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METRIC REGULARITY OF PARAMETRIC GENERALIZED INEQUALITY SYSTEMS

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Abstract. This paper is mainly devoted to applications of modern tools of variational analysis and generalized differentiation to the metric regularity of parametric generalized inequality systems in infinite-dimensional spaces. The basic tools of our analysis involve the Mordukhovich normal coderivatives of set-valued mappings, the limiting subgradient estimate for the marginal functions, and the Ekeland variational principle. Using these tools, we establish new sufficient conditions for the metric regularity of parametric generalized inequality systems. Our results extend the corresponding results in [23] which established some pointbased sufficient conditions for the metric regularity in the Robinson's sense of implicit multifunctions in finite-dimensional setting.

1. Introduction

In this paper we will focus on the study of stability of solutions to the parametric *generalized inequality system*:

$$0 \in F(x, y), \tag{1.1}$$

where $F\colon X \times Y \rightrightarrows Z$ is a set-valued mapping between Banach spaces.

In particular, if $F(x, y) = f(x, y) + Q(x, y)$ where $f\colon X \times Y \to Z$ is a single-valued mapping and $Q\colon X \times Y \rightrightarrows Z$ is a set-valued mapping between Banach spaces, then (1.1) becomes

$$0 \in f(x, y) + Q(x, y), \tag{1.2}$$

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It is well known that model (1.2) provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibrium, etc.; see, e.g., [3, 14, 15, 21] and the references in therein for more information and discussions. When $Q(x, y) = N(y; \Omega)$ is the normal cone operator for a convex set Ω, (1.2) is reduced to the *parametric variational inequality*:

$$\text{Find } y \in \Omega \text{ such that } \langle f(x, y), z - y\rangle \geq 0 \ \ \forall z \in \Omega, \tag{1.3}$$

which is of particular interest for applications.

The solution map $G\colon Y \rightrightarrows X$ associated with (1.1) is defined by

$$G(y) = \{x \in X \ : \ 0 \in F(x, y)\}. \tag{1.4}$$

Some interesting properties of G was examined such as lower (upper) semicontinuity, pseudo-Lipschitzian property, upper Lipschitzian continuity, metric regularity, and generalized differentiability... The characterizations of necessary and sufficient conditions for the pseudo-Lipschitzian property of the solution map of (1.2) was given in [10, 12, 13]. Together with the Lipschitzian stability, the metric regularity of (1.4) was intensively investigated in *implicit* and *inverse multifunctions* (see, e.g., [2–9, 11, 14–18, 20–23] and the references therein). Recently, Ledyaev and Zhu [8], Ngai and Théra [18] established sufficient conditions for the metric regularity property of (1.4) in terms of the Fréchet coderivatives. Another set of sufficient conditions for the same property was given by Lee, Tam and Yen [9] in terms of the Mordukhovich normal coderivatives. More recently, some *pointbased* sufficient conditions for the metric regularity property of implicit multifunctions was first established by Yen and Yao [23] in finite-dimensional setting. Also in this paper [23], Yen and Yao suggest a need for generalization of their corresponding results in infinite-dimensional setting.

The main objective of this paper is to establish pointbased sufficient conditions for the (local) metric regularity of the solution map (1.4) in the sense introduced by Robinson [20] in infinite-dimensional setting. Our results extend the corresponding results in [23].

The rest of this paper is as follows. In Section 2, we recall some basic definitions and preliminaries from the variational analysis and generalized differentiation. In Section 3, we derive pointbased sufficient conditions for the (local) metric regularity of (1.4) in finite-dimensional setting.

2. Preliminaries

Throughout the paper we use standard notations of the variational analysis and generalized differentiation. We refer the reader to the monographs by Mordukhovich

[14, 15] for more details and discussions. Unless otherwise stated, all the spaces under consideration are Banach spaces whose norms are always denoted by $\|\cdot\|$. For any X we consider its dual space X^* equipped with the weak* topology w^* where $\langle\cdot,\cdot\rangle$ means the canonical pairing. As usual, B_X and $B^*_{X^*}$ stand for the closed unit balls of the Banach space X and its dual, respectively. The symbol A^* is the adjoint operator to a linear continuous operator A. The closed ball with center x and radius ρ is denoted by $B_\rho(x)$.

For a subset $\Omega \subset X$, $\mathrm{cl}\,\Omega$, $\mathrm{int}\,\Omega$, $\mathrm{co}\,\Omega$ and $\mathrm{cone}\,\Omega$ denote, respectively, the *closure*, the *interior*, the *convex hull* and the *conical hull* of Ω. The *weak* topology* in the dual space X^* is denoted by w^*. Given a subset $\Omega \subset X$ and a point $u \in X$, we denote the set of the *metric projections* of u on the closure of Ω by $\mathcal{M}(u,\Omega)$, that is

$$\mathcal{M}(u,\Omega) = \{x \in \mathrm{cl}\,\Omega \mid \|x-u\| = \mathrm{dist}\,(u,\Omega)\},$$

where $\mathrm{dist}\,(u,\Omega) := \inf_{z\in\Omega}\|z-u\|$ is the *distance* from u to Ω.

Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between a Banach space X and its topological dual X^*, we denote by

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big|\ \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in I\!N\Big\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* with respect to the norm topology of X and the weak* topology of X^*, where $I\!N := \{1,2,\ldots\}$.

Given $\Omega \subset X$ and $\varepsilon \geq 0$, define the collection of *ε-normals* to Ω at $\bar{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{x^* \in X^* \Big|\ \limsup_{x\xrightarrow{\Omega}\bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq \varepsilon\Big\}, \tag{2.1}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}(\bar{x};\Omega) := \widehat{N}_0(\bar{x};\Omega)$ in (2.1) is a cone called the *prenormal cone* or the *Fréchet normal cone* to Ω at $\bar{x}$.

The *Mordukhovich normal cone* $N(\bar{x};\Omega)$ is obtained from $\widehat{N}_\varepsilon(x;\Omega)$ by taking the *sequential* Painlevé-Kuratowski upper limit in the weak* topology of X^* as

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{\substack{x\xrightarrow{\Omega}\bar{x}\\ \varepsilon\downarrow 0}} \widehat{N}_\varepsilon(x;\Omega), \tag{2.2}$$

where one can put $\varepsilon = 0$ when Ω is closed around $\bar{x}$ and the space X is *Asplund*, i.e., a Banach space whose separable subspaces have separable duals. The subset $\Omega \subset X$ is said to be *(locally) closed* around $\bar{x}$ if there is a neighborhood U of $\bar{x}$ such that $\Omega \cap \mathrm{cl}U$ is closed.

Let $F\colon X \rightrightarrows Y$ be a set-valued mapping between Banach spaces with the graph

$$\text{gph}\, F := \big\{(x,y) \in X \times Y \big|\; y \in F(x)\big\}.$$

The inverse multifunction $F^{-1} : Y \rightrightarrows X$ is defined by

$$F^{-1}(y) = \{x \in X \mid y \in F(x)\}.$$

The *Mordukhovich normal coderivative* $D^*F(\bar{x},\bar{y})\colon Y^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{y}) \in \text{gph}F$ is defined by

(2.3) $$D^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\; (x^*,-y^*) \in N((\bar{x},\bar{y}); \text{gph}\, F)\big\}, \quad y^* \in Y^*.$$

The Fréchet coderivative at $(\bar{x},\bar{y}) \in \text{gph}\, F$ is defined by

(2.4) $$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in \widehat{N}((\bar{x},\bar{y}); \text{gph}\, F))\}, \quad y^* \in Y^*.$$

F is said to be *graphically regular* at $(\bar{x},\bar{y}) \in \text{gph}\, F$ if

$$D^*F(\bar{x},\bar{y})(y^*) = \widehat{D}^*F(\bar{x},\bar{y})(y^*) \text{ for all } y^* \in Y^*.$$

A single-valued mapping $f\colon X \to Y$ is said to be strictly differentiable at $\bar{x}$ if there is a linear continuous operator $\nabla f(\bar{x})\colon X \to Y$ such that for any $\gamma > 0$ there exists $\nu > 0$ satisfying

$$\|f(x) - f(u) - \nabla f(\bar{x})(x-u)\| \le \gamma\|x-u\| \quad \forall x, u \in \bar{x} + \nu B_X.$$

We known that for such mappings one has

$$D^*f(\bar{x})(y^*) = \widehat{D}^*f(\bar{x})(y^*) = \{(\nabla f(\bar{x}))^* y^*\} \quad \forall y^* \in Y^*,$$

i.e., the Mordukhovich normal coderivative (resp., Fréchet coderivative) is a generalization of the adjoint operator to the classical Jacobian/strict derivative. For details, we refer the reader to [14].

A set Ω is *sequentially normally compact* (SNC) at $\bar{x}$ if for any sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ one has

$$\big[x_k^* \xrightarrow{w^*} 0\big] \Longrightarrow \big[\|x_k^*\| \to 0\big] \text{ as } k \to \infty,$$

where ε_k can be omitted if X is Asplund and if Ω is locally closed around $\bar{x}$. A set-valued mapping $F\colon X \rightrightarrows Y$ is *SNC* at $(\bar{x},\bar{y}) \in \text{gph}\, F$ if its graph enjoys this property.

For an extended real-valued function $\varphi : X \to \bar{\mathbb{R}} := [-\infty, \infty]$, we define

$$\text{dom}\, \varphi = \{x \in X \mid |\varphi(x)| < \infty\}, \quad \text{epi}\, \varphi(x) = \{(x,\mu) \in X \times \mathbb{R} \mid \mu \ge \varphi(x)\},$$

and say that φ is *lower semicontinuous* at $\bar{x} \in X$ if $\liminf\limits_{x\to\bar{x}} \varphi(x) \ge \varphi(\bar{x})$. Here $\liminf\limits_{x\to\bar{x}}$ denotes the lower limit of scalar functions in the classical sense.

The *subdifferential* $\partial\varphi(\bar{x})$ and the *singular subdifferential* $\partial^{\infty}\varphi(\bar{x})$ of φ at $\bar{x} \in \mathrm{dom}\,\varphi$ are defined by

$$\partial\varphi(\bar{x}) := \{x^* \in X \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\varphi)\},$$

$$\partial^{\infty}\varphi(\bar{x}) := \{x^* \in X \mid (x^*, 0) \in N((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\varphi)\}.$$

The *presubdifferential* or *Fréchet subdifferential* of φ at $\bar{x} \in \mathrm{dom}\,\varphi$ is denoted by

$$\widehat{\partial}\varphi(\bar{x}) := \{x^* \in X \mid (x^*, -1) \in \widehat{N}((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\varphi)\}.$$

If $\bar{x} \notin \mathrm{dom}\varphi$ then one puts $\widehat{\partial}\varphi(\bar{x}) = \partial\varphi(\bar{x}) = \partial^{\infty}\varphi(\bar{x}) = \emptyset$.

A multifunction $F : X \rightrightarrows Y$ is said to be *lower semicontinuous* at $x \in \mathrm{dom}\,F$ if for any open set $V \subset Y$ satisfying $V \cap F(x) \neq \emptyset$, there exists a neighborhood U of x such that $V \cap F(u) \neq \emptyset$ for all $u \in U$. One says that F is *inner semicompact* at $\bar{x} \in X$ if for any sequence $x_k \to \bar{x}$, there is a sequence $y_k \in F(x_k)$, $k = 1, 2, \ldots$ such that $\{y_k\}$ contains a convergent subsequence in the norm topology of Y. F is said to be inner semicompact around $\bar{x} \in X$ if it is inner semicompact at every point in a neighborhood of $\bar{x}$. It is clear that if F is *lower semicontinuous* around $\bar{x}$, i.e., F is lower semicontinuous at every point in a neighborhood of $\bar{x}$, then F is inner semicompact around $\bar{x}$.

We now consider the parametric minimization problem

$$\min\{\varphi(x, y) \mid y \in \Phi(x)\} \tag{2.5}$$

depending on the parameter x and the corresponding *marginal function*

$$m(x) := \inf\{\varphi(x, y) \,:\, y \in \Phi(x)\}, \tag{2.6}$$

where $\varphi : X \times Y \to \bar{\mathbb{R}}$ is an extended real-valued function and $\Phi : X \rightrightarrows Y$ is a multifunction between Banach spaces. Let

$$M(x) := \{y \in \Phi(x) \mid \varphi(x, y) = m(x)\} \tag{2.7}$$

be the parametric *solution set* of (2.5).

Theorem 2.1. ([14, Theorem 3.38]). *Let $\Phi : X \rightrightarrows Y$ be a closed-graph multifunction between Asplund spaces X and Y, let $\varphi : X \times Y \to \bar{\mathbb{R}}$ be lower semicontinuous on $\mathrm{gph}\,\Phi$. Suppose that for any $\bar{y} \in M(\bar{x})$, φ is locally Lipschitzian at $(\bar{x}, \bar{y})$, and the multifunction M in (2.7) is inner semicompact at $(\bar{x}, \bar{y})$. Then one has the inclusion*

$$\partial m(\bar{x}) \subset \bigcup \left\{x^* + D^*\Phi(\bar{x}, \bar{y})(y^*) \,:\, (x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y}),\ \bar{y} \in M(\bar{x})\right\}. \tag{2.8}$$

Given a single-valued mapping $f\colon X \to Y$ between Banach spaces. Let $\bar{x} \in X$. f is said to be *locally Lipschitzian* around $\bar{x}$ if there exist a neighborhood U of $\bar{x}$ and a number $\ell \geq 0$ such that

$$\|f(x_1) - f(x_2)\| \leq \ell \|x_1 - x_2\| \text{ for all } x_1, x_2 \in U.$$

Theorem 2.2. ([14, Theorem 3.36]). *Let X be an Asplund space, let $\varphi_i : X \to \bar{\mathbb{R}}$, $i = 1, 2$, be lower semicontinuous at $\bar{x}$, and one of these functions be locally Lipschitzian at $\bar{x}$. Then one has the inclusion*

$$\partial(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x}).$$

3. Main Results

In this section we establish some sufficient conditions for the metric regularity of (1.4). Let us recall the definitions of the (local) metric regularity and the pseudo-Lipschitzian property of multifunctions, which was introduced by Robinson [20] and Aubin [1], respectively.

Definition 3.1. ([14]). Let $\Phi\colon X \rightrightarrows Y$ be a multifunction between Asplund spaces. Let $(\bar{x}, \bar{y}) \in \operatorname{gph} \Phi$.

(a) Φ is said to be local-metrically regular around $(\bar{x}, \bar{y})$ with modulus $c > 0$ if there exist a neighborhood U of $\bar{x}$, a neighborhood V of $\bar{y}$, and a number $\mu > 0$ such that

$$\operatorname{dist}(x, \Phi^{-1}(y)) \leq c \operatorname{dist}(y, \Phi(x))$$

for any $x \in U$ and $y \in V$ satisfying $\operatorname{dist}(y, \Phi(x)) \leq \mu$.

(b) Φ is said to be pseudo-Lipschitzian around $(\bar{x}, \bar{y})$ with modulus $\ell > 0$ if there exist a neighborhood U of $\bar{x}$ and a neighborhood V of $\bar{y}$ such that

$$\Phi(x_1) \cap V \subset \Phi(x_2) + \ell\|x_1 - x_2\| B_X \ \ \forall x_1, x_2 \in U.$$

For the variational system defined as in (1.1) and (1.4), let $\omega_0 := (x_0, y_0, 0) \in \operatorname{gph} F$. G is said to be local-metrically regular around ω_0 with modulus $c > 0$ in the Robinson's sense if there exist a neighborhood U of x_0, a neighborhood V of y_0 and a number $\mu > 0$ such that

$$\operatorname{dist}(x, G(y)) \leq c \operatorname{dist}(0, F(x, y)) \tag{3.1}$$

for any $x \in U$ and $y \in V$ satisfying $\operatorname{dist}(0, F(x, y)) \leq \mu$.

We recall that a Banach space Z is *weakly compactly generated* (WCG), provided that there is a weakly compact set P such that $Z = \operatorname{cl}(\operatorname{span} P)$. This class of spaces is sufficiently large including, in particular, all reflexive space as well all separable Banach spaces; see the book by Phelps [19] for more information and references.

Proposition 3.2. ([14, Theorem 3.60]). *Let Z be a WCG Asplund space, and let $\Omega \subset Z$ be its closed subset that is SNC at $\bar{z}$. Then the multifunction $N(\cdot;\Omega)$ has closed graph around $\bar{z}$, i.e., there exists $\delta > 0$ such that the set*

$$(\operatorname{gph} N(\cdot;\Omega)) \cap ((\bar{z} + \delta B) \times Z^*)$$

is closed in the norm$\times$weak* topology of $Z \times Z^*$. In particular, for any sequences $z_k \to \bar{z}$ and $z_k^* \xrightarrow{w^*} z^*$ with $z_k^* \in N(z_k;\Omega)$, $k = 1, 2, \ldots$, one has $z^* \in N(\bar{z};\Omega)$.

Remark 3.3. Let Z be a WCG Asplund space and let Ω be a subset of Z. If Ω is closed around $\bar{z}$ and SNC at this point, then for any sequences

$$z_k^* \in N(z_k;\Omega) \text{ with } z_k \to \bar{z} \text{ and } z_k^* \xrightarrow{w^*} 0 \text{ as } k \to \infty$$

one has $z_k^* \to 0$ in the norm topology of Z^*.

We now state and prove our main result.

Theorem 3.4. *Let X, Y, Z be WCG Asplund spaces, a multifunction $F : X \times Y \rightrightarrows Z$ and a multifunction $G : Y \rightrightarrows X$ defined as in (1.1) and (1.4). Let $\omega_0 := (x_0, y_0, 0) \in \operatorname{gph} F$ and let $F_y(\cdot) := F(\cdot, y)$. Suppose that $\operatorname{gph} F$ is locally closed at ω_0 and SNC at this point, and there are a neighborhood U_0 of x_0, a neighborhood V_0 of y_0 such that for any $y \in V_0$ and for any $x \in U_0$, the multifunction $\mathcal{M}(0, F_y(\cdot))$ is inner semicompact at x and the following pointbased criteria holds:*

$$\forall (y^*, z^*) \in Y^* \times Z^*,\ (0, y^*) \in D^*F(\omega_0)(z^*) \Longrightarrow y^* = z^* = 0.$$

Then G is local-metrically regular around ω_0 with modulus $c > 0$, i.e., there exist a neighborhoods U of x_0, a neighborhood V of y_0, a number $c > 0$ and a number $\mu > 0$ such that

$$\operatorname{dist}(x, G(y)) \le c \operatorname{dist}(0, F(x, y)) \tag{3.2}$$

for any $x \in U$ and $y \in V$ satisfying $\operatorname{dist}(0, F(x, y)) \le \mu$. Moreover, for any $x^* \in X^*$,

$$D^*G(y_0, x_0)(x^*) = \bigcup_{z^* \in Z^*} \{y^* \ :\ (-x^*, y^*) \in D^*F(\omega_0)(z^*)\} \tag{3.3}$$

provided that F is graphically regular at ω_0.

For proving Theorem 3.4, we need the following auxiliary results.

Proposition 3.5. *Let X, Y, Z be WCG Asplund spaces and a multifunction $F : X \times Y \rightrightarrows Z$. Let $\bar{\omega} := (\bar{x}, \bar{y}, 0) \in \operatorname{gph} F$ and let $F_y(\cdot) := F(\cdot, y)$. Suppose that $\operatorname{gph} F$ is locally closed around $\bar{\omega}$ and SNC at this point. Consider the following statements:*

(i) *For any* $(y^*, z^*) \in Y^* \times Z^*$, $(0, y^*) \in D^*F(\bar\omega)(z^*) \Longrightarrow y^* = z^* = 0$;

(ii) *There exist a constant* $c > 0$, *a neighborhood* U *of* $\bar x$, *a neighborhood* V *of* $\bar y$ *and a neighborhood* W *of* 0 *such that for any point* $\omega = (x, y, z) \in \operatorname{gph} F \cap (U \times V \times W)$, *it holds*

$$\|z^*\| \le c\|x^*\| \quad \forall z^* \in Z^*, \forall x^* \in D^*F_y(x, z)(z^*); \tag{3.4}$$

(iii) *There exist a number* $\sigma > 0$, *a neighborhood* U *of* $\bar x$ *and a neighborhood* V *of* $\bar y$ *such that for any* $(x, y) \in U \times V$ *with* $0 \notin F(x, y)$

$$\sigma \le \inf\{\|x^*\| \ : \ x^* \in D^*F_y(x, z)(z^*), \ z \in \mathcal{M}(0, F_y(x)), \ \|z^*\| = 1\}. \tag{3.5}$$

Then (i)$\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. Obviously, (ii) $\Longrightarrow$ (iii). It remains to prove that (i)$\Longrightarrow$ (ii). Suppose that (i) holds. First we claim that there exist a neighborhood U of $\bar x$, a neighborhood V of $\bar y$ and a neighborhood W of 0 such that for any point $\omega = (x, y, z) \in \operatorname{gph} F \cap (U \times V \times W)$ and for any $(y^*, z^*) \in Y^* \times Z^*$ satisfying

$$(0, y^*) \in D^*F(\omega)(z^*) \Longrightarrow y^* = z^* = 0. \tag{3.6}$$

Indeed, if our claim is false, then there exist sequences $\omega_k = (x_k, y_k, z_k) \in \operatorname{gph} F$ and $(y_k^*, z_k^*) \in Y^* \times Z^* \setminus \{(0, 0)\}$ such that for every $k = 1, 2, \ldots$

$$(0, y_k^*) \in D^*F(\omega_k)(z_k^*) \ \text{ and } \ \omega_k \to \bar\omega \ \text{ as } \ k \to \infty. \tag{3.7}$$

Without loss of generality we can assume that $\|z_k^*\| = 1$ for every $k \in I\!N$. Consider the following two cases:

Case 1. $\{y_k^*\}$ is bounded. Since Y is an Asplund space, the unit ball of the dual space Y^* is sequentially weak* compact. Taking into account the boundedness of $\{(y_k^*, z_k^*)\}$, one may assume that $(y_k^*, z_k^*) \xrightarrow{w^*} (y^*, z^*) \in Y^* \times Z^*$. Clearly,

$$(0, y_k^*, -z_k^*) \in N(\omega_k; \operatorname{gph} F).$$

It follows from Proposition 3.2 and Remark 3.3 that $(y_k^*, z_k^*) \to (y^*, z^*)$ as $k \to \infty$ in the norm topology of $Y^* \times Z^*$ and $(0, y^*, -z^*) \in N(\bar\omega; \operatorname{gph} F)$. Then $\|z^*\| \ne 0$ and $(0, y^*) \in D^*F(\bar\omega)(z^*)$, contrary to (i).

Case 2. $\{y_k^*\}$ is not bounded. Then there is a subsequence $\{y_{k_j}^*\}$ of $\{y_k^*\}$ such that $\|y_{k_j}^*\| \to \infty$ as $j \to \infty$. Without loss of generality, we may assume that

$$\left(\frac{y_{k_j}^*}{\|y_{k_j}^*\|}, \frac{z_{k_j}^*}{\|y_{k_j}^*\|}\right) \xrightarrow{w^*} (y^*, z^*) \in Y^* \times Z^*.$$

Analysis similar to that as in Case 1 shows that $\|y^*\| \ne 0$ and $(0, y^*) \in D^*F(\bar\omega)(z^*)$ which contradicts (i). Therefore, our claim is proved.

Next we show that for any $\omega = (x, y, z) \in \operatorname{gph} F \cap (U \times V \times W)$, the inclusion

$$D^*F_y(x, z)(z^*) \subset \Big\{x^* \in X^* : \exists y^* \in Y^* \text{such that } (x^*, y^*) \in D^*F(\omega)(z^*)\Big\} \tag{3.8}$$

holds for every $z^* \in Z^*$. Indeed, let $\Phi : X \to X \times Y$ be a strictly differentiable map by setting $\Phi(x) = (x, y)$. Clearly, $F_y(x) = (F \circ \Phi)(x)$ for all $x \in X$. Let us examine the following constraint qualification:

$$D^*F(\Phi(x), z)(0) \cap \ker (\nabla\Phi(x))^* = \{0\}. \tag{3.9}$$

Let $(x^*, y^*) \in D^*F(\Phi(x), z)(0) \cap \ker (\nabla\Phi(x))^*$. It is easy to check that

$$(\nabla\Phi(x))^*(x^*, y^*) = x^* \text{ for every } (x^*, y^*) \in X^* \times Y^*. \tag{3.10}$$

Hence $x^* = 0$ and $(0, y^*) \in D^*F(\omega)(0)$, where $\omega = (x, y, z) = (\Phi(x), z)$. From (3.6) it follows that $y^* = 0$. Therefore, (3.9) holds. Applying Corollary 3.16 in [14], we have

$$D^*(F \circ \Phi)(x, z)(z^*) \subset (\nabla\Phi(x))^* D^*F(\Phi(x), z)(z^*) \quad \forall z^* \in Z^*.$$

Hence

$$D^*F_y(x, z)(z^*) \subset (\nabla\Phi(x))^* D^*F(\omega)(z^*) \quad \forall z^* \in Z^*. \tag{3.11}$$

By (3.10), $x^* \in (\nabla\Phi(x))^* D^*F(\omega)(z^*)$ if and only if there is some $y^* \in Y^*$ satisfying $(x^*, y^*) \in D^*F(\omega)(z^*)$. Combining this with (3.11), we get (3.8).

It remains to show that (ii) holds. On the contrary, suppose that the conclusion of (ii) is not true. Then we can find sequences $\omega_k = (x_k, y_k, z_k) \in \operatorname{gph} F$ and $(x_k^*, z_k^*) \in X^* \times Z^*$ such that $\omega_k \to \bar{\omega}$, $x_k^* \in D^*F_{y_k}(x_k, z_k)(z_k^*)$, and $\|z_k^*\| > k\|x_k^*\|$ for all $k \in I\!N$. There is no loss of generality in assuming that $\|z_k^*\| = 1$ for all $k \in I\!N$. Then

$$x_k^* \in D^*F_{y_k}(x_k, z_k)(z_k^*), \ \ \|z_k^*\| = 1, \ \text{ and } \|x_k^*\| \le \frac{1}{k} \ \ \forall k \in I\!N. \tag{3.12}$$

It follows from (3.8) that there exists a sequence $\{y_k^*\} \subset Y^*$ such that $(x_k^*, y_k^*) \in D^*F(\omega_k)(z_k^*)$, where $\omega_k := (x_k, y_k, z_k)$. Hence

$$(x_k^*, y_k^*, -z_k^*) \in N(\omega_k; \operatorname{gph} F).$$

If the sequence $\{y_k^*\}$ is bounded then, by the same analysis as in Case 1, it follows that there exists a subsequence $\{(x_{k_j}^*, y_{k_j}^*, z_{k_j}^*)\}$ of $\{(x_k^*, y_k^*, z_k^*)\}$ converges to $(0, y^*, z^*)$ in the norm topology of $X^* \times Y^* \times Z^*$ such that $\|z^*\| \neq 0$ and $(0, y^*, -z^*) \in N(\bar{\omega}; \operatorname{gph} F)$. Hence $\|z^*\| \neq 0$ and $(0, y^*) \in D^*F(\bar{\omega})(z^*)$ which contradicts (i). If the sequence $\{y_k^*\}$ is not bounded then, by the same method as in Case 2, it follows that there exists a subsequence $\left\{\left(\frac{x^*_{k_j}}{\|y^*_{k_j}\|}, \frac{y^*_{k_j}}{\|y^*_{k_j}\|}, \frac{z^*_{k_j}}{\|y^*_{k_j}\|}\right)\right\}$ of $\left\{\left(\frac{x^*_k}{\|y^*_k\|}, \frac{y^*_k}{\|y^*_k\|}, \frac{z^*_k}{\|y^*_k\|}\right)\right\}$ converges to $(0, y^*, z^*)$ in the norm topology of $X^* \times Y^* \times Z^*$ such that $\|y^*\| \neq 0$ and $(0, y^*) \in D^*F(\bar{\omega})(z^*)$ which is a contradiction to (i).

From what has already been proved, it follows that there must exist a constant $c > 0$, a neighborhood U of $\bar{x}$, a neighborhood V of $\bar{y}$ and a neighborhood W of 0 such that for any point $\omega = (x, y, z) \in \text{gph}\, F \cap (U \times V \times W)$, (3.4) is fulfilled. The proof is complete. ∎

Proposition 3.6. *Let X, Z be Asplund spaces, Y a topological space, $F : X \times Y \rightrightarrows Z$ and $G : Y \rightrightarrows X$ defined as in (1.1) and (1.4), respectively. Let $\omega_0 := (x_0, y_0, 0) \in \operatorname{gph} F$ and let $F_y(\cdot) := F(\cdot, y)$. Suppose that $\operatorname{gph} F$ is locally closed at ω_0, and there exist a neighborhood U_0 of x_0, a neighborhood V_0 of y_0 such that the following conditions hold:*

(i) *for any $y \in V_0$, for any $x \in U_0$, the multifunction $\mathcal{M}(0, F_y(\cdot))$ is inner semicompact at x;*

(ii) *there exists $\sigma > 0$ such that for any $(x, y) \in U_0 \times V_0$ with $0 \notin F(x, y)$*

$$\sigma \leq \inf\{\|x^*\| \ : \ x^* \in D^*F_y(x, z)(z^*), \ \ z \in \mathcal{M}(0, F_y(x)), \ \ \|z^*\| = 1\}.$$

Then G is local-metrically regular around ω_0 with modulus $\frac{1}{\sigma}$.

Proof. Let $\omega_0 \in \operatorname{gph} F$. By the assumption, there are a neighborhood U_0 of x_0 and a neighborhood V_0 of y_0 such that (i)–(ii) are satisfied. Choose a number $\mu > 0$ and a number $\rho > 0$ such that

$$\mu < \sigma\rho \ \text{ and } \ B_\rho(x_0) \subset U_0. \tag{3.13}$$

Now we just examine the case $(x, y) \in \operatorname{dom} F \cap (U_0 \times V_0)$ satisfying

$$\operatorname{dist}(0, F(x, y)) \leq \mu. \tag{3.14}$$

For convenience we will ignore $(x, y) \in \operatorname{dom} F$. Put $U := U_0$ and $V := V_0$. We want to show that U, V together with constants μ and σ satisfy the conclusion of the theorem. Fix any $x \in U$ and $y \in V$ and assume that (3.14) is fulfilled. Set $\alpha := \operatorname{dist}(0, F(x, y))$. By (3.13) and (3.14), $\alpha < \sigma\rho$. It remains to show that

$$\operatorname{dist}(x, G(y)) \leq \frac{\alpha}{\sigma}. \tag{3.15}$$

Obviously, (3.15) holds if $\alpha = 0$. Suppose that $\alpha > 0$. Consider the function

$$\upsilon_y(u) := \operatorname{dist}(0, F_y(u)), \quad u \in U.$$

We claim that $v_y(\cdot)$ is lower semicontinuous on U. Indeed, if there is a sequence $x_k \to u$ and $\varepsilon > 0$ such that $v_y(x_k) \le v_y(u) - \varepsilon$ for every $k \in I\!N$, then, by (i), there exist a subsequence $\{k_j\} \subset \{k\}$ and a sequence $z_{k_j} \in \mathcal{M}(0, F_y(x_{k_j}))$ such that z_{k_j} converges to some $z \in Z$ in the norm topology of Z. As $z_{k_j} \in F_y(x_{k_j})$ for every $j \in I\!N$, the closedness of gph F implies $z \in F_y(u)$. Hence, from the relation

$$\|z_{k_j}\| = v_y(x_{k_j}) \le v_y(u) - \varepsilon \quad \forall j \in I\!N,$$

it follows that $v_y(u) \le \|z\| \le v_y(u) - \varepsilon$, which is impossible and our claim is proved. Thus, $v(\cdot)$ is lower semicontinuous on $B_\rho(x_0)$. Fixing any $\delta \in (\frac{\alpha}{\rho}, \sigma)$, we have $v_y(x) = \alpha < \alpha\frac{\sigma}{\delta}$. Putting $t := \frac{\delta}{\alpha}\, v_y(x)$, we see that

$$v_y(x) = t\frac{\alpha}{\delta}, \quad \text{and } t \in (0, \sigma).$$

Clearly, $v_y(x) \le \inf\limits_{u \in B_\rho(x_0)} v_y(u) + t\frac{\alpha}{\delta}$. From the Ekeland variational principle [14, Theorem 2.26], it follows that there is $\bar{x} \in B_\rho(x_0)$ such that

$$v_y(\bar{x}) \le v_y(x), \quad \|\bar{x} - x\| \le \frac{\alpha}{\delta} \tag{3.16}$$

and

$$v_y(\bar{x}) \le v_y(u) + t\|u - \bar{x}\| \quad \forall u \in B_\rho(x_0). \tag{3.17}$$

We now claim that

$$0 \in F_y(\bar{x}).$$

Conversely, suppose that $0 \notin F_y(\bar{x})$. It follows from (3.17) that $\bar{x}$ is a local minimum of the function

$$\psi(u) := v_y(u) + \chi(u), \;\; u \in B_\rho(x_0),$$

where $\chi(u) := t\|u - \bar{x}\|$ is a local Lipschitzian function. From the nonsmooth version of Fermat's rule [14, Proposition 1.114], it follows that

$$0 \in \partial\psi(\bar{x}).$$

By Theorem 2.2, we have

$$0 \in \partial v_y(\bar{x}) + tB_{X^*}. \tag{3.18}$$

Let us now compute $\partial v_y(\bar{x})$. Define

$$\begin{aligned} &\Phi(u) := F_y(u), \quad \varphi(u, z) := \|z\|, \\ &m(u) := v_y(u) = \inf\{\varphi(u, z) \,:\, z \in \Phi(u)\}. \end{aligned}$$

Take arbitrary $\bar{z} \in M(\bar{x}) := \mathcal{M}(0, F_y(\bar{x}))$. Obviously, φ is locally Lipschitzian at $(\bar{x}, \bar{z})$. Define $\theta(z) := \|z\|$. It is easy to check that

$$\partial\varphi(\bar{x}, \bar{z}) = \{0\} \times \partial\theta(\bar{z}).$$

Applying Theorem 2.1, we obtain

$$\partial v_y(\bar{x}) \subset \bigcup \left[D^* F_y(\bar{x}, \bar{z})(z^*) \mid z^* \in \partial\theta(\bar{z}),\ \bar{z} \in \mathcal{M}(0, F_y(\bar{x}))\right]. \tag{3.19}$$

The condition $0 \notin F_y(\bar{x})$ implies $\bar{z} \neq 0$ for every $\bar{z} \in \mathcal{M}(0, F_y(\bar{x}))$. It follows that

$$\partial\theta(\bar{z}) = \{z^* \in Z^* \mid \|z^*\| = 1,\ \langle z^*, \bar{z}\rangle = \|\bar{z}\|\}. \tag{3.20}$$

By (3.18)–(3.20), there exist vectors $\bar{z} \in \mathcal{M}(0, F_y(\bar{x}))$, $z^* \in Z^*$ with $\|z^*\| = 1$, and $x^* \in D^* F_y(\bar{x}, \bar{z})(z^*)$ such that $\|x^*\| \le t$. Besides, by (ii), we can assert that $\sigma \le \|x^*\|$. Since $t \in (0, \sigma)$, this contradicts the inequality $\|x^*\| \le t$. We have thus shown that $0 \in F_y(\bar{x})$, i.e., $\bar{x} \in G(y)$. Hence, by (3.16),

$$\operatorname{dist}(x, G(y)) \le \|x - \bar{x}\| \le \frac{\alpha}{\delta}.$$

Letting $\delta \to \sigma$ we obtain $\operatorname{dist}(x, G(y)) \le \dfrac{\alpha}{\sigma}$. The proof is complete. ■

Remark 3.7. The condition (i) in Proposition 3.6 may be dropped if X, Y, Z are finite-dimensional spaces. The corresponding results for the metric regularity of (1.4) in [9, Theorem 3.2] always require the lower continuity of $F(x_0, \cdot)$ at y_0 and $F(\cdot)$ is lower semicontinuous at (x_0, y_0). Obviously, (A1) in [9, Theorem 3.1] implies (ii). Hence, Proposition 3.6 extends Theorem 3.1 in [9]. The condition (ii) is similar to the condition (iv') in [8, Theorem 3.6], but Ledyaev and Zhu [8] assumed that X and Z are Banach spaces with Fréchet-smooth Lipschitzian bump functions, $F(x_0, \cdot)$ is lower semicontinuous at y_0 and for any fixed $y \in V_0$, $F(\cdot, y)$ is upper semicontinuous.

Proof of Theorem 3.4. Obviously, (3.2) immediately follows from Propositions 3.5 and 3.6. Now let us examine the formula (3.3). We first observe that the graph of the mapping G under consideration can be represented as follows

$$\operatorname{gph} G = \{(y, x) \in X \times Y \mid g(x, y) \in \Theta \ \text{ with } \ \Theta := \operatorname{gph} F\},$$

where $g(x, y) := (x, y, 0)$. Obviously, g is a strictly differentiable function and

$$(\nabla g(x_0, y_0))^*(x^*, y^*, z^*) = (x^*, y^*) \ \ \forall (x^*, y^*, z^*) \in X^* \times Y^* \times Z^*. \tag{3.21}$$

We have

(3.22) $$N(\omega_0;\Theta)\cap\ker(\nabla g(x_0,y_0))^*=\{0\}.$$

Indeed, let $(x^*,y^*,z^*)\in N(\omega_0;\Theta)\cap\ker(\nabla g(x_0,y_0))^*$. Then, by (3.21), $x^*=y^*=0$ and $(0,0)\in D^*F(\omega_0)(z^*)$. It follows from (i) that $z^*=0$ and (3.22) is fulfilled. Applying Corollary 3.42 in [14], we have

$$N(\omega_0;g^{-1}(\Theta))\subset(\nabla g(\omega_0))^*N(\omega_0;\Theta).$$

From the graphical regularity of F at ω_0 and Corollary 1.15 in [14], it follows that

(3.23) $$N(\omega_0;g^{-1}(\Theta))=(\nabla g(\omega_0))^*N(\omega_0;\Theta).$$

For each $x^*\in X^*$, let $y^*\in D^*G(y_0,x_0)(x^*)$. Then $(y^*,-x^*)\in N((y_0,x_0);\operatorname{gph}G)$. It is easy to check that

(3.24) $$(y^*,-x^*)\in N((y_0,x_0);\operatorname{gph}G)\Longleftrightarrow(-x^*,y^*)\in N((x_0,y_0);g^{-1}(\Theta)).$$

From (3.21) and (3.23) it follows that

(3.25)

$$(-x^*,y^*)\in N((x_0,y_0);g^{-1}(\Theta))\Longleftrightarrow\exists z^*\in Z^*,\ (-x^*,y^*)\in D^*F(\omega_0)(z^*).$$

Thus, (3.3) immediately follows from (3.24) and (3.25). The proof is complete. ■

As the SNS property of $F(\cdot)$ and the inner semicompactness of $\mathcal{M}(0,F_y(\cdot))$ automatically holds in finite-dimensional setting, the following result is immediate from Theorem 3.4.

Corollary 3.8. ([23]). *Let X,Y,Z be finite-dimensional spaces, a multifunction $F:X\times Y\rightrightarrows Z$ and a multifunction $G:Y\rightrightarrows X$ defined as in (1.1) and (1.4). Let $\omega_0:=(x_0,y_0,0)\in\operatorname{gph}F$. Suppose that $\operatorname{gph}F$ is closed around ω_0, and the following pointbased criteria holds:*

$$\forall(y^*,z^*)\in Y^*\times Z^*,\ (0,y^*)\in D^*F(\omega_0)(z^*)\Longrightarrow y^*=z^*=0.$$

Then G is locally metric regular around ω_0 in the Robinson's sense.

The following corollary extends Theorem 3.3 in [9].

Corollary 3.9. *Under the assumption of Proposition 3.6 and suppose that $F:X\times Y\rightrightarrows Z$ is partially pseudo-Lipschitzian in y with rank ℓ around $\omega_0:=(x_0,y_0,0)$, i.e., there exist a neighborhood U_1 of x_0, a neighborhood V_1 of y_0 and a neighborhood W_1 of 0 such that, for any $(x,y)\in U_1\times V_1$ and $(x,y')\in U_1\times V_1$,*

$$F(x,y')\cap W_1\subset F(x,y)+\ell\|y'-y\|B_Y.$$

Then G is pseudo-Lipschitzian with rank $\dfrac{\ell}{\sigma}$ around (y_0, x_0), i.e., there exist a neighborhood V of y_0 and a neighborhood U of x_0 such that

$$G(y') \cap U \subset G(y) + \frac{\ell}{\sigma}\|y' - y\| B_X \ \ \forall y,\, y' \in V.$$

Proof. Since F is partially pseudo-Lipschitzian in y with rank ℓ around $\omega_0 := (x_0, y_0, 0)$, it follows that there exist a neighborhood U_1 of x_0, a neighborhood V_1 of y_0 and a neighborhood W_1 of 0 such that, for any $(x, y),\ (x, y') \in U_1 \times V_1$,

$$F(x, y') \cap W_1 \subset F(x, y) + \ell\|y' - y\| B_Y. \tag{3.26}$$

Choose $\mu > 0$ such that

$$B_\mu(0) \subset W_1.$$

It follows from Proposition 3.6 that there exist a neighborhood U_2 of x_0, a neighborhood V_2 of y_0 and a number $\sigma > 0$ such that

$$\operatorname{dist}(x, G(y)) \le \frac{1}{\sigma} \operatorname{dist}(0, F(x, y)) \tag{3.27}$$

for any $x \in U_2$ and $y \in V_2$ satisfying $\operatorname{dist}(0, F(x, y)) \le \mu$ for some $\mu > 0$. Let $U := U_1 \cap U_2$ and $V := V_1 \cap V_2$. Then

$$G(y') \cap U \subset G(y) + \frac{\ell}{\sigma}\|y' - y\| B_X \ \ \forall y,\, y' \in V. \tag{3.28}$$

Indeed, take arbitrary $x \in G(y') \cap U$. From (3.26) and (3.27), it follows that

$$\operatorname{dist}(x, G(y)) \le \frac{1}{\sigma} \operatorname{dist}(0, F(x, y)) \le \frac{\ell}{\sigma}\|y - y'\| \ \ \forall y,\, y' \in V.$$

Hence, $x \in G(y) + \dfrac{\ell}{\sigma}\|y' - y\| B_X \ \ \forall y,\, y' \in V$. So, (3.28) follows. The proof is complete. ∎

Corollary 3.10. *Let X, Y be Asplund spaces and $\Phi\colon X \rightrightarrows Y$ a closed-graphical multifunction. Let $(x_0, y_0) \in \operatorname{gph}\Phi$. Suppose that there exist a neighborhood U_0 of x_0, a neighborhood V_0 of y_0 and a number $\sigma > 0$ such that for any $x \in U_0$, Φ is inner semicompact at x, and for any $(x, y) \in U_0 \times V_0$ with $y \notin \Phi(x)$,*

$$\sigma \le \inf\{\|x^*\| \ :\ x^* \in D^*\Phi(x, y')(y^*),\ \ \|y^*\| = 1,\ \ y' \in \mathcal{M}(y, \Phi(x))\}.$$

Then

(a) *(Open Covering) there exists a neighborhood U of x_0 such that, for any $B_\rho(x_0) \subset U$,*

$$\operatorname{int} B_{\sigma\rho}(y_0) \subset \Phi(B_\rho(x_0));$$

(b) *(Metric Regularity) there exist a neighborhood U of x_0, a neighborhood V of y_0 and $\mu > 0$ such that*

$$\operatorname{dist}(x, \Phi^{-1}(y)) \leq \frac{1}{\sigma} \operatorname{dist}(y, \Phi(x)),$$

for any $x \in U$ and for any $y \in V$ satisfying $\operatorname{dist}(y, \Phi(x)) \leq \mu$*;*

(c) (*Aubin property*) Φ^{-1} *is pseudo-Lipschitzian at* (y_0, x_0).

Proof. (This proof is based on ideas of [8]) Let $Z := Y$, $F(x, y) := \Phi(x) - y$ and $G(y) := \{x \in X \ : \ 0 \in F(x, y)\}, \ (x, y) \in X \times Y$. Let $(x_0, y_0) \in \operatorname{gph} \Phi$. It is easy to check that the assumptions of Proposition 3.6 hold for F at $\omega_0 := (x_0, y_0, 0)$. Obviously, $G(y) = \Phi^{-1}(x)$ and $\operatorname{dist}(0, F(x, y)) = \operatorname{dist}(y, \Phi(x))$. From Proposition 3.6 it follows that there exist a neighborhood U of x_0, a neighborhood V of y_0, a number $\sigma > 0$ and a number $\mu > 0$ such that

$$\operatorname{dist}(x, \Phi^{-1}(y)) \leq \frac{1}{\sigma} \operatorname{dist}(y, \Phi(x)) \tag{3.29}$$

for any $x \in U$ and $y \in V$ satisfying $\operatorname{dist}(y, \Phi(x)) \leq \mu$. We now verify the conclusions of the corollary.

(a) Let $B_\rho(x_0) \subset U$. There is no loss of generality in assuming that $B_{\sigma\rho}(y_0) \subset V$. Taking arbitrary $y \in \operatorname{int} B_{\sigma\rho}(y_0)$, we have $y \in V$. From (3.29) and $y_0 \in \Phi(x_0)$ it follows that

$$\operatorname{dist}(x_0, \Phi^{-1}(y)) \leq \frac{1}{\sigma} \operatorname{dist}(y, \Phi(x_0)) \leq \frac{\operatorname{dist}(y, \Phi(x_0)) - \operatorname{dist}(y_0, \Phi(x_0))}{\sigma} \leq \rho.$$

Hence, $\operatorname{int} B_{\sigma\rho}(y_0) \subset \Phi(B_\rho(x_0))$.

(b) The proof is immediate from (3.29).

(c) Obviously, F is partially pseudo-Lipschitzian in y with rank 1 around $\omega_0 := (x_0, y_0, 0)$. The conclusion is immediate from Corollary 3.9. The proof is complete. ■

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