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ON APPROXIMATION OF INTEGRATED SEMIGROUPS

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Abstract. This paper is devoted to approximation of integrated semigroups in space and in time variables. The presentation is given in the abstract framework of discrete approximation scheme, which includes finite element methods, finite difference schemes and projection methods.

1. Introduction

Let $B(E)$ denote the Banach algebra of all linear bounded operators on a complex Banach space E. The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$.

Let A be the generator of a C_0-semigroup $\exp(tA),\ t \geq 0$, and consider in the Banach space E the Cauchy problem

$$\begin{aligned} &u'(t) = Au(t) + f(t),\ t \in\geq 0, \\ &u(0) = u^0, \end{aligned} \tag{1.1}$$

with some function $f(\cdot) \in C([0,T];E)$. Usually, one assumes $u^0 \in D(A)$ in order to obtain well-posedness.

Definition 1.1. *A function* $u(\cdot)$ *is called a solution of* (1.1) *in the classical sense if* $u(\cdot) \in C^1([0,T];E) \cap C([0,T];D(A))$ *and satisfies* (1.1).

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It is known that the mild solution of (1.1) is defined by the formula

$$u(t) = \exp(tA)u^0 + \int_0^t \exp((t-s)A)f(s)\,ds. \tag{1.2}$$

If $u(\cdot)$ in (1.2) is continuously differentiable, then such $u(\cdot)$ is a classical solution of (1.1). If one puts $f(s) = \exp(sA)x,\ s \geq 0$, in general boundedness of $Au(t) = \exp(tA)Au^0 + tA\exp(tA)x$ as $t \to 0$ implies that A generates an analytic C_0-semigroup. But even if A generates an analytic C_0-semigroup and if $f(\cdot) \in C([0,T];E)$, then $u(\cdot)$ from (1.2) is generally not a classical solution (see [30]). Therefore, in the numerical analysis of these equations one can only expect a maximal regularity inequality with a logarithm. Actually, if (1.1) is coercive well-posed in $C([0,T];E)$, then ([13]) either A is bounded or E contains the subspace which is isomorphic to c_0.

Therefore, in order to have well-posedness of (1.1) one needs to impose some smoothness assumption on $f(\cdot)$ in case of a C_0-semigroup. For instance, one may assume that $f(\cdot) \in C^1([0,T];E)$. The situation is dramatically changed if the operator A satisfies weaker conditions than those for a generator of a C_0-semigroup.

In the literature [1, 16, 19, 22] there has been quite interest in solving problem (1.1) under conditions weaker than those for a generator of a C_0-semigroup, namely under the condition that A generates an integrated semigroup $e_1^{tA},\ t \geq 0$, or C-semigroup $S(t),\ t \geq 0$. For example, the Schrödinger operator $i\Delta$ generates a C_0-semigroup on $L^p(I\!R^n)$ iff $p = 2$. Moreover, if $\alpha > n|\frac{1}{2} - \frac{1}{p}|$, then $i\Delta$ generates an α-times integrated semigroup. The starting point for this paper is the observation that there seems to be no systematically developed approximation theory for integrated semigroups, not even for the homogeneous case. In the mean time we have to mention the papers on the subject [2]-[7].

In recent years a rather general approach has been developed for studying the approximation of solutions of C_0-semigroups. We give here a short historical overview of some simplest general results from this approach. Let us consider a well-posed Cauchy problem in a Banach space E with some operator $A \in \mathcal{C}(E)$

$$\begin{aligned} u'(t) &= Au(t),\ t \in [0,\infty), \\ u(0) &= u^0 \in E. \end{aligned} \tag{1.3}$$

If A generates a C_0-semigroup $\exp(\cdot A)$, as is well known, the generalized solution of (1.3) is given by $u(t) = \exp(tA)u^0$ for $t \geq 0$. The theory of well-posed problems and the numerical analysis of these problems have been developed extensively, see for instance the papers [15, 17, 24, 26]. Let us consider a general discretization scheme obtained from the semidiscrete approximation of (1.3) in some Banach spaces E_n :

$$\begin{aligned} u_n'(t) &= A_n u_n(t),\ t \in [0,\infty), \\ u_n(0) &= u_n^0 \in E_n, \end{aligned} \quad . \tag{1.4}$$

Here we assume that $u_n^0 \xrightarrow{\mathcal{P}} u^0$ and operators $A_n \in \mathcal{C}(E_n)$ that generate C_0-semigroups, consistent with the operator $A \in \mathcal{C}(E)$ and $u_n^0 \xrightarrow{\mathcal{P}} u^0$. For a precise definition of discrete convergence of elements and operators see Section 3.

First, we state the following version of Trotter-Kato's Theorem for general approximation schemes:

Theorem 1.1. [28]. (Theorem ABC). *Assume that $A \in \mathcal{C}(E), A_n \in \mathcal{C}(E_n)$ and they generate C_0-semigroups. The following conditions (A) and (B) are equivalent to condition (C).*

(A) *Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge*

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{P}\mathcal{P}} (\lambda I - A)^{-1};$$

(B) *Stability. There are some constants $M \geq 1$ and ω, which are not depending on n and such that $\| \exp(tA_n)\| \leq M \exp(\omega t)$ for $t \geq 0$ and any $n \in I\!N$;*

(C) *Convergence. For any finite $T > 0$ one has $\max_{t \in [0,T]} \| \exp(tA_n)u_n^0 - p_n \exp(tA)u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.*

Theorem 1.2. [24]. *Let operators A and A_n generate analytic C_0-semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1).*

(A) *Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge*

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{P}\mathcal{P}} (\lambda I - A)^{-1};$$

(B_1) *Stability. There exist constants $M_2 \geq 1$ and ω_2 independent of n such that for any $Re\lambda > \omega_2$*

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_2}{|\lambda - \omega_2|} \text{ for all } n \in I\!N;$$

(C_1) *Convergence. For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$ we have*

$$\max_{\eta \in \Sigma(\theta,\mu)} \| \exp(\eta A_n)u_n^0 - p_n \exp(\eta A)u^0\| \to 0 \text{ as } n \to \infty \text{ whenever } u_n^0 \xrightarrow{\mathcal{P}} u^0.$$

Here we denote $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$ and $\Sigma(\theta) = \{z \in \mathbb{C} : |arg\, z| \leq \theta\}$.

Normally they assume that conditions (A) and (B) for the corresponding C_0-semigroup are satisfied without any restriction of generality if any discretization processes in time are considered. We denote by $T_n(\cdot)$ a family of discrete semigroups as in [17], i.e. $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = [\frac{t}{\tau_n}]$, as $\tau_n \to 0, n \to \infty$. The generator of discrete semigroup is defined by $\breve{A}_n = \frac{1}{\tau_n}(T_n(\tau_n) - I_n) \in B(E_n)$ and so $T_n(t) = (I_n + \tau_n \breve{A}_n)^{k_n}$, where $t = k_n \tau_n$.

Theorem 1.3. [28] (Theorem ABC-discr). *The following conditions* (A) *and* (B') *are equivalent to condition* (C').

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \bigcap_n \rho(\breve{A}_n)$ *such that the resolvents converge*

$(\lambda I_n - \breve{A}_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B') *Stability. There exist constants* $M_1 \geq 1$ *and* $\omega_1 \in I\!R$ *such that*

$$\|T_n(t)\| \leq M_1 \exp(\omega_1 t) \text{ for } t \in \overline{I\!R}_+ = [0, \infty), n \in I\!N;$$

(C') *Convergence. For any finite* $T > 0$ *one has* $\max_{t\in[0,T]} \|T_n(t)u_n^0 - p_n \exp(tA)u^0\| \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u^0 \in E, u_n^0 \in E_n$.

Theorem 1.4. [28]. *Assume that operators* $A \in \mathcal{C}(E), A_n \in \mathcal{C}(E_n)$ *and they generate* C_0*-semigroup. Assume also that conditions* (A) *and* (B) *of Theorem* 1.1 *hold. Then the implicit difference scheme*

$$\frac{\overline{U}_n(t+\tau_n) - \overline{U}_n(t)}{\tau_n} = A_n \overline{U}_n(t+\tau), \, \overline{U}_n(0) = u_n^0, \tag{1.5}$$

is stable, i.e. $\|(I_n - \tau_n A_n)^{-k_n}\| \leq M_1 e^{\omega_1 t}, t = k_n \tau_n \in \overline{I\!R}_+$, *and gives an approximation to the solution of the problem* (1.3), *i.e.* $\overline{U}_n(t) \equiv (I_n - \tau_n A_n)^{-k_n} u_n^0 \xrightarrow{\mathcal{P}} \exp(tA)u_n^0$ *uniformly with respect to* $t = k_n \tau_n \in [0, T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty, k_n \to \infty, \tau_n \to 0$.

Note that in this case $T_n(\tau_n) = (I_n - \tau_n A_n)^{-1}$ and $\breve{A}_n = ((I_n - \tau_n A_n)^{-1} - I_n)/\tau_n = A_n(I_n - \tau_n A_n)^{-1}$.

Theorem 1.5. [28]. *Assume that conditions* (A) *and* (B) *of Theorem* 1.1 *hold and condition*

$$\tau_n \|A_n^2\| \leq C, \, n \in I\!N, \tag{1.6}$$

is fulfilled. Then the difference scheme

$$\frac{U_n(t+\tau_n) - U_n(t)}{\tau_n} = A_n U_n(t), \ U_n(0) = u_n^0, \tag{1.7}$$

is stable and gives an approximation to the solution of the problem (1.3), *i.e.* $U_n(t) \equiv (I_n + \tau_n A_n)^{k_n} u_n^0 \xrightarrow{\mathcal{P}} u(t)$ *uniformly with respect to* $t = k_n \tau_n \in [0, T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty, k_n \to \infty, \tau_n \to 0$.

Theorem 1.6. $[14,\ 24]$. *Assume that conditions* (A) *and* (B_1) *of Theorem* 1.2 *hold and condition*

(1.8) $$\tau_n\|A_n\| \leq 1/(M+2),\ n \in I\!N,$$

is fulfilled. Then the difference scheme (1.7) *is stable and gives an approximation to the solution of the problem* (1.3), *i.e.* $U_n(t) \equiv (I_n + \tau_n A_n)^{k_n} u_n^0 \xrightarrow{\mathcal{P}} u(t)$ *uniformly with respect to* $t = k_n\tau_n \in [0,T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, n \to \infty, k_n \to \infty, \tau_n \to 0.$

In this case as we see $T_n(\tau_n) = I_n + \tau_n A_n$ and $\breve{A}_n = A_n$.

Let us recall that the constant M_2 in condition (B_1), which defines $\alpha \in (0, \frac{\pi}{2})$ by $M_2 \sin\alpha < 1$ [18] is such that

(1.9) $$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M}{|\lambda - \omega|} \text{ for any } \lambda \in \Sigma(\pi/2 + \alpha).$$

Recall that there exists a unique Padé approximation for e^{-z} of degree (p,q) given by the formula $R_{p,q}(z) = P_{p,q}(z)/Q_{p,q}(z) \in \pi_{p,q}$, where

$$P_{p,q}(z) = \sum_{j=0}^{p} \frac{(p+q-j)!p!(-z)^j}{(p+q)!j!(p-j)!},\ Q_{p,q}(z) = \sum_{j=0}^{q} \frac{(p+q-j)!q!z^j}{(p+q)!j!(q-j)!}.$$

Definition 1.2. *A rational approximation* $r_{\,p,q}(\cdot) \in \pi_{p,q}$ *for* e^{-z} *is said to be*

(a) *A-acceptable if* $|r_{p,q}(z)| < 1$ *for* $Re(z) > 0$;

(b) $A(\theta)$*-acceptable if* $|r_{p,q}(z)| < 1$ *for* $z \in \Sigma(\theta) \setminus \{0\}$.

Since $r(\cdot) \in \pi_{p,q}$ is an approximation of e^{-z}, it is natural to construct the operator-function $r(\tau_n A_n)^k$ which can be considered as an approximation of $\exp(tA_n)$ for $t = k\tau_n$. For simplicity, we assume in the following Theorems of this section that $\|\exp(tA_n)\| \leq M, t \in \overline{I\!R}_+$.

Theorem 1.7. [8]. *Let condition* (B) *be satisfied. There is a constant* C *depending on* $r(\cdot)$ *such that if* $r(\cdot)$ *is* A*-acceptable, then*

$$\|r(\tau_n A_n)^k\| \leq CM\sqrt{k} \textit{ for } \tau_n > 0, k \in I\!N.$$

Remark 1.1. The term $\sqrt{k}$ in Theorem 1.7 cannot be removed in general; moreover, there are examples (see [11]), which show that the inequality $\|r(\tau_n A_n)^k\| \geq c\sqrt{k}, k \in I\!N$, holds.

We say that $r(\cdot) \in \pi_{p,q}$ is accurate of order $1 \leq d \leq p+q$ if $|e^{-z} - r(z)| = O(|z|^{d+1})$ as $|z| \to 0$.

Theorem 1.8. [11]. *Let condition (B_1) be satisfied. Then there is a constant C depending on r, such that if r is $A(\theta)$-acceptable, accurate of order d, and $\theta \in (\pi/2-\alpha, \pi/2]$ for α from condition* (1.9)*, then*

$$\|r(\tau_n A_n)^k\| \le CM \text{ for } \tau_n > 0, k \in I\!N.$$

The difference scheme which corresponds to rational function $r(\cdot)$, which is Pade $R_{1,1}(z)$ (also called the Crank-Nicolson scheme) is given by the formula

$$\frac{U_n(t+\tau_n)-U_n(t)}{\tau_n} = A_n \frac{U_n(t+\tau)+U_n(t)}{2},\ U_n(0)=u_n^0, \tag{1.10}$$

It is easy to see that in such case $T_n(\tau_n) = \frac{I_n+\frac{\tau_n}{2}A_n}{I_n-\frac{\tau_n}{2}A_n}$ and $\breve{A}_n = (\frac{I_n+\frac{\tau_n}{2}A_n}{I_n-\frac{\tau_n}{2}A_n} - I_n)/\tau_n = A_n(I_n - \frac{\tau_n}{2}A_n)^{-1}$.

2. Preliminaries

Let us consider the Cauchy problem in the Banach space E

$$\begin{aligned} &u'(t) = Au(t) + f(t),\ t \in [0,T],\\ &u(0) = u^0 \in E, \end{aligned} \tag{2.1}$$

where the operator A generates a k-times integrated semigroup and $f(\cdot) \in L^1([0,T]; E)$. A function $u(\cdot)$ is called a classical solution of (2.1) if it belongs to $C^1([0,T];E) \cap C([0,T];D(A))$ and satisfies both equations in (2.1).

A k-times integrated semigroup is a family of bounded linear operators e_k^{tA}, that is strongly continuous in $t \in [0,\infty)$ and satisfies equation

$$e_k^{tA} = A\int_0^t e_k^{sA} ds + \frac{t^k}{k!}, \quad t \ge 0. \tag{2.2}$$

Let us define the function

$$v(t) = e_k^{tA}u^0 + \int_0^t e_k^{(t-s)A} f(s)ds, t \in [0,T].$$

If there is a classical solution of (2.1), then $v(\cdot) \in C^{k+1}([0,T];E)$ and $v^{(k)}(\cdot) = u(\cdot)$. If the k-times integrated semigroup is exponentially bounded, i.e. $\|e_k^{tA}\| \le Me^{\omega t}, t \in \overline{I\!R}_+$, then resolvent satisfies

$$(\lambda I - A)^{-1} = \lambda^k \int_0^\infty e^{-\lambda t} e_k^{tA} dt \quad \text{ for } \quad \lambda > \omega.$$

Let us consider the problem (2.1) when $k = 1$. If we have a once integrated semigroup and $f(\cdot) \equiv 0$, $u^0 \in D(A^2)$, then the solution of (2.1) is given by $u(t) = (e_1^{tA}u^0)'_t$. As an example, let us approximate (2.2) in this case as follows: $e_1^{\tau A} \approx \tau A e_1^{\tau A} + \tau I$. We can write the approximation to $e_1^{\tau A}$ can be written in the form

$$W(\tau) = \tau(I - \tau A)^{-1}. \tag{2.3}$$

It is also well-known that a once integrated semigroup satisfies the equation

$$e_1^{sA}e_1^{tA} = \int_0^s (e_1^{(r+t)A} - e_1^{rA})dr \quad \text{for any} \quad s,t \geq 0. \tag{2.4}$$

Setting $s = \tau, t = k\tau$, one obtains a discrete 1-times integrated semigroup using approximations

$$e_1^{k\tau A}e_1^{\tau A} \approx \tau e_1^{(k+1)\tau A} - \tau e_1^{\tau A}.$$

This leads to the difference scheme

$$W((k+1)\tau) = W(k\tau)W(\tau)/\tau + W(\tau), \quad W(\tau) = \tau(I - \tau A)^{-1}, \tag{2.5}$$

from which discrete approximation function of e_1^{tA} can be calculated. Expression (2.5) is an analogy to the scheme (1.5) for C_0-semigroups case (see Proposition 4.1 for details). In this paper we are going to construct a general approximation theory of exponentially bounded once integrated semigroups.

3. Discretisation of Integrated Semigroups in Space

The general approximation scheme can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n, p_n \in B(E, E_n), n \in I\!N = \{1, 2, \cdots\}$, with the property:

$$\|p_n x\|_{E_n} \to \|x\|_E \text{ as } n \to \infty \text{ for any } x \in E. \tag{3.1}$$

From (3.1) it follows (see [29]) that $\|p_n\| \leq C, n \in I\!N$.

Definition 3.1. *The sequence of elements $\{x_n\}, x_n \in E_n, n \in I\!N$, is said to be $\mathcal{P}$-convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$. We write this $x_n \xrightarrow{\mathcal{P}} x$.*

Definition 3.2. The sequence of bounded linear operators $B_n \in B(E_n), n \in I\!N$, is said to be $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in I\!N$, such that $x_n \xrightarrow{\mathcal{P}} x$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We then write $B_n \xrightarrow{\mathcal{PP}} B$.

Remark 3.1. If we set $E_n = E$ and $p_n = I$ for each $n \in \mathbb{N}$, where I is the identity operator on E, then Definition 3.1 leads to the traditional pointwise convergent bounded linear operators which we denote by $B_n \to B$.

By a similar argument as in the proof of Proposition 1 in [5], one can prove the following discrete version of approximation for Laplace transforms (Theorem 1.7.5 from [1]).

Theorem 3.1. *Let $f_n(\cdot) \in C(\mathbb{R}_+; E_n)$ with $\|f_n(t)\|_{E_n} \leq Me^{\omega t}$ for some $M > 0$, $\omega \in \mathbb{R}$ and all $n \in \mathbb{N}$. Let $\lambda_0 \geq \omega$. The following are equivalent:*

(i) The Laplace transforms $\hat{f}_n(\cdot)$ $\mathcal{P}$-converge pointwise on (λ_0, ∞) to $\hat{f}(\cdot)$ and the sequence $\{f_n(\cdot)\}, n \in \mathbb{N}$, is equicontinuous on compact subsets of $\mathbb{R}_+$;

(ii) The functions $f_n(\cdot)$ $\mathcal{P}$-converge uniformly on compact subsets of $\mathbb{R}_+$ to $f(\cdot)$.

Moreover, if (ii) holds, then $\hat{f}(\lambda) = \mathcal{P}\text{-}\lim_{n\to\infty} \hat{f}_n(\lambda)$ for all $\lambda > \lambda_0$, where $f(t) := \mathcal{P}\text{-}\lim_{n\to\infty} f_n(t)$.

Let us first consider a general discrete version of the ABC Theorem for integrated semigroups.

Theorem 3.2. (Theorem ABC - int). *Assume that closed operators A, A_n on E and E_n respectively generate exponentially bounded k-times integrated semigroups. The following conditions (A) and (B_{int}) are equivalent to condition (C_{int}).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B_{int}) Stability. There are some constants $M \geq 1$ and ω_1, which are independent of n and such that $\|e_k^{tA_n}\|_{B(E_n)} \leq M \exp(\omega_1 t)$ for $t \geq 0$ and any $n \in \mathbb{N}$, and the sequence $\{e_k^{tA_n} p_n u\}, n \in \mathbb{N}$, is equicontinuous on compact subsets of $\mathbb{R}_+$ for every $u \in E$.

(C_{int}) Convergence. For some finite $\omega > 0$ one has $\sup_{t\in[0,\infty)} e^{-\omega t}\|e_k^{tA_n}u_n^0 - p_n e_k^{tA}u^0\|_{E_n} \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Proof. Assume that conditions (A) and (B_{int}) hold. Put $f_n(t) = e_k^{tA_n}u_n^0 - p_n e_k^{tA}u^0$, where $u_n^0 \xrightarrow{\mathcal{P}} u^0$. From the convergence of resolvents (condition (A)), one has that the Laplace transforms of $f_n(t)$ converge to zero. Then for any $x \in E$ there is a sequence $\{u_n^0\}$ such that $u_n^0 \xrightarrow{\mathcal{P}} u^0$ and by Theorem 3.1 $\max_{t\in[0,T]} \|e_k^{tA_n}u_n^0 - p_n e_k^{tA}u^0\|_{E_n} \to 0$ as $n \to \infty$ for every $T > 0$. On the other hand, for any $\epsilon > 0$ and $u_n^0 \xrightarrow{\mathcal{P}} u^0$, there are t_0 and $n_0 \in \mathbb{N}$ such that $\sup_{t_0\leq t<\infty} e^{-\omega t}\|e_k^{tA_n}u_n^0 - p_n e_k^{tA}u^0\|_{E_n} < \epsilon$ for $n \geq n_0$. Combining these we have (C_{int}).

Conversely, assume that (C_{int}) holds. To prove (A) and (B_{int}), by Theorem 3.1 we only need to show (B_{int}). From (C_{int}) it follows that there is a constant $C > 0$ such that $\max_{t\in[0,\infty)} e^{-\omega t}\|e_k^{tA_n}\|_{B(E_n)} \le C$. If it is not true, then one can find sequences $\|u_n\|_{E_n} = 1$ and $t_n \in [0,\infty)$ such that $e^{-\omega t_n}\|e_k^{t_n A_n} u_n\|_{E_n} \to \infty$. Then the sequence $v_n = \frac{u_n}{e^{-\omega t_n}\|e_k^{t_n A_n} u_n\|_{E_n}} \xrightarrow{\mathcal{P}} 0$, and satisfies $e^{-\omega t_n} e_k^{t_n A_n} v_n \xrightarrow{\mathcal{P}} e^{-\omega t_n} e_k^{t_n A} 0 = 0$ uniformly in t_n, but this contradicts to $e^{-\omega t_n}\|e_k^{t_n A_n} v_n\|_{E_n} = 1$. The equicontinuity of $\{e_k^{tA_n} p_n u\}$ follows as in Theorem 3.1.

When $D(A)$ is dense, we have the following version of Theorem 3.2 but without equicontinuity condition:

Theorem 3.3. (Theorem ABC-int-dense). *Assume that a closed densely defined operator A on E generates an exponentially bounded k-times integrated semigroup, and closed operators A_n generate k-times integrated semigroups on E_n respectively. The following conditions (A) and (B'_{int}) are equivalent to condition (C'_{int}).*

(A) Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B'_{int}) Stability. There are some constants $M \ge 1$ and ω_1, which are not depending on n and such that $\|e_k^{tA_n}\|_{B(E_n)} \le M \exp(\omega_1 t)$ for $t \ge 0$ and any $n \in I\!N$;

(C'_{int}) Convergence. For some finite $\omega > 0$ one has $\sup_{t\in[0,\infty)} e^{-\omega t}\|e_k^{tA_n} u_n^0 - p_n e_k^{tA} u^0\|_{E_n} \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Proof. For any $u^0 \in D(A)$ there are $u_n^0 \in D(A_n)$ such that $u_n^0 \xrightarrow{\mathcal{P}} u^0$ and $A_n u_n^0 \xrightarrow{\mathcal{P}} A u^0$, so we have

$$\begin{aligned}\|e_k^{tA_n} u_n^0 - e_k^{sA_n} u_n^0\|_{E_n} &= \|\int_s^t e_k^{\tau A_n} A_n u_n^0 d\tau + \frac{t^k - s^k}{k!} u_n^0\|_{E_n} \\ &\le |t - s| C_k(T)(\|A_n u_n^0\|_{E_n} + \|u_n^0\|_{E_n}),\end{aligned}$$

where $C_k(T)$ is a constant depending only on M, ω, T and k. This implies the equicontinuity of $\{e_k^{tA_n} u_n^0\}$ on $[0,T]$ since $\|A_n u_n^0\|_{E_n}$ and $\|u_n^0\|_{E_n}$ are uniformly bounded. By Theorem 3.1 we have $\max_{t\in[0,T]} \|e_k^{tA_n} u_n^0 - p_n e_k^{tA} u^0\|_{E_n} \to 0$ as $n \to \infty$. This yields (C'_{int}), since $D(A)$ is dense in E.

If the integrated semigroups are uniformly locally Lipschitz continuous, we have

Corollary 3.1. *Assume that A, A_n generate locally Lipschitz continuous k-times integrated semigroups on E, E_n respectively. Then conditions (A) and (B''_{int}) imply (C''_{int}).*

(A) *Compatibility. There exists* $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ *such that the resolvents converge:*

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B''_{int}) *Stability-Uniformity. There are some constants* $M \geq 0$ *and* ω *such that*

$$\|e_k^{(t+h)A_n} - e_k^{tA_n}\|_{B(E_n)} \leq M e^{\omega(t+h)} h, \text{ for } t, h \geq 0, n \in I\!N;$$

(C''_{int}) *Convergence. For any finite* $T > 0$ *one has* $\max_{t\in[0,T]} \|e_k^{tA_n} u_n^0 - p_n e_k^{tA} u^0\|_{E_n} \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u_n^0 \in E_n, u^0 \in E$.

Proof. From condition (B''_{int}) the condition (B_{int}) of Theorem 3.2 follows.

From the proof of Theorem 3.3, we also obtain

Corollary 3.2. *Assume that closed operators* A, A_n *on* E *and* E_n *respectively generate exponentially bounded* k*-times integrated semigroups. Suppose that the following conditions hold:*

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ *such that the resolvents converge*

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B'_{int}) *Stability. There are some constants* $M \geq 1$ *and* ω_1, *which are independent of* n *and such that* $\|e_k^{tA_n}\|_{B(E_n)} \leq M \exp(\omega_1 t)$ *for* $t \geq 0$ *and any* $n \in I\!N$.
Then for some finite $\omega > 0$, *one has* $\sup_{t\in[0,\infty)} e^{-\omega t}\|e_k^{tA_n} u_n^0 - p_n e_k^{tA} u^0\|_{E_n} \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0, A_n u_n^0 \xrightarrow{\mathcal{P}} A u^0$ *for any* $u_n^0 \in D(A_n), u^0 \in D(A)$.

Suppose that the equicontinuity in (B_{int}) does not hold. Then the equicontinuity of the $(k+1)$-times integrated semigroup holds automatically, thus we have

Theorem 3.4. *Assume that closed operators* A, A_n *on* E *and* E_n *respectively generate exponentially bounded* k*-times integrated semigroups. Suppose that the following conditions hold:*

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ *such that the resolvents converge*

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B'_{int}) *Stability. There are some constants* $M \geq 1$ *and* ω_1, *which are independent of* n *and such that* $\|e_k^{tA_n}\|_{B(E_n)} \leq M \exp(\omega_1 t)$ *for* $t \geq 0$ *and any* $n \in I\!N$.
Then for some finite $\omega > 0$, *one has* $\sup_{t\in[0,\infty)} e^{-\omega t}\|e_{k+1}^{tA_n} u_n^0 - p_n e_{k+1}^{tA} u^0\|_{E_n} \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u_n^0 \in E_n, u^0 \in E$.

At last we give an ABC Theorem for analytic integrated semigroup.

Theorem 3.5. *Assume that closed operators A, A_n on E and E_n respectively generate exponentially bounded analytic k-times integrated semigroups. Suppose that the following conditions hold:*

(A) *Consistency. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge*

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B'''_{int}) *Stability. There are some constants $M \geq 1$, $0 < \theta \leq \pi/2$ and ω_1, which are independent of n and such that the sector $\omega_1 + \Sigma(\theta + \pi/2)$ is included in $\rho(A_n)$ and*

$$\sup_{\lambda \in \omega_1 + \Sigma(\beta + \pi/2)} \|(\lambda - \omega_1) R(\lambda, A_n)/\lambda^k\|_{B(E_n)} \leq M$$

for any $n \in I\!N$ and $0 < \beta < \theta$.
Then for every $0 < \beta < \theta$ and a compact set K of $\Sigma(\beta)$, which does not contain 0, one has $\max_{z \in K} \|e_k^{zA_n} u_n^0 - p_n e_k^{zA} u^0\|_{E_n} \to 0$ as $n \to \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}} u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Proof. By Theorem 4.3 in [10], the condition (B'''_{int}) is equivalent to the stability of the integrated semigroups. Moreover, in analytic cases, we are able to write

$$e_k^{zA_n} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda z} \frac{R(\lambda, A_n)}{\lambda^k} d\lambda,$$

where Γ is a positively oriented path which is the boundary of $\omega_1 + \Sigma(\beta + \frac{\pi}{2})$. Now we divide the contour $\Gamma = \Gamma_a \cup \Gamma_b$, where

$$\Gamma_a = \Gamma \cap \{z : |z| \leq a\}, \Gamma_b = \Gamma \setminus \Gamma_a.$$

One can make the integrals over Γ_b very small (less than $\epsilon > 0$) uniformly in n if a is large enough; for z in a compact subset K of $\Sigma(\beta)$, which does not contain 0, it is possible to find δ small enough such that

$$\left\| \int_{\Gamma_a} e^{\lambda z_1} \frac{R(\lambda, A_n)}{\lambda^k} d\lambda - \int_{\Gamma_a} e^{\lambda z_2} \frac{R(\lambda, A_n)}{\lambda^k} d\lambda \right\| = \int_{\Gamma_a} |e^{\lambda z_1} - e^{\lambda z_2}| \frac{\|R(\lambda, A_n)\|}{\lambda^k} d\lambda \leq \epsilon,$$

when $z_1, z_2 \in K$ such that $|z_1 - z_2| < \delta$. Thus we can show that condition of equicontinuity of $e_k^{zA_n} p_n u^0$ from [5], Proposition 1 is satisfied and we obtain the claim.

Remark 3.2. Let us introduce the condition. (C'''_{int}) *Convergence. For some finite* $\omega > 0$ *one has* $\max_{z\in\Sigma(\beta)} e^{-\omega Rez}\|e_k^{zA_n}u_n^0 - p_n e_k^{zA}u^0\|_{E_n} \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u_n^0 \in E_n, u^0 \in E$ *and* $0 < \beta < \theta$.
It is obvious that the condition (C'''_{int}) *implies the conditions* (A) *and* (B'''_{int}).

Remark 3.3. Trotter-Kato's Theorem involving integrated semigroups was considered in [2-4]. This subject was discussed also in [9, 20, 21, 24, 32-34].

4. Discretisation of Integrated Semigroups in Time

In case when the operator A generates an integrated semigroup, but does not generate C_0-semigroup, Theorems 1.1-1.5 from Introduction can not be applied. In the mean time there is a way to construct approximation of integrated semigroups of operators by analogy to the approach from Introduction. The solution of the original problem

$$u'(t) = Au(t),\ t \in [0,\infty), \qquad u(0) = u^0 \in E, \tag{4.1}$$

where the operator A generates once integrated semigroup e_1^{tA}, could be obtained by taking discrete derivative of discrete once integrated semigroup. Existence of derivatives is well-defined on smooth initial data. Therefore the main statement of convergence of difference schemes appears to be considered on smooth elements only. This approach will be the subject of the next paper. In this section we consider discrete once integrated semigroups.

As in Introduction, we will approximate A by a sequence of bounded operators $\breve{A}_n \in B(E_n)$, and then approximate e_1^{tA} by the discrete once integrated semigroups generated by $\breve{A}_n$. Let $T_n \in B(E_n)$, $\{\tau_n\}$ and $\tau_n > 0$, be a sequence converging to 0 as $n \to \infty$ and $\breve{A}_n = (T_n - I_n)/\tau_n \in B(E_n)$. A discrete once integrated semigroup can be defined as $\int_0^t T_n^{[s/\tau_n]}ds$, where $[s/\tau_n]$ is the integer part of the number s/τ_n, i.e. $\int_0^t T_n^{[s/\tau_n]}ds = \tau_n\Sigma_{j=0}^{[t/\tau_n]-1}(I_n+\tau_n\breve{A}_n)^j$. By definition we assume that $\tau_n\Sigma_{j=0}^{[t/\tau_n]-1}(I_n+\tau_n\breve{A}_n)^j = 0$ for $0 \le t < \tau_n$. We give an analogy of Theorem 1.3 for integrated semigroups.

Theorem 4.1. (Theorem ABC-discr-int). *Suppose that* A *generates an exponentially bounded integrated semigroup and* $\breve{A}_n \in B(E_n)$. *The following conditions* (A) *and* $(\tilde{B}_{int})$ *are equivalent to condition* $(\tilde{C}_{int})$.

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \bigcap_n \rho(\breve{A}_n)$ *such that the resolvents converge*

$$(\lambda I_n - \breve{A}_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

$(\tilde{B}_{int})$ *Stability. There are some constants* $M_1 \geq 1$ *and* $\omega_1 \in I\!R$ *such that the discrete once integrated semigroup* $\int_0^t T_n^{[s/\tau_n]}$ *is stable, i.e.*

$$\left\| \int_0^t T_n^{[s/\tau_n]} ds \right\| \leq M_1 \exp(\omega_1 t) \text{ for } t \in \overline{I\!R}_+ = [0, \infty), n \in I\!N,$$

and $\{\int_0^t T_n^{[s/\tau_n]} p_n x ds\}$ *is equicontinuous on bounded intervals of* $I\!R^+$ *for every* $x \in E$;

$(\tilde{C}_{int})$ *Convergence. For some finite* $\omega > 0$ *one has* $\sup_{t \in [0,\infty)} e^{-\omega t} \| \int_0^t T_n^{[s/\tau_n]} u_n^0 ds - p_n e_1^{tA} u^0 \| \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u^0 \in E, u_n^0 \in E_n$.

Proof. Since

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} T_n^{[t/\tau_n]} dt &= \sum_{k=0}^{\infty} T_n^k \int_{k\tau_n}^{(k+1)\tau_n} e^{-\lambda t} dt \\
&= \frac{e^{\lambda \tau_n} - 1}{\lambda} \left(e^{\lambda \tau_n} I_n - T_n \right)^{-1} \\
&= \frac{e^{\lambda \tau_n} - 1}{\lambda \tau_n} \left(\frac{e^{\lambda \tau_n} - 1}{\tau_n} I_n - \frac{T_n - I_n}{\tau_n} \right)^{-1} \\
&= \frac{e^{\lambda \tau_n} - 1}{\lambda \tau_n} \left(\frac{e^{\lambda \tau_n} - 1}{\tau_n} I_n - \breve{A}_n \right)^{-1},
\end{aligned}$$

we get by integration by parts and by the stability condition $(\tilde{B}_{int})$ that

$$\int_0^\infty e^{-\lambda t} \int_0^t T_n^{[s/\tau_n]} ds dt = \frac{e^{\lambda \tau_n} - 1}{\lambda^2 \tau_n} \left(\frac{e^{\lambda \tau_n} - 1}{\tau_n} I_n - \breve{A}_n \right)^{-1},$$

which $\mathcal{PP}$-converge to $\frac{1}{\lambda}(\lambda I - A)^{-1}$ by (A). Since $\{\int_0^t T_n^{[s/\tau_n]} ds\}$ are uniformly exponentially bounded and equicontinuous on bounded intervals, the rest of the proof is similar to the proof of Theorem 3.2.

Remark 4.1. (a) Let us compare condition $(\tilde{B}_{int})$ with condition
$(\tilde{B}'_{int})$ there are some constants $M \geq 1$ and $\omega \in I\!R$ such that

$$\|T_n^k\| \leq M e^{\omega k \tau_n} \text{ for all } n, k \in I\!N.$$

Condition $(\tilde{B}'_{int})$ is the main hypothesis in [27]. Condition $(\tilde{B}'_{int})$ implies condition $(\tilde{B}_{int})$, but not conversely. Indeed we have for $t = k\tau_n + r$ and $0 \leq r < \tau_n$,

$$\left\| \int_0^t T_n^{[s/\tau_n]} ds \right\| = \left\| \sum_{j=1}^{k-1} \int_{j\tau_n}^{(j+1)\tau_n} T_n^j ds + \int_0^r T_n^k ds \right\| \leq \sum_{j=1}^{k-1} \tau_n M e^{\omega j \tau_n} + r M e^{\omega k \tau_n}$$

$$\leq tMe^{\omega k\tau_n} \leq Me^{(\omega+1)t}.$$

Similarly one can show that in this case

$$\left\| \int_0^t T_n^{[\xi/\tau_n]} d\xi - \int_0^s T_n^{[\xi/\tau_n]} d\xi \right\| \leq Me^{\omega \max(t,s)}|t-s|. \tag{4.2}$$

Moreover, it is easy to prove that $(\tilde{B}'_{int})$ is equivalent to (4.2).

(b) If we assume that A is densely defined, then we can use Theorem 3.3 instead of Theorem 3.2, so we do not need the equicontinuity condition in $(\tilde{B}_{int})$ in such case. Also, with simplification of conditions in $(\tilde{B}_{int})$, we can show the convergence just on $D(A)$. Indeed, for any $u_0 \in D(A)$, we can find $u_n^0 \in D(A_n)$ such that $u_n^0 \xrightarrow{\mathcal{P}} u^0$ and $A_n u_n^0 \xrightarrow{\mathcal{P}} Au^0$. A direct calculation gives

$$\int_0^t T_n^{[r/\tau_n]} u_n^0 dr - \int_0^s T_n^{[r/\tau_n]} u_n^0 dr = \int_s^t \int_0^r T_n^{[v/\tau_n]} A_n u_n^0 dv dr + (t-s)u_n^0,$$

from this we get the equicontinuity of $\{\int_0^t T_n^{[r/\tau_n]} u_n^0 dr\}_{t\geq 0}$ for $u_n^0 \in D(A_n)$ such that $u_n^0 \xrightarrow{\mathcal{P}} u^0$ and $A_n u_n^0 \xrightarrow{\mathcal{P}} Au^0$.

(c) Without the equicontinuity condition in $(\tilde{B}_{int})$, we also have the convergence of the discrete twice integrated semigroups $\int_0^t \int_0^s T_n^{[r/\tau_n]} dr ds$ since the equicontinuity follows immediately from the stability in $(\tilde{B}_{int})$.

Remark 4.2. The equicontinuity of $\{\int_0^t T_n^{[r/\tau_n]} u_n^0 dr\}_{t\geq 0}$ by dicrete analogy of formula (4.13) (see Remark 4.8)

$$T_n^{[t/\tau]} R(\mu, A_n) = e^{\lambda\tau[t/\tau]} R(\mu, A_n)$$
$$-\mu \int_0^{\tau[t/\tau]-\tau} e^{\lambda\tau([t/\tau]-2-[u/\tau])} \int_0^{\tau[u/\tau]+\tau} T_n^{[s/\tau]} ds du - \int_0^{\tau[t/\tau]} T_n^{[u/\tau]} du,$$

where $\mu = \frac{e^{\lambda\tau}-1}{\tau}$, implies that $\{T_n^{[t/\tau]} R(\mu, A_n) u_n\}_{t\geq 0}$ is equicontinuous and conversely. This means that on smooth data one can investigate the same difference schemes as in case of C_0-semigroups, but on the smooth elements only.

4.1. Implicit scheme

Let us put now $T_n(\tau_n) = (I_n - \tau_n A_n)^{-1}$. In this subsection a discrete once integrated semigroup is defined as $\int_0^t T_n^{[s/\tau_n]} ds = \tau_n \Sigma_{j=1}^{[t/\tau_n]} (I_n - \tau_n A_n)^{-j}$ and we put by definition $\tau_n \Sigma_{j=1}^{[t/\tau_n]} (I_n - \tau_n A_n)^{-j} = 0$ for $0 \leq t < \tau_n$. So in this subsection we have a special choice of $\breve{A}_n = A_n(I_n - \tau_n A_n)^{-1}$, where the operator A_n is taken from Theorem 3.2. We can give the following definition for the implicit difference scheme (2.5).

Definition 4.1. The discrete family of operators $\{W_n^i(k\tau_n)\}, k = 0, 1, 2, \dots,$ is called discrete 1-times implicit integrated semigroup if $W_n^i(0) = 0, W_n^i(\tau_n) = \tau_n(I_n - \tau_n A_n)^{-1}$ and

$$W_n^i(k\tau_n)W_n^i(\tau_n) = \tau_n W_n^i((k+1)\tau_n) - \tau_n W_n^i(\tau_n).$$

Proposition 4.1. *If A_n^{-1} exist the discrete 1-times implicit integrated semigroup is given by the formulas*

$$W_n^i(0) = 0, \tag{4.3}$$

$$W_n^i((k+1)\tau_n) = W_n^i(k\tau_n)(I_n - \tau_n A_n)^{-1} + W_n^i(\tau_n),\ k = 1, 2, \dots, \tag{4.4}$$

$$W_n^i(k\tau_n) = \sum_{j=1}^{k} \tau_n(I_n - \tau_n A_n)^{-j} = ((I_n - \tau_n A_n)^{-k} - I_n)A_n^{-1},\ k = 1, 2, \dots. \tag{4.5}$$

Proof. From Definition 4.1 it follows that

$$W_n^i((k+1)\tau_n) = W_n^i(k\tau_n)W_n^i(\tau_n)/\tau_n + W_n^i(\tau_n).$$

One gets $W_n^i((k+1)\tau_n) = W_n^i(k\tau_n)(I_n - \tau_n A_n)^{-1} + W_n^i(\tau_n)$. Therefore

$$\begin{aligned} W_n^i((k+1)\tau_n) &= \frac{(I_n - \tau_n A_n)^{-(k+1)} - I_n}{(I_n - \tau_n A_n)^{-1} - I_n}\tau_n(I_n - \tau_n A_n)^{-1} \\ &= ((I_n - \tau_n A_n)^{-(k+1)} - I_n)A_n^{-1}. \end{aligned}$$

Theorem 4.2. *Suppose that conditions (A) and (B'_{int}) of Theorem 3.3 hold. Then discrete 1-times integrated semigroup is exponentially stable, i.e. $\| \sum_{j=0}^{k} \tau_n(I_n - \tau_n A_n)^{-j}\| \le M_1 e^{\omega_1 \tau_n k}$ and gives an approximation to once integrated semigroup, i.e. $\sum_{j=0}^{k_n} \tau_n(I_n - \tau_n A_n)^{-j} u_n^0 \xrightarrow{\mathcal{P}} e_1^{tA} u^0$ uniformly with respect to $t = k_n \tau_n \in [0, T]$ as $u_n^0 \xrightarrow{\mathcal{P}} u^0, A_n u_n^0 \xrightarrow{\mathcal{P}} Au^0, n \to \infty,$ for any $u^0 \in D(A)$.*

Proof. We only need to show that conditions (A) and (B_{int}) of Theorem 3.2 imply $(\check{B}_{int})$. By the stability condition (B_{int}) of Theorem 3.2, we know that

$$\Big\| \Big(\frac{R(\lambda, A_n)}{\lambda}\Big)_\lambda^{(m)} \Big\| \le \frac{Mm!}{|\lambda - \omega|^{m+1}},\ \lambda > \omega,$$

where $(\cdot)_\lambda^{(m)}$ denotes m-th derivative in λ. By the formulas for derivatives of resolvents $(R(\lambda, A_n))_\lambda^{(m)} = (-1)^m m! R(\lambda, A_n)^{m+1}$ and $(1/\lambda)_\lambda^{(m)} = (-1)^m m! (1/\lambda)^{m+1}$, one obtains

$$\begin{aligned}(R(\lambda, A_n)/\lambda)^{(m)}_\lambda &= \sum_{j=0}^{m} C^j_m (R(\lambda, A_n))^{(j)}_\lambda (1/\lambda)^{(m-j)}_\lambda \\ &= \sum_{j=0}^{m} C^j_m (-1)^j j! R(\lambda, A_n)^{j+1} (-1)^{m-j} (m-j)! (1/\lambda)^{m-j+1} \\ &= (-1)^m m! \sum_{j=1}^{m+1} R(\lambda, A_n)^j / \lambda^{m+2-j},\end{aligned}$$

and therefore

$$\|\frac{1}{\lambda} \sum_{j=1}^{m+1} \lambda^j R(\lambda, A_n)^j\| \le \frac{M}{|1-\frac{\omega}{\lambda}|^{m+1}}.$$

Now choosing $\lambda = 1/\tau_n$, we have

$$\left\|\tau_n \sum_{j=0}^{m} (I_n - \tau_n A_n)^{-j}\right\| \le \frac{M}{|1-\tau_n\omega|^m} \le M' e^{\omega' \tau_n m}.$$

To prove convergence one can apply Theorem 4.1 with $\breve{A}_n = A_n(I_n - \tau_n A_n)^{-1}$ and Remark 4.1 (b).

Remark 4.3. To get the convergence for any $u^0 \in E$ in Theorem 4.2 one could assume the condition of equicontinuity of $\sum_{j=0}^{k_n} \tau_n (I_n - \tau_n A_n)^{-j} u^0_n$ or density of $D(A)$.

4.2. Explicit scheme

In this subsection we put $T_n(\tau_n) = I_n + \tau_n A_n$, then $A_n = \breve{A}_n \in B(E_n)$. Consider the explicit difference scheme we give the following definition.

Definition 4.2. The discrete family of operators $\{W^e_n(k\tau_n)\}, k = 0, 1, 2, \ldots,$ is called discrete 1-times explicit integrated semigroup if $W^e_n(0) = 0, W^e_n(\tau_n) = \tau_n I_n, W^e_n(2\tau_n) = A_n W^e_n(\tau_n)\tau_n + 2\tau_n I_n$ and

$$W^e_n(k\tau_n) W^e_n(2\tau_n) = (W^e_n((k+1)\tau_n) + W^e_n(k\tau_n))\tau_n - \tau_n^2 I_n.$$

Proposition 4.2. *If A_n^{-1} exist the discrete 1-times explicit integrated semigroup is given by the formulas*

$$W^e_n(0) = 0,$$

$$W^e_n((k+1)\tau_n) = W^e_n(k\tau_n)(I_n + \tau_n A_n) + \tau_n I_n,\ k = 1, 2, \ldots,$$

$$W_n^e(k\tau_n) = \tau_n \sum_{j=0}^{k-1} (I_n + \tau_n A_n)^j = ((I_n + \tau_n A_n)^k - I_n)A_n^{-1},\ k = 1, 2, \ldots.$$

Proof. From Definition 4.2 it follows that

$$\begin{aligned} W_n^e((k+1)\tau_n) &= W_n^e(k\tau_n)(W_n^e(2\tau_n)/\tau_n - I_n) + \tau_n I_n \\ &= W_n^e((k-1)\tau_n)(I_n + \tau_n A_n)^2 + \tau_n(I_n + \tau_n A_n) + \tau_n I_n. \end{aligned}$$

One gets $W_n^e((k+1)\tau_n) = W_n^e(k\tau)(I_n + \tau_n A_n) + \tau I_n$. Therefore

$$W_n^e((k+1)\tau_n) = \frac{(I_n + \tau_n A_n)^{k+1} - I_n}{(I_n + \tau_n A_n) - I_n}\tau_n = ((I_n + \tau_n A_n)^{k+1} - I_n)A_n^{-1}. \tag{4.6}$$

Theorem 4.3. *Suppose that conditions (A) and (B'_{int}) of Theorem 3.3 hold and*

$$\tau_n\|A_n^2\|, \|A_n^{-1}\| \le C, n \in I\!N.$$

Then discrete explicit once integrated semigroup $\int_0^t (I_n + \tau_n A_n)^{[s/\tau_n]} ds$ is exponentially stable, i.e.

$$\|\sum_{j=0}^{k_n} \tau_n (I_n + \tau_n A_n)^j\| \le M_1\, e^{\omega_2 \tau_n k_n}, \tag{4.7}$$

for some $\omega_2 > 0$ and it gives an approximation of once integrated semigroup, i.e. $\tau_n \sum_{j=0}^{k_n-1} (I_n + \tau_n A_n)^j u_n^0 \xrightarrow{\mathcal{P}} p_n e_1^{tA} u^0$ uniformly with respect to $t = k_n\tau_n \in [0, T]$ as $u_n^0 \xrightarrow{\mathcal{P}} u^0, A_n u_n^0 \xrightarrow{\mathcal{P}} Au^0, n \to \infty$, for any $u^0 \in D(A)$.

Proof. Since

$$\begin{aligned} &\sum_{k=0}^{m-1} \tau_n (I_n + \tau_n A_n)^k \\ &= ((I_n + \tau_n A_n)^m - I_n)A_n^{-1} \\ &= \Big((I_n - \tau_n^2 A_n^2)^m (I_n - \tau_n A_n)^{-m} - I_n\Big)A_n^{-1} \\ &= (I_n - \tau_n^2 A_n^2)^m \Big((I_n - \tau_n A_n)^{-m} A_n^{-1} - A_n^{-1}\Big) + (I_n - \tau_n^2 A_n^2)^m A_n^{-1} - A_n^{-1} \\ &= (I_n - \tau_n^2 A_n^2)^m \sum_{k=1}^{m} \tau_n (I_n - \tau_n A_n)^{-k} + (I_n - \tau_n^2 A_n^2)^m A_n^{-1} - A_n^{-1}, \end{aligned}$$

then stability follows from estimates $(1 + \tau_n\|\tau_n A_n^2\|)^m \le Me^{\omega m \tau_n}$ and Theorem 4.2. To prove convergence one can apply Theorem 4.1 with $\breve{A}_n = A_n$ and Remark

4.1 (b). Indeed, to prove the equicontinuity condition for family $\{\sum_{k=0}^{m-1}\tau_n(I_n+\tau_nA_n)^k u_n^0\}_{t=k_n\tau_n>0}$ one has to remark that difference can be estimated **as** $\|\tau_n(I_n+\tau_nA_n)^{k_n}A_n^{-1}A_nu_n^0\| \le C\tau_n\|A_nu_n^0\|$.

Remark 4.4. We can give a different proof of Theorem 4.3. Let us define the operator $Q_n = A_n(I_n+\tau_nA_n)^{-1}$. It is clear that operators $A_n - Q_n = A_n(I_n+\tau_nA_n)^{-1}\tau_nA_n$ are uniformly in n bounded if $\|\tau_nA_n^2\| \le Constant$. Since Q_n commutes with A_n, it follows by Proposition 3.1 in [16] that the operators Q_n generate integrated semigroups and by Theorem 4.2 one gets $\|\tau_n\Sigma_{j=0}^{k-1}(I_n-\tau_nQ_n)^{-j}\| = \|\tau_n\Sigma_{j=0}^{k-1}(I_n+\tau_nA_n)^j\| \le constant.$

Theorem 4.4. *Suppose that conditions* (A) *and* (B_{int}''') *of Theorem* 3.5 *hold with* $\omega_1 = 0$ *and*

$$\sup_n \tau_n\|A_n\| < \mu < 2\sin\theta, 0 \in \rho(A_n), n \in I\!N.$$

Then discrete explicit once integrated semigroup $\int_0^t(I_n+\tau_nA_n)^{[s/\tau_n]}ds$ *is exponentially stable, i.e.*

$$\|\sum_{j=0}^{k_n}\tau_n(I_n+\tau_nA_n)^j\| \le M_1\, e^{\omega_3\tau_nk_n}, \tag{4.8}$$

for some $\omega_3 > 0$*, and it gives an approximation of once integrated semigroup, i.e.* $\tau_n\sum_{j=0}^{k_n-1}(I_n+\tau_nA_n)^ju_n^0 \xrightarrow{\mathcal{P}} e_1^{tA}u^0$ *uniformly with respect to* $t = k_n\tau_n \in [0,T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, A_nu_n^0 \xrightarrow{\mathcal{P}} Au^0, n \to \infty$*, for any* $u^0 \in D(A)$.

Proof. Since A_n are bounded operators, they generate C_0-semigroups and the once integrated semigroups are given by $e_1^{tA_n} = A_n^{-1}(e^{tA_n} - I)$. We know that $e_1^{tA_n}$ are uniformly bounded, i.e. $\|e_1^{zA_n}\| \le M, z \in \Sigma_\beta$. If we show that $\|(e^{tA_n} - I_n)A_n^{-1} - \sum_{k=0}^{m-1}\tau_n(I_n+\tau_nA_n)^k\| \le Ct$, then stability, i.e. condition $(\tilde{B}_{int})$ will be proved. One can write

$$\begin{aligned}
&(e^{tA_n} - I_n)A_n^{-1} - ((I_n+\tau_nA_n)^m - I_n)A_n^{-1}\\
&= -\int_0^{\tau_n}\frac{d}{ds}\Big(e^{m(\tau_n-s)A_n}(I_n+sA_n)^m\Big)dsA_n^{-1}\\
&= -\int_0^{\tau_n}\Big(-mA_ne^{m(\tau_n-s)A_n}(I_n+sA_n)^m + e^{m(\tau_n-s)A_n}(I_n+sA_n)^{m-1}mA_n\Big)dsA_n^{-1}\\
&= \int_0^{\tau_n} sme^{m(\tau_n-s)A_n}(I_n+sA_n)^{m-1}ds\,A_n\\
&= \frac{1}{2\pi i}\int_0^{\tau_n} sm\Big(\int_\Gamma e^{m(\tau_n-s)\lambda}(1+s\lambda)^{m-1}(\lambda I_n - A_n)^{-1}d\lambda\Big)ds\,A_n,
\end{aligned}$$

where the positively oriented contour Γ is composed of $\Gamma_1 = \{re^{\pm i(\theta+\pi/2)} : 0 \leq r \leq R_n\}$ and $\Gamma_2 = \{R_n e^{i\varphi} : \theta + \pi/2 \leq \varphi \leq -\theta + 3\pi/2)\}$ with R_n satisfying $\tau_n R_n = \mu$. First we can choose a positive γ, which depends only on θ and μ such that

$$|1 + s\lambda|^2 = 1 + s^2r^2 - 2sr\sin\theta = 1 - sr(2\sin\theta - sr) \leq 1 - \gamma sr,$$

for any $0 \leq s \leq \tau_n, 0 \leq r \leq R_n$. Thus for $2\sin\theta - \mu = \gamma > 0$

$$|1 + sz| \leq 1 - \gamma sr, \quad 0 \leq sr < \mu,\ z \in \Gamma_1.$$

For $z = re^{\pm i(\theta+\pi/2)} \in \Gamma_1$, the integral over Γ_1 can be estimated by

$$\int_0^{R_n} e^{-m(\tau_n - s)r\sin\theta}(1-\gamma sr)^{m-1}dr\|A_n\| \leq \int_0^{R_n} e^{-m(\tau_n-s)r\sin\theta}e^{-\gamma sr(m-1)}dr\|A_n\|$$

$$\leq e^{\gamma\tau_n R_n}\int_0^{R_n} e^{-[m(\tau_n-s)\sin\theta+\gamma sm]r}dr\|A_n\| \leq \frac{e^{\gamma\mu}}{m(\tau_n - s)\sin\theta + \gamma sm}\|A_n\|$$

since $(\tau_n - s)\sin\theta > 0$ and γ is positive. Moreover, since the function $[0, \tau_n] \ni s \mapsto m(\tau_n - s)\sin\theta + \gamma sm$ is linear, it reaches its minimum at 0 or τ_n. Thus one has

$$e^{\gamma\mu}\int_0^{\tau_n} \frac{sm}{m(\tau_n - s)\sin\theta + \gamma sm}ds\|A_n\|$$
$$\leq \frac{e^{\gamma\mu}\tau_n^2 m}{\min\{m\tau_n\sin\theta, \gamma\tau_n m\}}\|A_n\| \leq \frac{e^{\gamma\mu}\mu}{\min\{\sin\theta, \gamma\}},$$

for any $m \geq 1$. For $\lambda = R_n e^{i\varphi} \in \Gamma_2$, choose $\beta < 1$ such that $\tau_n\|A_n\| \leq \beta\mu$, then

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{1}{|\lambda|\cdot(1 - \|A_n\|/|\lambda|)} = \frac{1}{R_n\cdot(1 - \|A_n\|/R_n)} \leq \frac{1}{R_n(1-\beta)}.$$

Therefore, we can bound the integral over Γ_2 by

$$C\int_{\theta+\pi/2}^{3\pi/2-\theta} e^{-m\tau_n R_n\sin\theta}\frac{R_n d\varphi}{R_n(1-\beta)}\|A_n\| \leq C' e^{-m\tau_n R_n\sin\theta}\|A_n\|.$$

Also for $\lambda = \mu e^{i\varphi}/\tau_n \in \Gamma_2$, we have

$$|1+\tau_n\lambda| \leq |1+\mu e^{i(\theta+\pi/2)}| = (1+\mu^2 - 2\mu\sin\theta)^{1/2} = [1-\mu(2\sin\theta-\mu)]^{1/2} < 1;$$

combining this with the inequality $|1 + s\lambda| \leq 1$ for $\lambda \in \Gamma_1$ and $0 \leq s \leq \tau_n$, the Maximal Modulus Principle yields that $|1 + s\lambda| \leq 1$ for all $\lambda \in \Gamma$, $0 \leq s \leq \tau_n$. Thus we have

$$\left\|\int_0^{\tau_n} sm\Big(\int_{\Gamma_2} e^{m(\tau_n-s)\lambda}(1+s\lambda)^{m-1}(\lambda I_n - A_n)^{-1}d\lambda\Big)dsA_n\right\|$$
$$\leq \quad C'\int_0^{\tau_n} sme^{-m\tau_n R_n\sin\theta}ds\|A_n\| \leq C\frac{\tau_n^2 m}{2}\|A_n\| \leq Ct\mu.$$

Since $\tau_n\|A_n\| < \mu < \infty$ we get the stability (4.8), then the Theorem is proved by applying Theorem 4.1 with $\breve{A}_n = A_n$ and Remark 4.1 (b).

Remark 4.5. To get in Theorem 4.4 the convergence for any $u^0 \in E$ one could assume density of domain $D(A)$ or equicontinuity of family $\Sigma_{j=0}^{k_n}\tau_n(I_n+\tau_n A_n)^j u_n^0$.

Remark 4.6. In Theorems $4.3-4.4$ we assumed that there exist A_n^{-1}. However, if A generates exponentially bounded once integrated semigroup e_1^{tA}, i.e. $\|e_1^{tA}\| \leq Me^{\omega t}$, $t \geq 0$, then by [16] the once integrated semigroup generated by $A - \omega I$ is related to e_1^{tA} by formula

$$e_1^{tA} = e^{\omega t} e_1^{t(A-\omega I)} - \omega \int_0^t e^{\omega s} e_1^{s(A-\omega I)} ds.$$

One can show that $e_1^{t(A-\omega I)}$ is still exponentially bounded, but because of the choice of $\omega > 0$ one can achieve $0 \in \rho(A - \omega I)$. Because of Theorem 3.2 we can find $\omega_3 > 0$ such that $0 \in \rho(A_n - \omega_3 I_n)$ for any $n \geq n_0$. Now one can construct approximation of e_1^{tA} in the following way

$$e^{\omega_3 t} W_n^e(k_n\tau_n) - \omega_3\tau_n \Sigma_{j=0}^{k_n} e^{\omega_3 j\tau_n} W_n^e(j\tau_n) \xrightarrow{\mathcal{PP}} e_1^{tA},\ t = k_n\tau_n, \tag{4.9}$$

where $W_n^e(k_n\tau_n)$ is constructed by operators $A_n - \omega_3 I_n$. Of course, in (4.9) we can use different quadrature formulas for approximation of integral $\int_0^t e^{\omega s} e_1^{s(A-\omega I)} ds$.

4.3. Crank-Nicolson scheme

In this subsection let us put $T_n(\tau_n) = (I_n + \frac{\tau_n}{2} A_n)(I_n - \frac{\tau_n}{2} A_n)^{-1}$, then $\breve{A}_n = A_n(I_n - \frac{\tau_n}{2} A_n)^{-1}$. The following definition is concerned with central difference scheme.

Definition 4.3. The discrete family of operators $\{W_n^{cd}(k\tau_n)\}, k = 0, 1, 2, \ldots,$ is called central difference discrete 1-times integrated semigroup if $W_n^{cd}(0) = 0, W_n^{cd}(\tau_n) = \tau_n(I_n - \frac{\tau_n}{2} A_n)^{-1}$, and

$$W_n^{cd}(k\tau_n) W_n^{cd}(\tau_n) = \tau_n \frac{W_n^{cd}((k+1)\tau_n) + W_n^{cd}(k\tau_n)}{2} - \frac{\tau_n}{2} W_n^{cd}(\tau_n).$$

Proposition 4.3. *If A_n^{-1} exist the discrete central difference 1-times integrated semigroup is given by the formulas*

$$\begin{aligned} W_n^{cd}(0) &= 0, \\ W_n^{cd}((k+1)\tau_n) &= W_n^{cd}(k\tau_n) \frac{I_n + \frac{\tau_n}{2} A_n}{I_n - \frac{\tau_n}{2} A_n} + W_n^{cd}(\tau_n),\ k = 1, 2, \ldots, \\ W_n^{cd}(k\tau_n) &= \left((\frac{I_n + \frac{\tau_n}{2} A_n}{I_n - \frac{\tau_n}{2} A_n})^k - I_n \right) A_n^{-1},\ k = 0, 1, 2, \ldots. \end{aligned} \tag{4.10}$$

Proof. From Definition 4.3 it follows that

$$W_n^{cd}((k+1)\tau_n) = W_n^{cd}(k\tau_n)\Big(2W_n^{cd}(\tau_n)/\tau_n - I_n\Big) + W_n^{cd}(\tau_n).$$

One gets $W_n^{cd}((k+1)\tau_n) = W^{cd}(k\tau_n)\Big(2(I_n - \frac{\tau_n}{2}A_n)^{-1} - I_n\Big) + W^{cd}(\tau_n)$. Therefore

$$W_n^{cd}((k+1)\tau_n) = W_n^{cd}(k\tau_n)\frac{I_n + \frac{\tau}{2}A_n}{I_n - \frac{\tau}{2}A_n} + W_n^{cd}(\tau_n).$$

Using the identity $\frac{I_n + \frac{\tau_n}{2}A_n}{I_n - \frac{\tau_n}{2}A_n} - I_n = \Big(I_n + \frac{\tau_n}{2}A_n - (I_n - \frac{\tau_n}{2}A_n)\Big)(I_n - \frac{\tau_n}{2}A_n)^{-1}$ we get the following.

Theorem 4.5. *Suppose that conditions (A) and (B'_{int}) of Theorem 3.3 hold and*

$$\tau_n\|A_n^2\|, \|A_n^{-1}\| \le C,\ n \in I\!N.$$

Then the discrete central difference once integrated semigroup $\int_0^t (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^{[s/\tau_n]} ds$ is exponentially stable, i.e.

$$\|\tau_n \Sigma_{j=1}^{k_n} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j\| \le M_1\, e^{\omega_2 \tau_n k_n},\ 0 \le \tau_n k_n \le T, \tag{4.11}$$

and it provides an approximation of the once integrated semigroup, i.e.

$$\tau_n \Sigma_{j=0}^{k_n} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j u_n^0 \xrightarrow{\mathcal{P}} e_1^{tA} u^0,$$

uniformly with respect to $t = k_n\tau_n \in [0,T]$ as $u_n^0 \xrightarrow{\mathcal{P}} u^0, A_n u_n^0 \xrightarrow{\mathcal{P}} Au^0, n \to \infty$, for any $u^0 \in D(A)$.

Proof. To prove stability of (4.10) in the form $\tau_n \Sigma_{j=1}^{k} (\frac{I_n + \frac{\tau_n}{2}A_n}{I_n - \frac{\tau_n}{2}A_n})^j = \tau_n \Sigma_{j=1}^{k} (I_n - \tau_n Q_n)^{-j}$, where $Q_n = A_n(I_n + \tau_n A_n/2)^{-1}$, we apply Theorem 4.2. Let us consider the difference $(Q_n - A_n)x_n = A_n(I_n + \tau_n A_n/2)^{-1}\tau_n A_n x_n/2$. The operators $(Q_n - A_n)$ are uniformly bounded if $\|\tau_n A_n^2\| \le constant$ and they commute with A_n. By Proposition 3.1 in [16] we find that the operators Q_n generate exponentially bounded once integrated semigroups and under condition $\|\tau_n A_n^2\| \le constant$ we obtain stability by Theorem 4.2. So one has

$$\tau_n \sum_{j=0}^{k_n - 1} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j (I_n - \tau_n A_n/2)^{-1}$$
$$= \tau_n \sum_{j=1}^{k_n} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j (I_n + \tau_n A_n/2)^{-1},$$

and the estimate $\|\frac{1}{I_n+\tau_n A_n/2}\| = \|\frac{I_n-\tau_n A_n/2}{I_n-\tau_n^2 A_n^2/4}\| \leq \|I_n - \tau_n A_n^2 A_n^{-1}/2\| \frac{1}{1-\tau_n c} < \infty$, since $\tau_n\|A_n^2\| \leq constant$. To prove it we apply Theorem 4.1 with $\breve{A}_n = A_n(I_n - \tau_n A_n/2)^{-1}$ and Remark 4.1 (b).

Theorem 4.6. *Suppose that conditions* (A) *and* (B_{int}''') *of Theorem* 3.5 *hold with* $\omega_1 = 0$ *and*

$$\sup_n \tau_n\|A_n\| < \mu < 2\sin\theta, \quad 0 \in \rho(A_n), n \in I\!N. \tag{4.12}$$

Then the discrete central difference once integrated semigroup $\int_0^t T_n(\tau_n)^{[s/\tau_n]}ds$ *is exponentially stable, i.e.* (4.7) *holds and it gives an approximation of the once integrated semigroup in the sense that*

$$\tau_n \sum_{k=0}^{k_n} (\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^j u_n^0 \xrightarrow{\mathcal{P}} e_1^{tA} u^0 \text{ as } n \to \infty,$$

uniformly with respect to $t = k_n\tau_n \in [0, T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0, A_n u_n^0 \xrightarrow{\mathcal{P}} Au^0, n \to \infty$, *for any* $u^0 \in D(A)$.

Proof. One can write

$$\begin{aligned}
&(e^{tA_n} - I_n)A_n^{-1} - ((\frac{I_n + \tau_n A_n/2}{I_n - \tau_n A_n/2})^m - I_n)A_n^{-1} \\
&= -\int_0^{\tau_n} \frac{d}{ds}\Big(e^{m(\tau_n - s)A_n}(\frac{I_n + sA_n/2}{I_n - sA_n/2})^m\Big) ds\, A_n^{-1} \\
&= \int_0^{\tau_n} \frac{ms^2}{4} e^{m(\tau_n - s)A_n}\Big(\frac{I_n + sA_n/2}{I_n - sA_n/2}\Big)^{m-1} \frac{A_n^3}{(I_n - sA_n/2)^2} ds A_n^{-1} \\
&= \frac{1}{2\pi i}\int_0^{\tau_n} \frac{ms^2}{4}\Big(\int_\Gamma e^{m(\tau_n - s)\lambda}\frac{(1 + s\lambda/2)^{m-1}}{(1 - s\lambda/2)^{m+1}}(\lambda I_n - A_n)^{-1} d\lambda\Big) ds\, A_n^2,
\end{aligned}$$

where the positively oriented contour Γ is composed of $\Gamma_1 = \{re^{\pm i(\theta+\pi/2)} : 0 \leq r \leq R_n\}$ and $\Gamma_2 = \{R_n e^{i\varphi} : \theta + \pi/2 \leq \varphi \leq -\theta + 3\pi/2)\}$. Since for $\lambda \in \Gamma, 0 \leq s \leq \tau_n$, $|1 - s\lambda/2| \geq 1$, we have $\frac{|1+s\lambda/2|^{m-1}}{|1-s\lambda/2|^{m+1}} \leq |1 + s\lambda/2|^{m-1}$. The rest of the proof is similar to that of Theorem 4.4.

Remark 4.7. To get in Theorem 4.6 the convergence for any $u^0 \in E$ one could assume density of domain or equicontinuity of family $\tau_n \sum_{k=0}^{k_n}(\frac{I_n+\tau_n A_n/2}{I_n-\tau_n A_n/2})^j u_n^0$.

Remark 4.8. (a) Note that the analogy of Theorem 4.6 for the case of analytic C_0-semigroups, namely Theorem 1.8, does not involve stability condition like (4.12). Here we follow the idea of [25]. The proof of Theorem 1.8 is based on the fact that if A generates a bounded analytic C_0-semigroup and A^{-1} exists, then this inverse A^{-1} also generates a bounded analytic C_0-semigroup. Unfortunately, for analytic integrated semigroups such a statement does not make sense. As was shown in [12] if one assumes that A and A^{-1} both generate bounded analytic integrated semigroups, then A generates in fact a bounded analytic C_0-semigroup.

(b) If A generates an exponentially bounded integrated semigroup e_1^{tA}, then by

$$e^{tA}R(\lambda,A)x = e^{\lambda t}R(\lambda,A)x - \lambda\int_0^t e^{\lambda(t-s)}e_1^{sA}x\,ds - e_1^{tA}x \tag{4.13}$$

we give a semigroup e^{tA} on $D(A)$. Some sufficient conditions are given in [6] to guarantee that $\{e^{tA}\}_{t\geq 0}$ is a semigroup on E of class $(1.A)$.

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