TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 6, pp. 2163-2180, December 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

STRONG CONVERGENCE OF MODIFIED ITERATION PROCESSES FOR RELATIVELY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. Ishikawa and Halpern's iterations are modified to prove the strong convergence problems of such iteration processes for uniformly Lipschitzian mappings which are relatively asymptotically nonexpansive in Banach spaces, which extend the result due to Matsushita and Takahashi [*J. Approx. Theory*, 134 (2005), 257–266] for relatively nonexpansive mappings, and also some recent results due to Martinez-Yanez and Xu [*Nonlinear Anal.*, 64 (2006), 2400–2411], and Kim and Xu [*Nonlinear Anal.*, 64 (2006), 1140–1152] for nonexpansive mappings and asymptotically nonexpansive mappings, respectively, which are considered in the Hilbert space frameworks.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping. Then T is said to be a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$. A Lipschitzian mapping T is called *uniformly k-Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* [9] if $\lim_{n\to\infty} k_n = 1$, respectively. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an *asymptotic fixed point* of T [22] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

Received September 23, 2006, accepted April 5, 2009.
Communicated by Mau-Hsiang Shih.
2000 *Mathematics Subject Classification*: Primary 47H09; Secondary 65J15.
Key words and phrases: Strong convergence, Modified Ishikawa's iteration, Modified Halpern's iteration, Relatively asymptotically nonexpansive mapping.
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Let X be a smooth Banach space and let X^* be the dual of X. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in X$, where J is the normalized duality mapping from X to X^*. We say that a mapping $T : C \to C$ is *relatively asymptotically nonexpansive* if $F(T)$ is nonempty, $\hat{F}(T) = F(T)$ and, for each $n \geq 1$ there exists a constant $k_n > 0$ such that $\phi(p, T^n x) \leq k_n^2 \phi(p, x)$ for $x \in C$ and $p \in F(T)$, where $\lim_{n\to\infty} k_n = 1$. In particular, T is called *relatively nonexpansive* [18] if $k_n = 1$ for all n; see also [3-5].

Construction of approximating fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing. However, the sequence $\{T^n x\}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus three averaged iteration methods often prevail to approximate a fixed point of a nonexpansive mapping T. The first one is introduced by Halpern [10] and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n) T x_n, \quad n \geq 0, \tag{1.1}$$

where $\{t_n\}$ is a sequence in the interval $[0, 1]$.

The second iteration process is now known as Mann's iteration process [16] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$.

The third iteration process is referred to as Ishikawa's iteration process [11] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \quad n \geq 0, \tag{1.3}$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$. By taking $\beta_n = 1$ for all $n \geq 0$ in (1.3), Ishikawa's iteration process reduces to the Mann's iteration process (1.2). It is known in [6] that the process (1.2) may fail to converge while the process (1.3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space.

In general, the iteration process (1.1) has been proved to be strongly convergent in both Hilbert spaces [10, 15, 25] and uniformly smooth Banach spaces [20, 23, 27], while Mann's iteration (1.2) has only weak convergence even in a Hilbert space [8].

Attempts to modify the Mann iteration method (1.2) or the Ishikawa iteration method (1.3) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [19] proposed the following modification of Mann's iteration process(1.2) for a single nonexpansive mapping T with $F(T) \neq \emptyset$ in a Hilbert space H:

$$(1.4) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}x_0$. Recently, Kim and Xu [14] generalized Nakajo and Takahashi's iteration process (1.4) to the following iteration process for an asymptotically nonexpansive mapping T in a Hilbert space, under the hypothesis of boundedness of C:

$$(1.5) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0 \quad \text{as } n \to \infty.$$

They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of T provided the sequence $\{\alpha_n\}$ is bounded above from one.

Very recently, Martinez-Yanez and Xu [17] generalized Nakajo and Takahashi's iteration process (1.4) to the following modification of Ishikawa's iteration process (1.3) for a nonexpansive mapping $T : C \to C$ with $F(T) \neq \emptyset$ in a Hilbert space H:

$$(1.6) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 \\ \qquad +(1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v \rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

and proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to $P_{F(T)}x_0$ provided the sequence $\{\alpha_n\}$ is bounded above from one and $\lim_{n\to\infty}\beta_n = 1$.

On the other hand, Matsushita and Takahashi [18] extended Nakajo and Takahashi's iteration process (1.4) to the following modification of Mann's iteration process (1.2) using the hybrid method in mathematical programming for a relatively nonexpansive mapping $T : C \to C$ in a uniformly convex and uniformly smooth Banach space X:

$$(1.7)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0, \end{cases}$$

where J is the normalized duality mapping. Then they proved that if the sequence $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\limsup_{n\to\infty}\alpha_n < 1$, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to $Q_{F(T)}x_0$, where Q_K denotes the generalized projection from X onto a closed convex subset K of X.

The purpose of this paper is to employ the idea due to Martinez-Yanez and Xu [17] and Matsushita and Takahashi [18] to prove some strong convergence theorems for uniformly Lipschitzian mappings which are relatively asymptotically nonexpansive. The paper is organized as follows. In the next section we introduce some lemmas and propositions studied recently in [12] and [13] which play crucial roles for our argument. In Section 3, motivated by [17, 18] and [14], we extend Matsushita and Takahashi's iteration process (1.7) to the Ishikawa iteration process for such a uniformly Lipschitzian mapping which is relatively asymptotically nonexpansive. In the final section, we develop a similar modification for the process (1.1) and discuss the problem of strong convergence concerning such a mapping in a Banach space.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle\cdot,\cdot\rangle$ the duality product. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}$. The normalized duality mapping J from X to X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$.

A Banach space X is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all $x,\, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if $\|x_n - y_n\| \to 0$ for any two sequences $\{x_n\},\, \{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \to 1$.

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then the Banach space X is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x,\, y \in U$. It is also said to be *uniformly smooth* if the limit in (2.1) is attained uniformly for $x, y \in U$. It is well known that if X is smooth, then the duality mapping J is single-valued. It is also known that if X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X. Some properties of the duality mapping have been given in [7, 21, 24]. A Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [7, 24] for more details.

Let X be a smooth Banach space. Recall that the function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in X$. It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2 \tag{2.2}$$

for all $x, y \in X$. Further, we have that for any $x, y, z \in X$,

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, J(z) - J(y)\rangle.$$

In particular, it is easy to see that if X is strictly convex, for $x, y \in X$, $\phi(y,x) = 0$ if and only if $y = x$ (see, for example, Remark 2.1 of [18]).

Let X be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed convex subset of X. Then, for any $x \in X$, there exists a unique element $\tilde{x} \in C$ such that

$$\phi(\tilde{x}, x) = \inf_{z\in C} \phi(z,x).$$

Then a mapping $Q_C : X \to C$ defined by $Q_C x = \tilde{x}$ is called the *generalized projection* (see [1, 2, 12]). In Hilbert spaces, notice that the generalized projection is clearly coincident with the metric projection.

The following result is well known (see, for example, [1, 2, 12]).

Proposition 2.1. ([1, 2, 12]). *Let K be a nonempty closed convex subset of a real Banach space X and let $x \in X$.*

(a) *If X is smooth, then, $\tilde{x} = Q_K x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$ for $y \in K$.*

(b) *If X is reflexive, strictly convex and smooth, then $\phi(y, Q_K x) + \phi(Q_K x, x) \leq \phi(y, x)$ for all $y \in K$.*

The following subsequent two lemmas are motivated by Lemmas 1.3 and 1.5 of Martinez-Yanes and Xu [17] in Hilbert spaces, respectively; for detailed proofs, see [13].

Lemma 2.2. ([13]). *Let C be a nonempty closed convex subset of a smooth Banach space X, $x, y, z \in X$ and $\lambda \in [0, 1]$. Given also a real number $a \in \mathbb{R}$, the set*

$$D := \{v \in C : \phi(v, z) \leq \lambda\phi(v, x) + (1 - \lambda)\phi(v, y) + a\}$$

is closed and convex.

Lemma 2.3. ([13]). *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, and let K be a nonempty closed convex subset of X. Let $x_0 \in X$ and $q := Q_K x_0$, where Q_K denotes the generalized projection from X onto K. If $\{x_n\}$ is a sequence in X such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\phi(x_n, x_0) \leq \phi(q, x_0)$$

for all n. Then $x_n \to q \; (= Q_K x_0)$.

Recently, Kamimura and Takahashi [12] proved the following result, which plays a crucial role in our discussion.

Proposition 2.4. ([12]). *Let X be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of X. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Finally, concerning the set of fixed points of a relatively asymptotically nonexpansive mapping, we can prove the following result.

Proposition 2.5. *Let X be a uniformly convex and smooth Banach space, let C be a nonempty closed convex subset of X, and let $T : C \to C$ be a continuous mapping which is relatively asymptotically nonexpansive. Then $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence of $F(T)$ such that $x_n \to x \in C$. Since T is relatively asymptotically nonexpansive, we have that

$$\phi(x_n, Tx) \leq k_1^2 \phi(x_n, x)$$

for each $n \geq 1$. Taking the limit on both sides as $n \to \infty$, we have

$$\phi(x, Tx) = \lim_{n\to\infty} \phi(x_n, Tx) \leq k_1^2 \lim_{n\to\infty} \phi(x_n, x) = k_1^2 \phi(x, x) = 0,$$

which implies $x = Tx$ and so $x \in F(T)$. Next, we show that $F(T)$ is convex. For $x, y \in F(T)$ and $\lambda \in (0,1)$, put $z = \lambda x + (1-\lambda)y$. It suffices to show that $z \in F(T)$. Indeed, as in [18], we have that for $n \geq 1$,

$$\begin{aligned} \phi(z, T^n z) &= \|z\|^2 - 2\langle \lambda x + (1-\lambda)y, JT^n z\rangle + \|T^n z\|^2 \\ &= \|z\|^2 - 2\lambda\langle x, JT^n z\rangle - 2(1-\lambda)\langle y, JT^n z\rangle + \|T^n z\|^2 \\ &= \|z\|^2 + \lambda\phi(x, T^n z) + (1-\lambda)\phi(y, T^n z) - \lambda\|x\|^2 - (1-\lambda)\|y\|^2 \\ &\leq \|z\|^2 + k_n^2[\lambda\phi(x, z) + (1-\lambda)\phi(y, z)] - \lambda\|x\|^2 - (1-\lambda)\|y\|^2. \end{aligned}$$

Since $k_n \to 1$, the right hand side of the above inequality converges to 0 because

$$\begin{aligned} &\|z\|^2 + \lambda\phi(x, z) + (1-\lambda)\phi(y, z) - \lambda\|x\|^2 - (1-\lambda)\|y\|^2 \\ = &\|z\|^2 - 2\langle \lambda x + (1-\lambda)y, Jz\rangle + \|z\|^2 \\ = &\|z\|^2 - 2\langle z, Jz\rangle + \|z\|^2 = 0. \end{aligned}$$

By Proposition 2.4, we have $T^n z \to z$ and hence $z \in F(T)$ by the continuity of T. ∎

3. Strong Convergence of Modified Ishikawa's Iteration Processes

In this section we propose a modification of Ishikawa's iteration process (1.3), motivated by the idea due to [17, 18], to have strong convergence for uniformly Lipschitzian mappings which are relatively asymptotically nonexpansive.

Theorem 3.1. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a uniformly k-Lipschitzian mapping which is relatively asymptotically nonexpansive. Assume that $F(T)$ is a nonempty bounded subset of C and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n), \\ z_n = \beta_n x_n + (1-\beta_n)T^n x_n, \\ H_n = \{v \in C : \phi(v, y_n) \leq \alpha_n\phi(v, x_n) + (1-\alpha_n)\phi(v, z_n) + \eta_n\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0\rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0, \end{cases}$$

where J is the normalized duality mapping and

$$\eta_n = (1-\alpha_n)(k_n^2-1)\cdot\sup\{\phi(p,z_n) : p \in F(T)\}.$$

Then $\{x_n\}$ converges in norm to $Q_{F(T)}x_0$, where $Q_{F(T)}$ is the generalized projection from X onto $F(T)$.

Proof. First, observe that H_n is closed and convex by Lemma 2.2, and that W_n is obviously closed and convex for each $n \geq 0$. Next we show that $F(T) \subset H_n$ for all n. Indeed, for $p \in F(T)$, using the convexity of $\|\cdot\|^2$ for the first inequality and relative asymptotic nonexpansivity of T for the second inequality, we get

$$\begin{aligned}
&\phi(p,y_n) = \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n))\\
=\ &\|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JT^n z_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JT^n z_n\|^2\\
\leq\ &\|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, JT^n z_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|T^n z_n\|^2\\
=\ &\alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,T^n z_n)\\
\leq\ &\alpha_n\phi(p,x_n) + (1-\alpha_n)k_n^2\phi(p,z_n)\\
=\ &\alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,z_n) + (1-\alpha_n)(k_n^2-1)\phi(p,z_n)\\
\leq\ &\alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,z_n) + \eta_n.
\end{aligned}$$

So $p \in H_n$ for all n. Moreover, we show that

$$F(T) \subset H_n \cap W_n \tag{3.1}$$

for all $n \geq 0$. It suffices to show that $F(T) \subset W_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F(T) \subset C = W_0$. Assume that $F(T) \subset W_k$ for some $k \geq 1$. Since x_{k+1} is the generalized projection of x_0 onto $H_k \cap W_k$, by Proposition 2.1 (a) we have

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1}\rangle \geq 0$$

for all $z \in H_k \cap W_k$. As $F(T) \subset H_k \cap W_k$, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of W_{k+1} implies that $F(T) \subset W_{k+1}$. Hence (3.1) holds for all $n \geq 0$. So, $\{x_n\}$ is well defined. Obviously, since $x_n = Q_{W_n}x_0$ by the definition of W_n and Proposition 2.1 (a), and since $F(T) \subset W_n$, we have $\phi(x_n, x_0) \leq \phi(p, x_0)$ for all $p \in F(T)$. In particular, we obtain, for all $n \geq 0$,

$$\phi(x_n, x_0) \leq \phi(q, x_0), \qquad \text{where } q := Q_{F(T)}x_0. \tag{3.2}$$

Therefore, $\{\phi(x_n, x_0)\}$ is bounded; so is $\{x_n\}$ by (2.2). Consequently, $\{T^n x_n\}$ is bounded, and so is $\{z_n\}$.

Noticing that $x_n = Q_{W_n} x_0$ again and the fact that $x_{n+1} \in H_n \cap W_n \subset W_n$, we get

$$\phi(x_n, x_0) = \min_{z \in W_n} \phi(z, x_0) \le \phi(x_{n+1}, x_0),$$

which shows that the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing and $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Simultaneously, from Proposition 2.1 (b), we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi\left(x_{n+1}, Q_{W_n} x_0\right) \le \phi(x_{n+1}, x_0) - \phi(Q_{W_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \to 0. \end{aligned} \tag{3.3}$$

By Proposition 2.4, we have

$$\|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

Now since $x_{n+1} \in H_n$, we have

$$\begin{aligned} &\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_n) + (1-\alpha_n)\phi(x_{n+1}, z_n) + \eta_n \\ &= \phi(x_{n+1}, x_n) + (1-\alpha_n)(\phi(x_{n+1}, z_n) - \phi(x_{n+1}, x_n)) + \eta_n \\ &= \phi(x_{n+1}, x_n) + (1-\alpha_n)(2\langle x_{n+1}, Jx_n - Jz_n\rangle + \|z_n\|^2 - \|x_n\|^2) + \eta_n. \end{aligned} \tag{3.5}$$

On the other hand, since $\beta_n \to 1$, and $\{x_n\}, \{T^n x_n\}$ are bounded, we have

$$\|z_n - x_n\| = (1-\beta_n)\|x_n - T^n x_n\| \to 0. \tag{3.6}$$

Recalling that $\{z_n\}$ is also bounded, by (2.2), we see that

$$\begin{aligned} \eta_n &= (1-\alpha_n)(k_n^2 - 1) \cdot \sup\{\phi(p, z_n) : p \in F(T)\} \\ &\le (1-\alpha_n)(k_n^2 - 1) \cdot (\delta + \tau)^2 \to 0 \end{aligned}$$

as $k_n \to 1$, where $\delta := \sup\{\|p\| : p \in F(T)\}$ and $\tau := \sup\{\|z_n\| : n \ge 0\}$. Since J is uniformly norm-to-norm continuous on bounded sets, we have $\|Jx_n - Jz_n\| \to 0$. Hence, we have

$$\begin{aligned} &|2\langle x_{n+1}, Jx_n - Jz_n\rangle + \|z_n\|^2 - \|x_n\|^2| \\ &\le 2\|x_{n+1}\| \cdot \|Jx_n - Jz_n\| + (\|z_n\| + \|x_n\|)(\|z_n - x_n\|) \to 0. \end{aligned} \tag{3.7}$$

Using (3.3), (3.7) and $\eta_n \to 0$, we readily see that the right hand of (3.5) converges to 0; hence $\phi(x_{n+1}, y_n) \to 0$. Using Proposition 2.4 again, we obtain $\|x_{n+1} - y_n\| \to 0$. This, together with (3.4), yields that $\|x_n - y_n\| \to 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have $\|Jx_n - Jy_n\| \to 0$. Combining with $\limsup_{n\to\infty} \alpha_n < 1$ and

$$\begin{aligned} Jx_n - Jy_n &= Jx_n - JJ^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n) \\ &= Jx_n - (\alpha_n Jx_n + (1-\alpha_n)JT^n z_n) \\ &= (1-\alpha_n)(Jx_n - JT^n z_n) \end{aligned}$$

(from the definition of y_n) yields

$$\|Jx_n - JT^n z_n\| = \frac{1}{1-\alpha_n}\|Jx_n - Jy_n\| \to 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have $\|x_n - T^n z_n\| \to 0$. Since T is uniformly k-Lipschitzian, this combined with (3.6) gives that

$$\begin{aligned} \|x_n - T^n x_n\| &\le \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| \\ &\le \|x_n - T^n z_n\| + k\|z_n - x_n\| \to 0. \end{aligned} \tag{3.8}$$

Then, it follows from (3.4) and (3.8) that

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\le (1+k)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + k\|T^n x_n - x_n\| \to 0. \end{aligned} \tag{3.9}$$

By (3.9), $\omega_w(x_n) \subset \hat{F}(T) = F(T)$. This, combined with (3.2) and Lemma 2.3 (with $K = F(T)$), guarantees that $x_n \to q = Q_{F(T)}x_0$. ∎

Remark 3.2. Overlooking the processes of the proof of Theorem 3.1, we readily see that if $\{\eta'_n\}$ is a sequence of real numbers such that $\eta_n \le \eta'_n$ for all n and $\eta'_n \to 0$, the conclusion of Theorem 3.1 still remains true with $\{\eta'_n\}$ instead of $\{\eta_n\}$. Note also that the hypothesis of boundedness of $F(T)$ is abundant in case $\eta_n = 0$ for all n; see [18, 13].

Immediately, taking $\beta_n = 1$ for all $n \ge 0$ in Theorem 3.1, we obtain strong convergence of the modified Mann iteration process in a Banach space.

Theorem 3.3. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a uniformly k-Lipschitzian mapping which is relatively asymptotically nonexpansive. Assume that $F(T)$ is a nonempty bounded subset of C and $\{\alpha_n\}$ is a sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ H_n = \{v \in C : \phi(v, y_n) \le \phi(v, x_n) + \eta_n\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0\rangle \le 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0, \end{cases}$$

where J is the normalized duality mapping and

$$\eta_n = (1-\alpha_n)(k_n^2-1)\cdot \sup\{\phi(p,x_n) : p \in F(T)\}.$$

Then $\{x_n\}$ converges in norm to $Q_{F(T)}x_0$, where $Q_{F(T)}$ is the generalized projection from X onto $F(T)$.

With the help of Theorem 3.3 and Remark 3.2, we immediately have the following result due to Matsushita and Takahashi [18].

Corollary 3.4. ([18]). *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $[0,1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then the sequence $\{x_n\}$ generated by the algorithm (1.7) converges in norm to $Q_{F(T)}x_0$, where $Q_{F(T)}$ is the generalized projection from X onto $F(T)$.*

Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then, after noticing that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$, we see that $\|T^n x - T^n y\| \leq k_n\|x-y\|$ is equivalent to $\phi(T^n x, T^n y) \leq k_n^2\phi(x,y)$. It is therefore easy to show that every asymptotically nonexpansive mapping is both uniformly k-Lipschitzian and relatively asymptotically nonexpansive. In fact, it suffices to show that $\hat{F}(T) \subset F(T)$. The inclusion follows easily from the well-known demiclosedness at zero of $I - T$ (c.f., [26]), where I denotes the identity operator.

Now applying Theorem 3.3 again, we have the following Hilbert space's version.

Theorem 3.5. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping. Assume that $F(T)$ is a nonempty bounded subset of C and $\{\alpha_n\}$ is a sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \eta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where

$$\eta_n = (1-\alpha_n)(k_n^2-1)\sup\{\|p-x_n\|^2 : p \in F(T)\}.$$

Then the sequence $\{x_n\}$ converges in norm to $P_{F(T)}x_0$, where P_K denotes the metric projection from H onto a closed convex subset K of H.

As a direct consequence of Theorem 3.5, we have the following result due to Kim and Xu [14].

Corollary 3.6. ([14]). *Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \leq a$ for all n and for some $0 < a < 1$. Then the sequence $\{x_n\}$ generated by the algorithm (1.5) converges in norm to $P_{F(T)}x_0$.*

Proof. Noticing that $\sup\{\|p-x_n\|^2 : p\in F(T)\} \leq (diam\, C)^2$, we see that $\eta_n \leq \theta_n$ for all n. The conclusion follows easily from Remark 3.2 and Theorem 3.5. ■

Now we propose another modification of Ishikawa's iteration process (1.3) to have strong convergence for a uniformly Lipschitzian mapping which is relatively asymptotically nonexpansive defined on a Banach space.

Theorem 3.7. *Let X be a uniformly convex and uniformly smooth Banach space, and let $T : X \to X$ be a uniformly k-Lipschitzian mapping which is relatively asymptotically nonexpansive. Assume that $F(T)$ is a nonempty bounded subset of X and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ by the algorithm:*

$$\begin{cases} x_0 \in X \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n), \\ H_n = \{v \in X : \phi(v, y_n) \leq \alpha_n\phi(v, x_n) + (1-\alpha_n)\phi(v, z_n) + \eta_n\}, \\ W_n = \{v \in X : \langle x_n - v, Jx_n - Jx_0\rangle \leq 0\}, \\ x_{n+1} = Q_{H_n\cap W_n}x_0, \end{cases}$$

where J is the normalized duality mapping and

$$\eta_n = (1-\alpha_n)(k_n^2 - 1)\cdot \sup\{\phi(p, z_n) : p \in F(T)\}.$$

Then $\{x_n\}$ converges in norm to $Q_{F(T)}x_0$, where $Q_{F(T)}$ is the generalized projection from X onto $F(T)$.

Proof. Use the following (3.10)-(3.12) instead of (3.5)-(3.7) in the proof of Theorem 3.1. Since $x_{n+1} \in H_n$, we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n\phi(x_{n+1}, x_n) + (1-\alpha_n)\phi(x_{n+1}, z_n) + \eta_n. \tag{3.10}$$

On the other hand, recalling that $\{x_n\}$ and $\{T^n x_n\}$ are bounded in the proof of Theorem 3.1, we see that the sequence $\{\phi(x_{n+1}, T^n x_n)\}$ is also bounded by (2.2). Now using the convexity of $\|\cdot\|^2$ for the first inequality, $\beta_n \to 1$, and $\phi(x_{n+1}, x_n) \to 0$, we get

$$
\begin{aligned}
\phi(x_{n+1}, z_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n Jx_n + (1-\beta_n)JT^n x_n\rangle \\
&\quad + \|\beta_n Jx_n + (1-\beta_n)JT^n x_n\|^2 \\
&\leq \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, Jx_n\rangle - 2(1-\beta_n)\langle x_{n+1}, JT^n x_n\rangle \\
&\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|T^n x_n\|^2 \\
&= \beta_n\phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, T^n x_n) \to 0.
\end{aligned} \tag{3.11}
$$

Therefore, the right hand of (3.10) converges to 0; hence $\phi(x_{n+1}, y_n) \to 0$. Also, from Proposition 2.6, $\phi(x_{n+1}, z_n) \to 0$ implies that $\|x_{n+1} - z_n\| \to 0$, and this, together with (3.4), gives that

$$\|x_n - z_n\| \to 0. \tag{3.12}$$

Now repeating the remaining part of the proof of Theorem 3.1, we conclude that $x_n \to Q_{F(T)}x_0$. ■

Here, we shall give an example of a uniformly Lipschitzian mapping which is relatively asymptotically nonexpansive as in the hypotheses of Theorem 3.1, but not relatively nonexpansive.

Example 3.7. Let $X = \ell^p$, where $1 < p < \infty$, and $C = \{x = (x_1, x_2, \ldots) \in X; x_n \geq 0\}$. Then C is a closed convex subset of X. Note that C is not bounded. Obviously, X is uniformly convex and uniformly smooth. Let $\{\lambda_n\}$ and $\{\bar{\lambda}_n\}$ be sequences of real numbers satisfying the following properties:

(i) $0 < \lambda_n < 1$, $\bar{\lambda}_n > 1$, $\lambda_n \uparrow 1$ and $\bar{\lambda}_n \downarrow 1$,

(ii) $\lambda_{n+1}\,\bar{\lambda}_n = 1$ and $\bar{\lambda}_{n+j}\,\lambda_{j+1} < 1$ for all n and j.

(for examples, consider either $\lambda_n = 1 - \frac{1}{n+1}$, $\bar{\lambda}_n = 1 + \frac{1}{n+1}$ or $\lambda_n = e^{-1/n}$, $\bar{\lambda}_n = e^{1/(n+1)}$). Then we define $T : C \to C$ by

$$Tx = (0, \bar{\lambda}_1|\sin x_1|, \lambda_2 x_2, \bar{\lambda}_2 x_3, \lambda_3 x_4, \bar{\lambda}_3 x_5, \cdots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in C$. Obviously, $F(T) = \{0\}$, where $0 = (0, 0, \ldots) \in C$. Since $\lambda_{n+1}\,\bar{\lambda}_n = 1$, it is easy to check that, for $x = (x_1, x_2, \ldots) \in C$,

$$T^{2n-1}x = \Big(\overbrace{0, \cdots, 0}^{2n-1}, \bar{\lambda}_n|\sin x_1|, \lambda_2 x_2, \bar{\lambda}_{n+1}x_3, \lambda_3 x_4, \bar{\lambda}_{n+2}x_5, \ldots\Big)$$

and

$$T^{2n}x = \Big(\overbrace{0, \cdots, 0}^{2n}, |\sin x_1|, \bar{\lambda}_{n+1}\lambda_2 x_2, x_3, \bar{\lambda}_{n+2}\lambda_3 x_4, x_5, \ldots\Big).$$

Letting $k_{2n-1} = \bar{\lambda}_n$ and $k_{2n} = 1$ for all $n \geq 1$, by (i) and (ii), an easy calculation yields that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$. Then, since $k_n \to 1$, T is asymptotically nonexpansive and so, uniformly Lipschitzian. Moreover, it is relatively asymptotically nonexpansive. Indeed,

$$\phi(0, T^n x) = \|T^n x\|^2 \leq k_n^2 \|x\|^2 = k_n^2 \phi(0, x)$$

for all $x \in C$. From the demiclosedness principle of the asymptotically nonexpansive mapping T (see Theorem 2 of [26]) it immediately follows that $\hat{F}(T) \subset F(T)$. Since the converse inclusion always holds true, it must be $\hat{F}(T) = F(T)$. Therefore, T is relatively asymptotically nonexpansive. However, for $e_3 = (0, 0, 1, 0, 0, \ldots) \in C$, since $\|Te_3\| = \bar{\lambda}_2 > 1 = \|e_3\|$ (hence T is not nonexpansive), we have

$$\phi(0, Te_3) = \|Te_3\|^2 = \bar{\lambda}_2^2 > 1 = \|e_3\|^2 = \phi(0, e_3)$$

and thus T is not relatively nonexpansive.

4. Strong Convergence of Modified Halpern's Iteration Processes

In this section, by modifying the proof of Theorem 3.1 slightly, we also study the following strong convergence problem of the Halpern's iteration process (1.1) in a Banach space.

Theorem 4.1. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a uniformly k-Lipschitzian mapping which is relatively asymptotically nonexpansive. Assume that $F(T)$ is a nonempty bounded subset of X and $\{t_n\}$ is a sequence in $(0, 1]$ such that $t_n \to 0$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(t_n J x_0 + (1 - t_n) J T^n x_n), \\ H_n = \{v \in C : \phi(v, y_n) \leq (1 - t_n)\phi(v, x_n) + t_n \phi(v, x_0) + \eta_n\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0, \end{cases}$$

where J is the normalized duality mapping and

$$\eta_n = (1 - t_n)(k_n^2 - 1) \cdot \sup\{\phi(p, x_n) : p \in F(T)\}.$$

Then $\{x_n\}$ converges in norm to $Q_{F(T)}x$, where $Q_{F(T)}$ is the generalized projection from X onto $F(T)$.

Proof. By Lemma 2.2, we see that H_n is closed and convex. For any $p \in F(T)$, we have, using the convexity of $\|\cdot\|^2$ and relatively asymptotic nonexpansivity of T,

$$\begin{aligned}
&\phi(p, y_n) = \phi(p, J^{-1}(t_n Jx_0 + (1-t_n)JT^n x_n)) \\
&= \|p\|^2 - 2\langle p, t_n Jx_0 + (1-t_n)JT^n x_n\rangle + \|t_n Jx_0 + (1-t_n)JT^n x_n\|^2 \\
&\leq \|p\|^2 - 2t_n\langle p, Jx_0\rangle - 2(1-t_n)\langle p, JT^n x_n\rangle + t_n\|x_0\|^2 + (1-t_n)\|T^n x_n\|^2 \\
&= t_n\phi(p, x_0) + (1-t_n)\phi(p, T^n x_n) \\
&\leq (1-t_n)\phi(p, x_n) + t_n\phi(p, x_0) + (1-t_n)(k_n^2 - 1)\phi(p, x_n) \\
&\leq (1-t_n)\phi(p, x_n) + t_n\phi(p, x_0) + \eta_n.
\end{aligned}$$

So $p \in H_n$ for all n. As in the proof of Theorem 3.1, we also have $F(T) \subset W_n$ for all $n \geq 0$ and hence x_n is well-defined for all $n \geq 0$.

The definition of W_n and Proposition 2.1 (a) imply that $x_n = Q_{W_n} x_0$ which in turn implies that $\phi(x_n, x_0) \leq \phi(p, x_0)$ for all $p \in F(T)$; in particular, we obtain

$$\phi(x_n, x_0) \leq \phi(q, x_0) \qquad (\text{with } q := Q_{F(T)} x_0) \tag{4.1}$$

for all $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is bounded; so is $\{x_n\}$ by (2.2). It is obvious that $\{T^k x_n : k, n \geq 0\}$ is bounded. Also, since $x_{n+1} \in H_n \cap W_n \subset W_n$, we have

$$\phi(x_n, x_0) = \min_{z \in W_n} \phi(z, x_0) \leq \phi(x_{n+1}, x_0).$$

That shows that the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing and so the $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. From Proposition 2.1 (b), we have

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, Q_{W_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(Q_{W_n} x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \to 0.
\end{aligned}$$

By Proposition 2.4, we have

$$\|x_{n+1} - x_n\| \to 0. \tag{4.2}$$

Since $\{x_n\}$ is bounded and $k_n \to 1$, as in the proof of Theorem 3.1 we see that $\eta_n \to 0$. Since $x_{n+1} \in H_n$, this, together with $\phi(x_{n+1}, x_n) \to 0$ and $t_n \to 0$, implies that

$$\phi(x_{n+1}, y_n) \leq (1-t_n)\phi(x_{n+1}, x_n) + t_n\phi(x_{n+1}, x_0) + \eta_n \to 0.$$

Using Proposition 2.4 again, it follows that

$$\|x_{n+1} - y_n\| \to 0. \tag{4.3}$$

On the other hand, by the definition of y_n we have

$$\begin{aligned}\|Jy_n - JT^n x_n\| &= \|JJ^{-1}(t_n Jx_0 + (1-t_n)JT^n x_n) - JT^n x_n\| \\ &= t_n \|Jx_0 - JT^n x_n\| \to 0.\end{aligned}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\|y_n - T^n x_n\| \to 0. \tag{4.4}$$

Then it follows from (4.2)-(4.4) that

$$\|x_n - T^n x_n\| \le \|x_n - y_n\| + \|y_n - T^n x_n\| \to 0. \tag{4.5}$$

Then, since T is uniformly k-Lipschitzian, combining (4.2) and (4.5) gets

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\le (1+k)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + k\|T^n x_n - x_n\| \to 0.\end{aligned} \tag{4.6}$$

By (4.6), $\omega_w(x_n) \subset \hat{F}(T) = F(T)$. This, combined with (4.1) and Lemma 2.3 (with $K := F(T)$), gives that $x_n \to q = Q_{F(T)}x_0$. ■

As a direct consequence of Theorem 4.1, we obtain the following result in Hilbert spaces.

Corollary 4.2. *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a asymptotically nonexpansive mapping. Assume that $F(T)$ is a nonempty bounded subset of C and $\{t_n\} \subset (0, 1]$ is such that $t_n \to 0$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = t_n x_0 + (1-t_n)T^n x_n, \\ C_n = \{v \in C : \|v - y_n\|^2 \le (1-t_n)\|v - x_n\|^2 + t_n\|v - x_0\|^2 + \eta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

$$\eta_n = (1-t_n)(k_n^2 - 1)\cdot \sup\{\|p - x_n\|^2 : p \in F(T)\}.$$

Then $\{x_n\}$ converges in norm to $P_{F(T)}x$, where P_K is the metric projection from H onto a closed convex subset K of H.

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