(\cdot)$ is a Lebesgue measure on $S_0(1)$. Then for each $x \in H$ we have

$$\lim_{n\to\infty} T^n x = 0 \quad \text{and} \quad \lim_{n\to\infty} T^{*n} x = 0.$$

The method of treating this problem given in [13] was picked up in 1988 by Lyubich and Vu Phong [8] who brought in it some new non-trivial elements from isometric semigroups theory and obtained this way a proof in the general case. Independently in 1988 Theorem 1 was obtained by Arendt and Batty [1].

In the present paper within the development of the approach proposed in [13, 8] we obtain a more general result on nonnexistence of a maximal asymptotics (the fastest growing solution) for equation (1). First we recall the main lines of the proof of Theorem 1 from [13, 8].

1. We introduce in X the seminorm $l(\cdot)$

   $$l(x) = \limsup_{t\to+\infty} \left\| e^{At} x \right\|, \quad x \in X,$$

   where $\left\{ e^{At},\ t \geq 0 \right\}$ is the semigroup generated by A, which satisfies $l(x) \leq C\|x\|$. Then $L = \ker l$ is a subspace of X. Our goal is to show that, actually, $L = X$.

   If that is not the case, we consider the nontrivial quotient space $\widehat{X} = X/L$ where the seminorm l generates a norm $\tilde{l}$ dominated by the natural quotient norm $\|\cdot\|_F$

   $$\tilde{l}(\hat{x}) \leq C\|\hat{x}\|_F.$$

2. Then we consider the completion $\widetilde{X}$ of $\widehat{X}$ w.r.t. the norm $\tilde{l}(x)$ and observe that the extensions to $\widetilde{X}$ of the quotient operators $\left\{ (\widehat{e^{At}}),\ t \geq 0 \right\}$ form a C_0-semigroup in $\tilde{l}(\cdot)$ which is, obviously, isometric. We denote this semigroup by $\left\{ e^{\tilde{A}t},\ t \geq 0 \right\}$ and its generator by $\tilde{A}$.

3. We prove the following inclusion for the spectrum $\sigma(\tilde{A})$ of the operator $\tilde{A}$:

   $$\sigma(\tilde{A}) \subset \sigma(A) \cap (i\mathbb{R}).$$

   Next, we infer that

   (a) the semigroup $e^{\tilde{A}t}$ is extended to a C_0-group of isometries $\left\{ e^{\tilde{A}t}, -\infty < t < +\infty \right\}$;

   (b) the spectrum $\sigma(\tilde{A})$ is at most countable set and, moreover, it is not empty (the latter fact is nontrivial only for the case of an unbounded A).

4. Finally we notice that the spectrum $\sigma(\tilde{A})$ possesses an isolated point, say $i\lambda_0$, and the operator $\tilde{A}$ has the invariant subspace, say Λ, corresponding to this point, i.e. $\Lambda \subset D(\tilde{A})$, $\tilde{A}|\Lambda$ is bounded and $\sigma(\tilde{A}|\Lambda) = \{i\lambda_0\}$. Since $\left\{e^{(\tilde{A}|\Lambda)t}, -\infty < t < +\infty\right\}$ is a group of isometries we conclude that $\tilde{A}|\Lambda = (i\lambda_0)I|\Lambda$. Then $i\lambda_0$ is an eigenvalue of $\tilde{A}$ (but not necessarily of A). The same argument concerning the operator $\tilde{A}^*$ gives that $i\lambda_0$ is also an eigenvalue of $\tilde{A}^*$. But in this case that fact implies that $i\lambda_0$ is also an eigenvalue of A^*. Contradiction.

In 1993 Vu Phong proposed an extension of this scheme considering the asymptotic behavior of semigroups restricted by so-called weight functions. In this work we give further development. We introduce a concept of maximal asymptotics and show that our approach allows to solve the problem of its existence for a wide class of semigroups.

Definition 1. We say that equation (1) (or the semigroup $\left\{e^{At},\ t \geq 0\right\}$) has a maximal asymptotics if there exists a real positive function, say $f(t), t \geq 0$, such that

(i) for some $a \geq 0$ and for any initial vector $x \in X$ the function $\frac{\|e^{At}x\|}{f(t)}$ is bounded on $[a, +\infty]$,

(ii) there exists at least one $x_0 \in X$ such that

$$\lim_{t \to +\infty} \frac{\|e^{At}x_0\|}{f(t)} = 1.$$

We call each such function *a maximal asymptotics* for (1). Note that in the finite-dimensional case the maximal asymptotics always exists. More exactly, a function $f(t)$ from Definition 1 can be chosen as

$$f(t) = t^{p-1}e^{\mu t},$$

where $\mu = \max_{\lambda \in \sigma(A)} \operatorname{Re} \lambda$ and p is the maximal size of Jordan boxes corresponding to the eigenvalues of A with real part μ. In the infinite-dimensional case it is relatively easy to give an example of the equation (even with a bounded A) for which the maximal asymptotics does not exist. In this context Theorem 1 may be interpreted in the following way:

Let the semigroup $\left\{e^{At},\ t \geq 0\right\}$ be bounded and let $\sigma(A) \cap (i\mathbb{R})$ be at most countable set. Then the asymptotics $f(t) \equiv 1$ is maximal for this semigroup iff A^* possess a pure imaginary eigenvalue.

In particular, this means that if $\sigma(A) \cap (i\mathbb{R})$ is, in addition, nonempty but does not contain eigenvalues then the semigroup has no maximal asymptotics at all. In fact, in this case we have for some $0 < c_0 < C_0 < \infty$

$$c_0 \le \left\| e^{At} \right\| \le C_0, \quad t \ge 0.$$

With this inequality, nonexistence of the maximal asymptotics follows from the following assertion.

Assertion 2. Equation (1) has a maximal asymptotics iff there exists $x_0 \in X$ such that for some $C > 0$

$$C \left\| e^{At} \right\| \le \left\| e^{At} x_0 \right\|, \quad t \ge 0. \tag{2}$$

Proof. Necessity. Let $f(t)$ be a maximal asymptotic. Consider the operator family $B_t = e^{At}/f(t)$, $t \ge 0$. Since for any $x \in X$ the set $\{B_t x\}_{t \ge 0}$ is bounded then (due to Banach – Steinhaus theorem) $\{B_t x\}_{t \ge 0}$ is uniformly bounded. That yields for some $C_1 > 0$:

$$C_1 \left\| e^{At} \right\| \le f(t), \quad t \ge 0.$$

Taking into account the relation

$$\lim_{t \to +\infty} \frac{\left\| e^{At} x_0 \right\|}{f(t)} = 1$$

we obtain (2).

Sufficiency. Assume (2) holds. Denote $f(t) = \left\| e^{At} x_0 \right\|$. Then for any $x \in X$ one has

$$\left\| e^{At} x \right\| / f(t) \le \frac{\left\| e^{At} \right\| \|x\|}{C \left\| e^{At} \right\|} = \|x\| / C.$$

So *(i)* is valid. The validity of *(ii)* is obvious.

Remark 3. From (2) and (ii) one can conclude that any maximal asymptotics (if exists) satisfies the estimate

$$c' \le \frac{\left\| e^{At} \right\|}{f(t)} \le C', \quad t \ge t_0, \tag{3}$$

where $0 < c' < C' < \infty$.

Before we formulate our main result (Theorem 5) let us recall that one of the most important characteristics of the semigroups growth is [9, 5, 4]

$$\omega_0 = \lim_{t \to +\infty} \frac{\ln \|e^{At}\|}{t}.$$

It is well known [5] that this limit exists and the following estimate is valid: for any $\varepsilon > 0$ there exists $M_1 \geq 1$ such that

$$\|e^{At}\| \leq M_1 e^{(\omega_0+\varepsilon)t}, \quad t \geq 0. \tag{4}$$

On the other hand, it is easy to see [9] that the spectral radius of the operator e^{At} equals $e^{\omega_0 t}$. That yields the estimate

$$\|e^{At}\| \geq e^{\omega_0 t}, \quad t \geq 0. \tag{5}$$

Comparing (3), (4), (5) we get

Assertion 4. If equation (1) possesses a maximal asymptotics then it can be chosen so that the following relations are valid:

(i) for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$f(t) \leq M_\varepsilon e^{(\omega_0+\varepsilon)t}, \quad t \geq 0;$$

(ii) there exists $m > 0$ such that

$$f(t) \geq m e^{\omega_0 t}, \quad t \geq 0.$$

Note that in the case of bounded A it is easy to show that

$$\omega_0 = \sup_{\lambda \in \sigma(A)} \operatorname{Re} \lambda,$$

but in the general case we have only [9, 16]

$$\omega_0 \geq \sup_{\lambda \in \sigma(A)} \operatorname{Re} \lambda.$$

See [9] for more details.

The main contribution of the present work is the following theorem.

Theorem 5. *Assume that*

(i) $\sigma(A) \cap \{\lambda : Re\, \lambda = \omega_0\}$ *is at most countable;*

(ii) *Operator* A^* *does not possess eigenvalues with real part* ω_0.

Then equation (1) *(the semigroup* $\{e^{At},\ t \geq 0\}$*) does not have any maximal asymptotics.*

Our proof relies on the following fact from the real analysis.

Lemma 6. *Let $h(t)$ be a real nonnegative function defined on the positive semiaxis $\mathbb{R}^+ = \{t : t \geq 0\}$ and such that*

(a) *for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$h(t) \leq C_\varepsilon + \varepsilon\, t, \quad t \geq 0;$$

(b) *h is concave, i.e.*

$$\alpha h(t_1) + (1-\alpha)h(t_2) \leq h(\alpha t_1 + (1-\alpha)t_2), \quad t_1, t_2 \in \mathbb{R}^+,\ 0 \leq \alpha \leq 1.$$

Then for any $\Delta > 0$ it is valid

$$\lim_{t\to+\infty} (h(t+\Delta) - h(t)) = 0.$$

Proof of Lemma 6. Let $0 < t_1 < t_2 < \infty$ and let $y = l(t)$ be the straight line passing through the points $(t_1, h(t_1))$ and $(t_2, h(t_2))$. Then from assumption (*b*) we have

$$\begin{aligned} &h(t) \leq l(t), \quad t \in \mathbb{R}^+ \setminus (t_1, t_2), \\ &h(t) \geq l(t), \quad t \in (t_1, t_2). \end{aligned} \tag{6}$$

From (6) and positivity of h it follows that h is a nondecreasing function. Besides, from (6) and assumption (*b*) we observe that for any $\Delta > 0$ the function

$$g_\Delta(t) = h(t+\Delta) - h(t), \quad t \geq 0$$

is nonincreasing. On the other hand, from the assumption (*a*) we infer that for any $\Delta, \delta > 0$ there exists $t_0 > 0$ such that

$$h(t+\Delta) - h(t) < \delta, \quad t \geq t_0.$$

This fact completes the proof.

Proof of Theorem 5. Let us observe that without loss of generality it suffices to prove the theorem for $\omega_0 = 0$ (otherwise we consider $(A - \omega_0 I)$ instead of A). We argue by contradiction. Let $f(t)$ be a maximal asymptotics for equation (1) chosen according to Assertion 4 and let

$$\varphi(t) = \log \max\{f(t), 1\}, \quad t \geq 0.$$

Then it follows from Assertion 4 that $\varphi(t)$ is a positive function satisfying the relation: for any $\varepsilon > 0$ there exists $C > 0$ such that

$$\varphi(t) \leq C + \varepsilon t, \quad t \geq 0.$$

Denote

$$C_\varepsilon = \inf\{C : \varphi(t) \le C + \varepsilon t, \quad t \ge 0\}$$

and consider the convex set

$$\Gamma = \bigcap_{\varepsilon>0} \{(t,y) : t \ge 0,\, y \le C_\varepsilon + \varepsilon t\}.$$

Finally put

$$h(t) = \max_{(t,y)\in\Gamma} y, \quad t \ge 0.$$

Then $h(t)$ is a positive concave function such that

$$0 \le \varphi(t) \le h(t) \le C_\varepsilon + \varepsilon t, \quad t \ge 0, \quad \varepsilon > 0,$$

i.e., h satisfies the assumptions of Lemma 6. Besides, one can observe that for any ε there exists $t_\varepsilon > 0$ such that $h(t_\varepsilon) = \limsup_{t\to t_\varepsilon} \varphi(t)$. Moreover, t_ε can be chosen so that $\lim_{\varepsilon\to 0} t_\varepsilon = +\infty$. This means that

$$\limsup_{t\to+\infty} e^{\varphi(t)}/e^{h(t)} = 1$$

and, therefore, the function $\overline{f}(t) = e^{h(t)}$ satisfies condition (i) of Definition 1 and also the condition

(ii$'$) there exists at least one $x_0 \in X$ such that

$$\limsup_{t\to+\infty} \frac{\|e^{At}x_0\|}{\overline{f}(t)} = 1.$$

On the other hand, applying Lemma 6 we get

$$\frac{\overline{f}(t+s)}{\overline{f}(s)} = e^{h(t+s)-h(t)} \to 1 \quad \text{as } s \to +\infty,\ t \ge 0. \tag{7}$$

The further part of our proof is a direct development of the proof from [8, 13] (see the above mentioned scheme). We give it here in detail in order the paper to be self-contained and also to point out those particular items that were added in the case of unbounded operator A.

Let us introduce the seminorm[1] $l(\cdot) = l_{\overline{f}}(\cdot)$ in X defined by the rule

$$l(x) = \limsup_{t\to+\infty} \left(\|e^{At}x\|/\overline{f}(t)\right), \quad x \in X.$$

[1] A similar seminorm was considered in [10] for semigroups restricted by weight functions

Since $\overline{f}(t)$ satisfies condition *(i)* of Definition 1 then there exists $C > 0$ such that

(8) $$l(x) \le C\|x\|, \quad x \in X.$$

Let $L = L_{\overline{f}} = \ker l$. Using (8) we conclude that L is a closed subspace of X. On the other hand, it follows from *(ii′)* that there exists $x \in X$ with $l(x) = 0$, so L is nontrivial. Then we can consider the quotient space $\hat{X} = X/L$ which is also nontrivial. The seminorm l generates the norm $\tilde{l}$ in $\hat{X}$ defined by

$$\tilde{l}(\hat{x}) = l(x), \quad \text{where } x \in \hat{x}.$$

It is dominated by the natural quotient norm $\|\cdot\|_F$ since (8) implies

(9) $$\tilde{l}(\hat{x}) = l(x) \le C \cdot \inf_{x\in\hat{x}} \|x\| = C\|\hat{x}\|_F.$$

So one can consider the completion $\tilde{X}$ of the space $\hat{X}$ w.r.t the norm $\tilde{l}(\cdot)$. Let us now observe that the subspace L is invariant w.r.t. the semigroup $\{e^{At}, t \ge 0\}$. Indeed, for any $x \in X$ we have

$$l(e^{At}x) = \limsup_{s\to+\infty} \frac{\|e^{A(t+s)}x\|}{\overline{f}(s)} = \limsup_{s\to+\infty} \frac{\|e^{A(t+s)}x\|}{\overline{f}(t+s)} \frac{\overline{f}(t+s)}{\overline{f}(s)}.$$

From here and (7) we obtain

(10) $$l(e^{At}x) = \limsup_{s\to+\infty} \frac{\|e^{A(t+s)}x\|}{\overline{f}(t+s)} = l(x).$$

So, if $x \in L$ then $e^{At}x \in L$. Now we consider the quotient semigroup $\hat{T}(t) : \hat{X} \to \hat{X}$, $t \ge 0$, $\hat{T}(t) = e^{At}/L$. It follows from (9) that $\{\hat{T}(t), t \ge 0\}$ is strongly continuous also in the norm $\tilde{l}$. Besides, it is easy to see from (10) that for any $t \ge 0$ the operator $\hat{T}(t)$ is an isometry in the norm $\tilde{l}$. Further on we consider the extension $\tilde{T}(t)$ of the semigroup $\{\hat{T}(t), t \ge 0\}$ to the space $\tilde{X}$. This semigroup is also isometric. Denote by $\tilde{A}$ the generator of the semigroup $\tilde{T}(t)$. Our next goal is to show that

(11) $$\sigma(\tilde{A}) \subset \sigma(A) \cap (i\mathbb{R}).$$

To this end we use the lemma on a boundary point of the spectrum.

Lemma 7. *Let S be a closed operator. If μ is a point of the boundary of the spectrum $\sigma(S)$ then there exists $\{x_k\} \subset D(S)$ such that $\|x_k\| = 1$, $k \in \mathbb{N}$ and $(S - \mu I)x_k \to 0$, $k \to \infty$.*

This lemma is proved in [2] for the case of a bounded operator. Here we give a short proof for the general case.

Proof of Lemma 7. Assume the contrary. Then there exist $\Delta > 0$, $M > 0$ such that

$$\|(S - \lambda I)x\| \geq M\|x\|, \quad x \in D(S) \quad \text{as } |\lambda - \mu| < \Delta.$$

Let $\lambda_k \to \mu$, $k \to \infty$ and $\lambda_k \notin \sigma(S)$. Then $\|R(S, \lambda_k)\| \leq M^{-1}$ if $|\lambda_k - \mu| < \Delta$. The latter property yields that the sequence of operators $R(S, \lambda_k)$ is convergent because

$$\|R(S, \lambda_k) - R(S, \lambda_m)\| \leq |\lambda_k - \lambda_m| \|R(S, \lambda_k)\| \|R(S, \lambda_m)\| \leq |\lambda_k - \lambda_m| M^{-2}$$

as $|\lambda_k - \mu|, |\lambda_m - \mu| < \Delta$. It remains to check directly that the limit of this sequence is the inverse operator to $S - \mu I$. So we arrive at contradiction. Lemma is proved.

Denote by $\partial(\sigma(\tilde{A}))$ the boundary of $\sigma(\tilde{A})$. It follows from Lemma 7 that

$$\partial(\sigma(\tilde{A})) \subset \sigma(A). \tag{12}$$

In fact, let $\mu \notin \sigma(A)$. Then for some $d > 0$

$$\|(A - \mu I)x\| \geq d\|x\|, \quad x \in D(A).$$

From here we get for any $x \in D(A)$

$$\begin{aligned} l((A - \mu I)x) &= \limsup_{t \to +\infty} \left(\|e^{At}(A - \mu I)x\| / \overline{f}(t)\right) \\ &= \limsup_{t \to +\infty} \left(\|(A - \mu I)e^{At}x\| / \overline{f}(t)\right) \\ &\geq \limsup_{t \to +\infty} \left(d\|e^{At}x\| / \overline{f}(t)\right) = d\, l(x). \end{aligned}$$

This immediately yields $\tilde{l}((\tilde{A} - \mu I)y) \geq d\, \tilde{l}(y)$ and, due to Lemma 7, $\mu \notin \partial\sigma(\tilde{A})$. That proves (12).

In the case when the operator A is bounded it is almost obvious that

$$\sigma(\tilde{A}) \subset i\mathbb{R}. \tag{13}$$

In fact, in this case $\tilde{T}(t)$, $t \geq 0$ are invertible isometric operators, so $\sigma(\tilde{T}(t)) \subset \{\lambda : |\lambda| = 1\}$. Then (13) follows from the spectral mapping theorem.

In the case when A is unbounded the validity of inclusion (13) follows from comparing (12), Lemma 7 and the following

Lemma 8. [7] *If S generates a semigroup of isometries in a Banach space then*

$$\|Sx - \lambda x\| \geq |Re\lambda| \, \|x\|$$

for all $x \in D(S)$, $\lambda \in \mathbb{C}$.

The proof of the lemma is contained in [8].

Let us observe that (12) and (13) implies (11). From (11) and Lemma 8 it follows, in turn, due to Hille-Yosida inequality, that the operator $-A$ also generates a semigroup and, therefore, the semigroup $\tilde{T}(t)$, $t \geq 0$ is extended to the group of isometries $\{\tilde{T}(t), -\infty < t < +\infty\}$.

Given this we conclude, following authors of [8], that the set $\sigma(\tilde{A})$ is nonempty (see [9]). Actually this fact and also the application of Lemma 8 are the only additional points in the proof in the case of an unbounded A. Thus, the spectrum $\sigma(\tilde{A})$ is a nonempty closed at most countable set on the imaginary axis. So it possesses an isolated point, say $i\lambda_0$, $\lambda_0 \in \mathbb{R}$. Then $i\lambda_0$ is also an isolated point of the spectrum $\sigma(\tilde{A}^*)$ of the adjoint operator $\tilde{A}^*$. This operator has the invariant subspace, say Ω, corresponding to $i\lambda_0$, i.e. $\Omega \subset D(A^*)$, $A^*|\Omega$ is bounded and $\sigma(\tilde{A}^*|\Omega) = \{i\lambda_0\}$. Since $\{e^{(\tilde{A}^*|\Omega)t}, -\infty < t < +\infty\}$ is a group of isometries we conclude, following [8, 13], that $\tilde{A}^*|\Omega = (i\lambda_0)I|\Omega$. Note that due to relation (3) we have the inclusion $\tilde{X}^* \subset \hat{X}^*$, therefore $\Omega \subset \tilde{X}^* \subset \hat{X}^*$. That means that if $\hat{f} \in \Omega \subset \hat{X}^*$ then

$$(\widehat{e^{At}})^* \hat{f} = e^{i\lambda_0 t} \hat{f}, \quad t \in \mathbb{R}.$$

Finally observe that the latter relation implies that

$$(e^{At})^* f = e^{i\lambda_0 t} f, \quad t \in \mathbb{R},$$

where functionals $f \in X^*$ are extensions of functionals $\hat{f} \in \Omega$ given by

$$f(x) = \hat{f}(\hat{x}), \quad x \in \hat{x}.$$

Therefore we get that $i\lambda_0$ is an eigenvalue of A^*. This contradiction completes the proof of Theorem 5.

Corollary. *If the set* $\sigma(A) \cap \{\lambda : Re\lambda = \omega_0\}$ *is empty then equation* (1) *(or semigroup* $\{e^{At}, t \geq 0\}$*) does not have any maximal asymptotic.*

Using the idea of the above proof one can also obtain the following Theorem that complements the results of [10].

Theorem 9. *Let the assumptions of Theorem 5 be satisfied and let* $f(t)$, $t \geq 0$ *be a positive function such that*

(a) $\log f(t)$ *is concave,*

(b) *for any* $x \in X$ *the function* $\|e^{At}x\|/f(t)$ *is bounded.*

Then

(14) $$\lim_{t\to+\infty} \|e^{At}x\|/f(t) = 0, \quad x \in X.$$

Proof. Let $f_0(t) = f(t)e^{-\omega_0 t}$ and $A_0 = A - \omega_0 I$. It follows from the assumption (*b*) and from (5) that $f_0(t) \geq d > 0$, $t \geq 0$. On the other hand $\log f_0(t) = \log f(t) - \omega_0 t$ is a concave function. Denote $h(t) = \log f_0(t) - \log d \geq 0$ and $\overline{f}(t) = e^{h(t)}$. Next we introduce the seminorm

$$l(x) = \limsup_{t\to+\infty} \left(\|e^{A_0 t}x\|/\overline{f}(t)\right)$$

and repeat all arguments of the proof of Theorem 5 with respect to the operator A_0 and the function $\overline{f}(t)$. That gives $\|e^{A_0 t}x\|/\overline{f}(t) \to 0$, $t \to +\infty$ for any $x \in X$ and finally leads to (14). The proof is completed.

Remark 10. Let us observe that the main statement of the theorem on asymptotic stability ([13, 8, 1]) follows from Theorem 9 in the case when $f(t) \equiv 1$ and $\omega_0 = 0$. On the other hand, the theorem from [13] also states the inverse (see above):

"If $\omega_0 = 0$, $f(t) \equiv 1$ and conditions (*i*) of Theorem 5 and (*b*) of Theorem 9 hold then the existence of a pure imaginary eigenvalue for A^* guarantees that $f(t)$ is a maximal asymptotics". This statement remains true for arbitrary ω_0 and $f(t) = e^{\omega_0 t}$. However, the general statement:

"If $\sigma(A) \cap \{\lambda : Re\lambda = \omega_0\}$ is at most countable and A^* has an eigenvalue with real part ω_0 then a maximal asymptotics for (1) exists" turns out to be false (see Example 1 below).

Example 1. In [13] we considered the example of the operator A:

$$Ax(\cdot) = -\int_0^s x(\tau)d\tau, \qquad s \in [0,1],$$

$x(\cdot) \in X = L_2[0,1]$. This operator satisfies the assumptions of the theorem on asymptotic stability and then equation (1) is asymptotically stable. That means that the function $f(t) \equiv 1$ is not a maximal asymptotics of (1). Let us consider now a more general case:

(15) $$Ax(\cdot) = k\int_0^s x(\tau)d\tau, \qquad s \in [0,1],$$

$x(\cdot) \in X = L_p[0,1]$, where $k \in \mathbb{C}$, $k \neq 0$, $1 \leq p < \infty$ and observe that A satisfies the assumptions of Theorem 5. Indeed, $\sigma(A) = \{0\}$ and the adjoint operator

$$A^*y(\cdot) = k\int_s^1 y(\tau)d\tau,$$

$y(\cdot) \in L_q[0,1]$ as $1 < p < \infty$ or $y(\cdot) \in L_\infty[0,1]$ as $p = 1$, has no eigenvalues. Thus, equation (1) has no maximal asymptotics.

On the other hand, if $X = C[0,1]$, $k = 1$ then equation (1) with the operator given by (15) has a maximal asymptotics

$$f_{max}(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2}.$$

This asymptotics is achieved for the solution corresponding to the initial data $x_0(s) \equiv 1$:

$$e^{At}x_0(s) = \sum_{n=0}^{\infty} \frac{(st)^n}{(n!)^2}, \quad s \in [0,1].$$

This fact is explained by the existence of an eigenvalue of the operator A^*. Indeed, it is easy to see that for the functional $\varphi \in X^*$ defined by $\varphi(x(\cdot)) = x(0)$ one has $A^*\varphi = 0$. Of course, the same situation occurs for $X = L_\infty[0,1]$, but in this case the determining of the eigenvector for A^* is slightly more complicated (see [6]).

Note that the function $f_{max}(t)$ is connected with the Bessel function of imaginary argument $I_0(z)$ as

$$f_{max}(t) = I_0(2\sqrt{t}).$$

This means (see [15]) that a maximal asymptotics can be also chosen by

$$\tilde{f}_{max}(t) = \frac{e^{2\sqrt{t}}}{2\sqrt{\pi}t^{\frac{1}{4}}}.$$

Finally, let $\mathcal{X} = L_p[0,1] \times \mathbb{C}$, $1 \le p < \infty$, and $\mathcal{A} \in [\mathcal{X}, \mathcal{X}]$ be defined by $\mathcal{A}(x(\cdot), y) = (Ax(\cdot), 0) = (\int_0^s x(\tau)d\tau, 0)$, where A is given by (15) with $k = 1$. Then obviously $\|e^{\mathcal{A}t}\| = \|e^{At}\| \to +\infty$ as $t \to +\infty$. This shows that the semigroup $e^{\mathcal{A}t}$ does not have maximal asymptotics though 0 is an eigenvalue of A^*.

Example 2. In [11] we considered the following neutral type system:

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)d\theta,$$

where A_{-1} is a constant $n \times n$-matrix with $\det A_{-1} \neq 0$; A_2, A_3 are $n \times n$-matrices whose elements belong to $L_2(-1,0)$. This equation is reduced to the form

(16) $$\dot{x} = \mathcal{A}x,$$

where $\mathcal{A}$ is a certain infinitesimal operator acting in the space $\mathcal{X} = \mathbb{C}^n \times L_2(-1,0;\mathbb{C}^n)$. It is shown in [11] that the spectral properties of the operator $\mathcal{A}$ are asymptotically defined by the matrix A_{-1}. To illustrate this point let us consider for simplicity the special case when A_{-1} is a Jordan box of order $p = n$ corresponding to the eigenvalue μ, $|\mu| \geq 1$, $\mu \neq 1$, and let $p \geq 2$. Denote by $\overline{\mathcal{A}}$ the operator $\mathcal{A}$ in the special case when $A_2 = A_3 = 0$. Then $\sigma(\overline{\mathcal{A}}) = \{\lambda_{00} = 0, \lambda_k = \log|\mu| + i(\arg\mu + 2\pi k), k \in \mathbb{Z}\}$ and the following orthogonal decomposition holds

$$\mathcal{X} = \bigoplus_{k\in\mathbb{Z}} \overline{V}_k \oplus \overline{W}_0,$$

where the invariant subspace $\overline{W}_0$ corresponds to $\lambda = \lambda_{00}$, $\dim\overline{W}_0 = 2$, $\mathcal{A}|\overline{W}_0$=0, invariant subspaces $\overline{V}_k$, $k \in \mathbb{Z}$, correspond to $\lambda = \lambda_k$, $\dim\overline{V}_k = p$, and $\mathcal{A}|\overline{V}_k$ are Jordan boxes of order p. In particular, this means that the semigroup $\{e^{\mathcal{A}t}, t \geq 0\}$ has a maximal asymptotics

$$f_{max}(t) = t^{p-1}|\mu|^t.$$

In general case, the operator $\mathcal{A}$ possesses a Riesz basis of finite dimensional invariant subspaces (see [11]). More exactly, for an arbitrarily small $r_0 > 0$ there exists $N \in \mathbb{N}$ such that the infinite part of $\sigma(\mathcal{A})$ is located inside the circles $L_k(\lambda_k) = \{\lambda : |\lambda - \lambda_k| < r_0\}$, $|k| > N$, and the only finite number of eigenvalues are outside of these circles. Moreover,

$$\mathcal{X} = \sum_{|k|>N} V_k + W_N,$$

where V_k are images of Riesz projectors corresponding to the spectrum concentrated in $L_k(\lambda_k)$, $|k| > N$, $\dim V_k = p$ and W_N is the invariant subspace corresponding to the spectrum located outside of these circles, $\dim W_N = 2(N+1)p$. Besides, it can be shown that

$$\mathcal{A}|V_k \to \overline{\mathcal{A}}|\overline{V_k}, \quad k \to \infty. \tag{17}$$

Now let us assume that the matrices $A_2(\cdot)$ and $A_3(\cdot)$ are chosen in such a way that

$$\operatorname{Re}\sigma(\mathcal{A}) < \log|\mu|. \tag{18}$$

Then, due to Theorem 5, equation (16) does not have any maximal asymptotics. Moreover, one can derive from (17), (18) that the function

$$\varphi(t) = \|e^{\mathcal{A}t}\|/t^{p-1}|\mu|^t$$

is bounded on the semiaxis $(0, +\infty)$. Thus, applying Theorem 9, we conclude that for any $x \in \mathcal{X}$

$$\|e^{\mathcal{A}t}x\|/t^{p-1}|\mu|^t \to 0, \quad t \to +\infty.$$

On the other hand, it is shown in [11] that there exists a solution $e^{\mathcal{A}t}x_0$ for which

$$\|e^{\mathcal{A}t}x_0\|/|\mu|^t \to \infty,$$

i.e. if, for example, $|\mu| = 1$ then equation (16) is not asymptotically stable.

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