TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 5, pp. 2091-2105, October 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

# MODULI SPACE FOR GAUSSIAN TERM STRUCTURE MODELS WITH FINITE DIMENSIONAL REALIZATIONS

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**Abstract.** In this paper, we show that the set of deterministic volatility [10] term structure models with finite dimensional realizations (fdrs) considered in [2] can be identified with an open subset of a Euclidean space, and hence be equipped with the topological and analytical properties of the latter. In particular, the notions of *distance*, and *differentiability* of functions defined on this set, can be defined which have important implications for parameter estimation and risk analysis. It is also shown that Lie algebras, which play a key role in the characterization of term structure models with fdrs in [3] and [7], do not separate, and are hence unable to parameterize, these models.

## 1. Introduction

The so called term structure of interest rates is roughly speaking the relation between the interest rate (or cost of borrowing money) and the time to maturity of the debt. Since interest comes into play in all transactions involving money even when we are in depression, it is indeed important to model the interest rates ([5, 6, 11, 12]). Money aside interesting mathematical problems comes up in term structure models.

This paper deals with the famous [10] (henceforth HJM) term structure model (see also [1], Chap. 23; [4] Chap. 5). HJM models the evolution of forward rates (see for example [4] section 1.4) assuming that the drifts of the no-arbitrage evolution of certain variables can be expressed as functions of their volatilities and the correlations among themselves and so captures the full dynamics of the entire forward rate curve.

Received January 19, 2009, accepted May 21, 2009.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: Primary: 91B24, 91B28, 91B70; Secondary: 14J10, 14D20, 17B45, 55R10.
*Key words and phrases*: Classification of HJM models, Term structures, Moduli space, Lie algebra.

Using the geometric theory of *moduli spaces* of time invariant linear systems as developed by [8, 9] and [16], we solve in this paper the classification problem for those HJM models which are Gaussian and have finite dimensional realizations. According to [3] the forward rate model has a finite dimensional realization if and only if the Lie algebra generated by the drift and the volatility of the forward rate equation is finite dimensional. We give in section 3 a simple direct proof establishing the structure of the *Lie algebra* of any Gaussian HJM with fdr. We construct a covering of the set $\mathcal{M}$ of all Gaussian HJM models with fdr by Euclidean open patches $\mathcal{M}^{r,o}_{d,n,1}$ and study these patches via a *general linear group fibre bundle* $\mathbb{M}^{r,o}_{d,n,1} \to \mathcal{M}^{r,o}_{d,n,1}$. An explicit description of $\mathcal{M}^{r,o}_{d,n,1}$ is provided by a simple method transforming any model to its canonical matrix representative. Moreover, $\mathcal{M}^{r,o}_{d,n,1}$ will be identified with an open subset of $\mathbb{R}^{dn} \times \mathbb{R}^{d}$, and so they naturally inherit the topological and analytical properties of $\mathbb{R}^{dn} \times \mathbb{R}^{d}$. In particular, $\mathcal{M}^{r,o}_{d,n,1}$ inherits from $\mathbb{R}^{dn} \times \mathbb{R}^{d}$ a natural notion of *distance* that enables the differences between the various term structure models to be quantified and thus allowing comparison which was absent in previous discussions.

The problem of determining the necessary and sufficient conditions under which a [10] term structure model admits a finite dimensional realization (fdr) was initiated in [2], in which a solution was obtained for the special case of Gaussian HJM models. Although this problem has now been solved under quite general conditions in [3] and [7], the associated problem of *classifying* the set of HJM models that admit fdr, in the sense of finding an explicit parametrization of these models, remains unsolved. More specifically, although it is known from [3] that an HJM model admits an fdr if and only if the Lie algebra generated by the vector fields corresponding to forward rate volatilities and the Stratonovich forward rate drift is finite dimensional, and from [7] that forward rate volatilities must be linear combinations of “constant directions” that satisfy certain *Ricatti differential equations* in Banach spaces, a concrete *classification* that enables systematic enumeration of these models is unavailable.

A key result from [2] is that Gaussian HJM models with fdr can be characterized in terms of matrix triples that satisfy certain rank conditions. Consequently, the classification of these models reduces to the classification of the corresponding matrix triples. However, since the correspondence between the matrix triples and the term structure models is not one-to-one, the problem becomes that of classifying the *equivalence class* of matrix triples that correspond to the same term structure model. In its simplest form, the classification problem is then to find a suitable set, $\mathcal{M}$, called the *moduli space,* in bijection with the equivalence class of matrix triples. Additionally, one seeks to determine a map that assigns to each matrix triple the canonical representative of its equivalence class in $\mathcal{M}$, since this will enable the matrix triples to be compared for equivalence. Finally, it is useful to equip $\mathcal{M}$

with a metric to quantify the differences between the various equivalence classes of matrix triples, and hence the corresponding term structure models.

Since the existence of fdrs for HJM models is characterized in terms of associated Lie algebras, it may be believed that an appropriate set of finite dimensional Lie algebras parameterizes such models. However, by computing these Lie algebras explicitly, and determining the necessary and sufficient conditions under which they are isomorphic, we show that this is in fact *false*. It turns out that the isomorphism class of Lie algebras depends only on the first component of the matrix triple associated to a given term structure model, and is hence unable to distinguish these models in general.

The remainder of the paper is structured as follows. The main results from [2] on Gaussian HJM models with fdr is briefly reviewed in Section 2 along with a description of the connection between matrix triples and term structure models. This is followed by a detailed investigation of the associated Lie algebras in Section 3. The main results of the paper are contained in Section 4 where the solution to the classification problem for Gaussian HJM models with fdr is given. The paper concludes with Section 5.

## 2. Minimal Realizations of Gaussian HJM Models with FDR

In this section, we briefly review the main results from [2] on the minimal realizations of HJM models with fdr. Write $\mathbb{R}_+$ for the interval $[0,\infty)$. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P})$ be a filtered probability space satisfying the usual conditions, where the filtration, $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is generated by a standard $n$-dimensional $(\mathcal{F}_t,\mathbb{P})$-Wiener process $w_t$. For any $(t,x)\in\mathbb{R}_+^2$, let $r_t(x)$ be the $(t+x)$-maturity forward rate at time $t$, and suppose the initial forward rate curve $r^\star$ is in the space $\mathcal{C}(\mathbb{R}_+)$ of continuous functions on $\mathbb{R}_+$, and *deterministic* and *time-homogeneous* forward rate volatility, $\varsigma=(\varsigma_1,\ldots,\varsigma_n)^T\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}^n)$, are given, where the superscript $^T$ denotes matrix transpose. Then under the [13] parametrization, the dynamics of the forward rate curve determined by $r^*$ and $\varsigma$ is given by

$$\tag{2.1}\begin{cases} dr_t(x)=\mu(x)dt+\varsigma(x)^T dw_t, \\ \mu(x)=\dfrac{\partial}{\partial x}\left[r_t(x)+\dfrac{1}{2}\left|\displaystyle\int_0^x \varsigma(u)du\right|^2\right], \\ r_0(x)=r^*(x), \end{cases}$$

for all $t,x\in\mathbb{R}_+$. Note that if $\sigma$ denotes the forward rate volatility in the standard HJM notation, then $\varsigma(x)=\sigma(t,t+x)$ and $\varsigma$ is assumed to be independent of the running time $t\in\mathbb{R}_+$.

For any $k, l \in \mathbb{N}_+$, denote by $\mathbb{M}_{k\times l}$ the set of $k \times l$ matrices with entries in $\mathbb{R}$. Then the forward rate model in (2.1) is said to admit a *finite dimensional realization* (fdr) if there exists $d \in \mathbb{N}_+$, $(A, B) \in \mathbb{M}_{d\times d} \times \mathbb{M}_{d\times n}$, and $\gamma \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ such that the forward rate curve can be represented in the form

$$r_t(x) = r^\star(t+x) + \frac{1}{2}\left[\left|\int_0^{t+x} \varsigma(u)du\right|^2 - \left|\int_0^{x} \varsigma(u)du\right|^2\right] + \gamma(x)^T z_t, \tag{2.2}$$

where $z_t$ is a $d$-dimensional process satisfying the stochastic differential equation

$$\begin{cases} dz_t = Az_t dt + Bdw_t, \\ z_0 = 0. \end{cases} \tag{2.3}$$

Note that the first two terms in (2.2) are easily obtained from the corresponding terms in (2.1), and so the key content of the above definition is the requirement

$$\int_0^t \varsigma(t-s+x)^T dw_s = \gamma(x)^T z_t. \tag{2.4}$$

The *dimension* of the fdr (2.2)-(2.3) is the dimension of $z_t$, and a realization is said to be *minimal* if there are no realizations of smaller dimension.

For any $(A, B) \in \mathbb{M}_{d\times d} \times \mathbb{M}_{d\times n}$ and $(A, C) \in \mathbb{M}_{d\times d} \times \mathbb{M}_{1\times d}$, define

$$Q(A, C) = (C; CA; CA^2; \cdots; CA^d), \tag{2.5}$$

$$R(A, B) = (B, AB, A^2B, \ldots, A^dB), \tag{2.6}$$

where the semicolons in $Q(A, C)$ indicate row breaks. Then $(A, C)$ is said to be *observable* if $Q(A, C)$ has full rank, and $(A, B)$ is said to be *reachable* if $R(B, C)$ has full rank. Similarly, for any $\Sigma = (A, B, C) \in \mathbb{M}_{d,n,1} := \mathbb{M}_{d\times d} \times \mathbb{M}_{d\times n} \times \mathbb{M}_{1\times d}$, define

$$Q(\Sigma) = Q(A, C) \quad \text{and} \quad R(\Sigma) = R(A, B). \tag{2.7}$$

Then $\Sigma$ is said to be *observable* if $(A, C)$ is observable, and *reachable* if $(A, B)$ is reachable. Denote by $\mathbb{M}^r_{d,n,1}$ and $\mathbb{M}^o_{d,n,1}$ the subsets of $\mathbb{M}_{d,n,1}$ consisting of reachable and observable triples respectively, and let $\mathbb{M}^{r,o}_{d,n,1} = \mathbb{M}^r_{d,n,1} \cap \mathbb{M}^o_{d,n,1}$.

**Theorem 2.1.** [2]. *The forward rate model* $(2.1)$ *admits an* fdr *if and only if* $\varsigma(x) = Ce^{Ax}B$ *for some* $(A, B, C) \in \mathbb{M}_{d,n,1}$. *The realization is minimal if and only if* $(A, B, C)$ *is reachable and observable.*

Hence, Gaussian HJM models with fdr are *characterized* as those for which the forward rate volatilities have the form $\varsigma(x) = Ce^{Ax}B$ for some $(A, B, C) \in \mathbb{M}_{d,n,1}$. It must be noted, however, that the above theorem does not *classify* Gaussian HJM models with fdr in the sense that it does not provide a *unique parametrization* for such models. In fact, it is the case that, for any given initial curve, many such triples $(A, B, C)$ determine the *same* HJM model. The main objective of this paper is to determine a unique set of parameters for each Gaussian model with fdr.

Note that although the minimal realization is not uniquely determined in general, the *dimension* of the minimal realization is unique. So to classify HJM models with fdr, it suffices to classify the minimal realizations by determining the equivalence classes within $\mathbb{M}^{r,o}_{d,n,1}$ for each $(d, n) \in \mathbb{N}^2_+$. Since an HJM model is completely determined by the forward rate volatilities, for a fixed initial curve $r^*$, it follows that $\Sigma_1, \Sigma_2 \in \mathbb{M}^{r,o}_{d,n,1}$ determine the same HJM model, and are hence *equivalent*, if and only if

$$C_1 e^{A_1 x} B_1 = C_2 e^{A_2 x} B_2, \tag{2.8}$$

where $\Sigma_i = (A_i, B_i, C_i)$. In this case, we will write $\Sigma_1 \sim \Sigma_2$.

Denote by $\mathrm{GL}_d$ the set of $d \times d$ invertible matrices with entries in $\mathbb{R}$. Then $\mathrm{GL}_d$ acts on $\mathbb{M}_{d,n,1}$ according to the rule

$$g(A, B, C) = (gAg^{-1}, gB, Cg^{-1}), \tag{2.9}$$

where $g \in \mathrm{GL}_d$ and $(A, B, C) \in \mathbb{M}_{d,n,1}$. The transformation $(A, B, C) \mapsto g(A, B, C)$ corresponds to the basis change in the state space, $\mathbb{R}^d$, for the process $z_t$ by $g$. The following lemma is trivial.

**Lemma 2.1.** *The subsets $\mathbb{M}^{r}_{d,n,1}$, $\mathbb{M}^{o}_{d,n,1}$, and $\mathbb{M}^{r,o}_{d,n,1}$ of $\mathbb{M}_{d,n,1}$ are invariant under the GL$_n$-action.*

It is known in the theory of linear systems that $\Sigma_1, \Sigma_2 \in \mathbb{M}_{d,n,1}$ define the same *linear system* if and only if there exists $g \in \mathrm{GL}_n$ such that $\Sigma_2 = g\Sigma_1$. In the context of term structure models, $\Sigma_1$ and $\Sigma_2$ are equivalent if and only if they determine the same HJM model or, equivalently, satisfy (2.8). It will now be shown that $\Sigma_1, \Sigma_2 \in \mathbb{M}^{r,o}_{d,n,1}$ are equivalent in this sense if and only if $\Sigma_2 = g\Sigma_1$ for some $g \in \mathrm{GL}_d$, which coincides with the notion of equivalence in the linear systems sense.

For any $\Sigma \in \mathbb{M}_{d,n,1}$, let $Q(\Sigma, d) = (C; CA; \cdots; CA^{d-1})$ be the submatrix of $Q(\Sigma)$ consisting of the first $d$ rows, and let $Q(\Sigma)_i$ be the $i$-th row of $Q(\Sigma)$ for $1 \le i \le d+1$.

**Lemma 2.2.** *Let* $\Sigma \in \mathbb{M}_{d,n,1}$. *Then* $\Sigma \in \mathbb{M}^o_{d,n,1}$ *if and only if* $Q(\Sigma, d)$ *is invertible.*

*Proof.* If $Q(\Sigma, d)$ is invertible, then $Q(\Sigma)$ has the full rank and so $Q(\Sigma) \in \mathbb{M}^o_{d,n,1}$. Conversely, suppose $Q(\Sigma) \in \mathbb{M}^o_{d,n,1}$. Let $k \in \{1, 2, \ldots, d\}$ be the maximal element with the property $Q(\Sigma)_1, \ldots, Q(\Sigma)_k$ are linearly independent, and let

$$V_{\Sigma,k} = \text{span}\{Q(\Sigma)_1, \ldots, Q(\Sigma)_k\}.$$

We show by induction that $Q(\Sigma)_i \in V_{\Sigma,k}$ for all $1 \leq i \leq d+1$. The claim is clearly true for $1 \leq i \leq k$, and by maximality of $k$, it is also true for $i = k+1$. Now, suppose the claim is true for some $i$, where $k+1 \leq i \leq d$. Then there exist $a_l \in \mathbb{R}$ such that $Q(\Sigma)_i = \sum_{l=1}^k a_l Q(\Sigma)_l$. But then, by inductive hypothesis,

$$Q(\Sigma)_{i+1} = Q(\Sigma)_i A = \left[\sum_{l=1}^k a_l Q(\Sigma)_l\right] A = \sum_{l=1}^k a_l Q(\Sigma)_{l+1} \in V_{\Sigma,k}.$$

So by induction $\text{span}\{Q(\Sigma)_1, \ldots, Q(\Sigma)_{d+1}\} \subset V_{\Sigma,k}$, and since $\Sigma \in \mathbb{M}^o_{d,n,1}$ we have $k = \text{rank} Q(\Sigma) = d$. It follows that $Q(\Sigma, d)$ is invertible. ■

**Lemma 2.3.** *Let* $\Sigma_1, \Sigma_2 \in \mathbb{M}^{r,o}_{d,n,1}$. *Then* $Q(\Sigma_i, d)$ *and* $R(\Sigma_i)R(\Sigma_i)^T$ *belong to* $GL_d$. *If* $\Sigma_1 \sim \Sigma_2$, *then for any* $i, j \in \{1, 2\}$,

$$Q(\Sigma_i, d)R(\Sigma_i)R(\Sigma_i)^T = Q(\Sigma_j, d)R(\Sigma_j)R(\Sigma_i)^T. \tag{2.10}$$

*Proof.* Invertibility of $Q(\Sigma_i, d)$ follows from the previous lemma. Next, since $\Sigma_i$ is reachable, $R(\Sigma_i)$ has full rank and so $R(\Sigma_i)R(\Sigma_i)^T \in \text{GL}_d$ for $i = 1, 2$. Finally, if $\Sigma_1 \sim \Sigma_2$, then $C_1 A_1^k B_1 = C_2 A_2^k B_2$ for all $k \in \mathbb{N}$. It follows that

$$\begin{aligned} Q(\Sigma_1)_{\boldsymbol{d}} B_1 &= Q(\Sigma_2)_{\boldsymbol{d}} B_2, \\ C_1 R(\Sigma_1) &= C_2 R(\Sigma_2), \\ Q(\Sigma_1, d)R(\Sigma_1) &= Q(\Sigma_2, d)R(\Sigma_2). \end{aligned} \tag{2.11}$$

Multiplying the final equation on the right by $R(\Sigma_i)^T$ gives (2.10). ■

**Proposition 2.4.** *Let* $\Sigma_1, \Sigma_2 \in \mathbb{M}^{r,o}_{d,n,1}$. *Then* $\Sigma_1 \sim \Sigma_2$ *in the sense of (2.11) if and only if there exists* $g \in GL_d$ *such that* $\Sigma_2 = g\Sigma_1$.

*Proof.* Suppose $\Sigma_1 \sim \Sigma_2$. Firstly, we have by the above lemma that $Q(\Sigma_i)_{\boldsymbol{d}} \in \text{GL}_d$ and so we can define $g \in \text{GL}_d$ by $g = Q(\Sigma_2, d)^{-1} Q(\Sigma_1, d)$. It is now claimed that $\Sigma_2 = g\Sigma_1$. The first equation in (2.11) implies $B_2 = gB_1$, while the second equation implies

$$C_2 = C_1[R(\Sigma_1)R(\Sigma_2)^T][R(\Sigma_2)R(\Sigma_2)^T]^{-1}.$$

But, $g^{-1} = [R(\Sigma_1)R(\Sigma_2)^T][R(\Sigma_2)R(\Sigma_2)^T]^{-1}$ by (2.10) and so $C_2 = C_1 g^{-1}$. Finally, we have $Q(\Sigma_1, d)A_1R(\Sigma_1) = Q(\Sigma_2, d)A_2R(\Sigma_2)$, and since this can be rewritten

$$A_2 = [Q(\Sigma_2, d)^{-1}Q(\Sigma_1, d)]A_1[R(\Sigma_1)R(\Sigma_2)^T][R(\Sigma_2)R(\Sigma_2)^T]^{-1} = gA_1g^{-1},$$

it follows that $\Sigma_2 = g\Sigma_1$. The reverse implication is trivial. ∎

The above proposition implies that the classification of Gaussian HJM models with fdr is equivalent to the classification of the $\mathrm{GL}_d$*-orbits* in $\mathbb{M}^{r,o}_{d,n,1}$. Before embarking on a detailed study of the $\mathrm{GL}_d$-orbits of $\mathbb{M}^{r,o}_{d,n,1}$, we investigate whether or not the Lie algebra associated with HJM models is *fine* enough to separate these models.

## 3. Lie Algebras for Gaussian HJM Models

Recall from above that the characterization of HJM models with fdr in [3] is in terms of the Lie algebra generated by forward rate volatilities and the forward rate drift. For any $\Sigma \in \mathbb{M}^{r,o}_{d,n,1}$ denote by $\mathfrak{L}(\Sigma)$ the Lie algebra associated with the forward rate model corresponding to $\Sigma$. Now, if $\Sigma_1 \sim \Sigma_2$ then the corresponding HJM models have the same forward rate volatilities and drift, and so, in particular, $\mathfrak{L}(\Sigma_1) \cong \mathfrak{L}(\Sigma_2)$. In this section, we determine the necessary and sufficient conditions under which $\mathfrak{L}(\Sigma_1) \cong \mathfrak{L}(\Sigma_2)$, and establish that the converse is *false*. That is, $\mathfrak{L}(\Sigma_1) \cong \mathfrak{L}(\Sigma_2)$ does *not* imply $\Sigma_1 \sim \Sigma_2$ in general.

**Lemma 3.1.** *Let $\Sigma \in \mathbb{M}^{r,o}_{d,n,1}$, and let $V(\Sigma) = span\{A^iB_j \mid i \in \mathbb{N},\ 1 \le j \le n\} = \mathbb{R}^d$ and $W(\Sigma) = span\{Ce^{Ax}A^iB_j \mid i \in \mathbb{N},\ 1 \le j \le n\}$, where $B_j$ denotes the $j$-th column of $B$. Then there exists a vector space isomorphism $\phi\colon V(\Sigma) \to W(\Sigma)$ under which $\phi(v) = Ce^{Ax}v$ for all $v \in \mathbb{R}^d$. In particular, $\phi(A^iB_j) = Ce^{Ax}A^iB_j$ for all $i \in \mathbb{N}$ and $i \le j \le n$.*

*Proof.* Firstly, since $R(\Sigma)$ has full rank, there exist $(i_k(\Sigma), j_k(\Sigma))$, where $1 \le k \le d$, $0 \le i_k(\Sigma) \le d$ and $1 \le j_k(\Sigma) \le n$, such that $\mathcal{B} = \{A^{i_k(\Sigma)}B_{j_k(\Sigma)} \mid 1 \le k \le d\}$ is a basis for $\mathbb{R}^d$. Define $\phi(A^{i_k(\Sigma)}B_{j_k(\Sigma)}) = Ce^{Ax}A^{i_k(\Sigma)}B_{j_k(\Sigma)}$ for $1 \le k \le d$, and extend $\phi$ by linearity to $V(\Sigma) = \mathbb{R}^d$. Then since $\mathcal{B}$ is a basis of $V(\Sigma)$, for any $v \in \mathbb{R}^d$, there exists $\alpha^v_k \in \mathbb{R}$ such that

$$v = \sum_{1 \le k \le d} \alpha^v_k A^{i_k(\Sigma)}B_{j_k(\Sigma)},$$

and by definition of $\phi$, we have

$$\begin{aligned}\phi(v) &= \sum_{1\le k\le d} \alpha_k^v \phi(A^{i_k(\Sigma)} B_{j_k(\Sigma)}) \\ &= \sum_{1\le k\le d} \alpha_k^v C e^{Ax} A^{i_k(\Sigma)} B_{j_k(\Sigma)} \\ &= C e^{Ax} \sum_{1\le k\le d} \alpha_k^v A^{i_k(\Sigma)} B_{j_k(\Sigma)} \\ &= C e^{Ax} v.\end{aligned}$$

It is clear that $\phi$ is surjective. To show $\phi$ is injective, suppose $\phi(v) = 0$ for some $v \in V(\Sigma)$. Then from above, we have $Ce^{Ax}v = 0$. Differentiating with respect to $x$ gives $CA^k e^{Ax} v = 0$, and setting $x = 0$ gives $CA^k v = 0$ for all $k \in \mathbb{N}$. In particular, $Q(\Sigma, d)v = 0$, and since $Q(\Sigma, d) \in \mathrm{GL}_d$, it follows that $v = 0$ and so $\phi$ is injective. ■

Given $d \in \mathbb{N}_+$ and $A \in \mathbb{M}_{d\times d}$, let $\mathfrak{L}(A, d) = \mathbb{R}A \oplus \mathbb{R}^d$. If we define $[v_1, v_2] = 0$ for all $v_1, v_2 \in \mathbb{R}^d$ and $[A, v] = Av$ for any $v \in \mathbb{R}^d$, then $\mathfrak{L}(A, d)$ is a Lie algebra and $\mathbb{R}^d$ is a commutative ideal. Although it is not explicitly stated, it follows from [7] that the Lie algebra of *any* HJM model with fdr is necessarily of the form $\mathfrak{L}(A, d)$. We now give a (simple) direct proof of this result for the Gaussian HJM models. For any $\Sigma \in \mathbb{M}_{d,n,1}$, let

$$\varsigma_\Sigma(x) = Ce^{Ax}B = (Ce^{Ax}B_1, \ldots, Ce^{Ax}B_n). \tag{3.1}$$

**Proposition 3.2.** *Let* $\Sigma = (A, B, C) \in \mathbb{M}^{r,o}_{d,n,1}$. *Then* $\mathfrak{L}(\Sigma) \cong \mathfrak{L}(A, d)$.

*Proof.* Firstly, recall from [3] that since $\varsigma_\Sigma$ is deterministic, $\mathfrak{L}(\Sigma)$ is generated by $\mu_\Sigma$ and $\varsigma_{\Sigma,i}$, for $1 \le i \le n$, where

$$\mu_\Sigma(r_t) = \frac{\partial r_t}{\partial x} + \frac{1}{2}\varsigma_\Sigma(x)^T \int_0^x \varsigma_\Sigma(u)du$$

is the Itô forward rate drift. Next, since the second term in $\mu_\Sigma(r_t)$ is deterministic, the Fréchet derivative of $\mu_\Sigma(r_t)$ with respect to $r_t$ is given by

$$\partial_{r_t}\mu_\Sigma(r_t) = \frac{\partial}{\partial x},$$

and since $\varsigma_{\Sigma,i}$ are deterministic $\partial_{r_t}\varsigma_{\Sigma,i} = 0$ for all $1 \le i \le n$. Note that

$$[\mu_\Sigma, \varsigma_{\Sigma,i}] = \partial_{r_t}\mu_\Sigma(\varsigma_{\Sigma,i}) = \frac{\partial}{\partial x}\varsigma_{\Sigma,i}(x) = \frac{\partial}{\partial x}(Ce^{Ax}B_i) = Ce^{Ax}AB_i,$$

and more generally, $(\mathrm{ad}\mu_\Sigma)^k \varsigma_{\Sigma,i} = Ce^{Ax}A^kB_i$ for all $1 \le i \le n$ and $k \in \mathbb{N}$. Define $\varphi\colon \mathfrak{L}(A,d) \to \mathfrak{L}(\Sigma)$ by $\varphi(A) = \mu_\Sigma$ and $\varphi(A^iB_j) = \phi(A^iB_j) = Ce^{Ax}A^iB_j$, where $\phi\colon V(\Sigma) \to W(\Sigma)$ is the linear isomorphism defined in Lemma 3.1. We now show that $\varphi$ is in fact a Lie algebra homomorphism. For this it suffices to show that $[\varphi(A), \varphi(A^iB_j)] = \varphi([A, A^iB_j])$ for all $i \in \mathbb{N}$ and $1 \le j \le n$. But we have

$$\begin{aligned}[\varphi(A), \varphi(A^iB_j)] &= [\mu_\Sigma, Ce^{Ax}A^iB_j] = \frac{\partial}{\partial x}(Ce^{Ax}A^iB_j)\\ &= Ce^{Ax}A^{i+1}B_j = Ce^{Ax}(A(A^iB_j)) = \varphi([A, A^iB_j]).\end{aligned}$$

Hence, $\varphi$ is a Lie algebra homomorphism, and since $\dim \mathfrak{L}(\Sigma) = d + 1 = \dim \mathfrak{L}(A,d)$ it follows that $\varphi$ is in fact an isomorphism. ∎

Though it is not used here we note that the above result does not require that $\Sigma$ is reachable and observable. A consequence of the above proposition is that the Lie algebra, $\mathfrak{L}(\Sigma)$, depends *only* on $A$, and is completely independent of $B$ and $C$. We now determine the necessary and sufficient conditions under which $\mathfrak{L}(\Sigma_1) \cong \mathfrak{L}(\Sigma_2)$ for $\Sigma_1, \Sigma_2 \in \mathbb{M}_{d,n,1}$.

**Proposition 3.3.** *Let $\Sigma_i = (A_i, B_i, C_i) \in \mathbb{M}^{r,o}_{d,n,1}$ for $1 \le i \le 2$. Then we have $\mathfrak{L}(\Sigma_1) \cong \mathfrak{L}(\Sigma_2)$ if and only if $A_1 \sim \alpha A_2$ for some $0 \ne \alpha \in \mathbb{R}$.*

*Proof.* Since $\mathfrak{L}(\Sigma_i) \cong \mathfrak{L}(A_i, d)$ by the previous proposition, it suffices to prove that $\mathfrak{L}(A_1, d) \cong \mathfrak{L}(A_2, d)$ if and only if $A_1 \sim \alpha A_2$ for some $0 \ne \alpha \in \mathbb{R}$. So suppose firstly that $\mathfrak{L}(A_1, d) \cong \mathfrak{L}(A_2, d)$, and let $\varphi\colon \mathfrak{L}(A_1, d) \to \mathfrak{L}(A_2, d)$ be a Lie algebra isomorphism. Then since $\varphi$ maps the commutative part of $\mathfrak{L}(A_1, d)$ to the commutative part of $\mathfrak{L}(A_2, d)$, it follows that $g_\varphi = \varphi|_{\mathbb{R}^d}$ is a linear isomorphism. Moreover, without loss of generality, we may assume that $\varphi(A_1) = \alpha A_2$ for some $0 \ne \alpha \in \mathbb{R}$ so that the following diagram

$$\begin{tikzcd} \mathbb{R}^d \arrow[r, "g_\varphi"] \arrow[d, "A_1"'] & \mathbb{R}^d \arrow[d, "\alpha A_2"] \\ \mathbb{R}^d \arrow[r, "g_\varphi"'] & \mathbb{R}^d \end{tikzcd}$$

commutes. To see this, suppose $\varphi(A_1) = \alpha A_2 + u$ with $\alpha \in \mathbb{R}$ and $u \in \mathbb{R}^d$. Then since $\varphi$ is an isomorphism, we have $\alpha \ne 0$. Define $\bar\varphi\colon \mathfrak{L}(A_1, d) \to \mathfrak{L}(A_2, d)$ by $\bar\varphi(A_1) = \alpha A_2$ and $\bar\varphi|_{\mathbb{R}^d} = \varphi|_{\mathbb{R}^d}$. Then $\bar\varphi$ is a vector space isomorphism, and to show that $\bar\varphi$ is a Lie algebra isomorphism we only need to show $\bar\varphi([A_1, v]) = [\bar\varphi(A_1), \bar\varphi(v)]$ for all $v \in \mathbb{R}^d$. But since $\varphi$ is a Lie algebra homomorphism, we have from definitions

$$\begin{aligned}\bar{\varphi}([A_1, v]) &= \bar{\varphi}(A_1 v) = \bar{\varphi}|_{\mathbb{R}^d}(A_1 v) = \varphi|_{\mathbb{R}^d}(A_1 v) = \varphi(A_1 v) = \varphi([A_1, v]) \\ &= [\varphi(A_1), \varphi(v)] = [\alpha A_2 + u, \varphi|_{\mathbb{R}^d}(v)] = [\alpha A_2, \bar{\varphi}|_{\mathbb{R}^d}(v)] = [\bar{\varphi}(A_1), \bar{\varphi}(v)],\end{aligned}$$

and so replacing $\varphi$ with $\bar{\varphi}$ if necessary, we may assume that $g(A_1) = \alpha A_2$ for some $0 \neq \alpha \in \mathbb{R}$ as claimed. It then follows that $\alpha A_2 = g_\varphi A_1 g_\varphi^{-1}$ and so $A_1 \sim \alpha A_2$. Conversely, if $A_1 \sim \alpha A_2$ for some $0 \neq \alpha \in \mathbb{R}$, then there exists $g \in \mathrm{GL}_d$ such that $\alpha A_2 = g A_1 g^{-1}$. It is easily verified that the map $\varphi_g \colon \mathfrak{L}(A_1, d) \to \mathfrak{L}(A_2, d)$ defined by $\varphi_g(A_1) = \alpha A_2$ and $\varphi_g|_{\mathbb{R}^d} = g$ is then a Lie algebra isomorphism. ∎

It was observed above that $\mathfrak{L}(\Sigma)$ depends only on the matrix $A$, and it may have been hoped that $\mathfrak{L}(\Sigma)$ determined, at least, the conjugacy class of $A$. The previous proposition shows that even this is *not* the case, and that the Lie algebra only determines the conjugacy class of $A$ *up to a scalar*. So although they play a key role in determining whether or not a term structure model admits an fdr, their usefulness in parameterizing term structure models is somewhat limited by the fact that completely different term structure models can have isomorphic Lie algebras. We now turn to the description of the $\mathrm{GL}_d$-orbits in $\mathbb{M}^{r,o}_{d,n,1}$.

## 4. Description of the Moduli Space

A characterization of the moduli, or parameter, space for the $\mathrm{GL}_d$-orbits in $\mathbb{M}^{r,o}_{d,n,1}$ is known from the theory of moduli spaces for time invariant linear systems. More specifically, a simple method for computing the canonical representative of each $\mathrm{GL}_d$-orbit is known, and the set of these orbits in $\mathbb{M}^{r,o}_{d,n,1}$ is known to be homeomorphic to an open subset of $\mathbb{R}^{dn} \times \mathbb{R}^d$. In particular, it follows from the latter that the moduli space, $\mathbb{M}^{r,o}_{d,n,1}/\mathrm{GL}_d$, is equipped with a natural metric. The results in this section are based on ([8, 9]) and [16]. Note that since $C \in \mathbb{M}_{1\times d}$ for the matrix triples considered in this paper, the description of $\mathbb{M}^{o}_{d,n,1}$, and the subsequent description of $\mathbb{M}^{r,o}_{d,n,1}/\mathrm{GL}_d$, is much simpler than the dual description of $\mathbb{M}^{r}_{d,n,1}$, which involves $B \in \mathbb{M}_{d\times n}$ with $n > 1$ in general. In particular, the discussion of the so-called nice selections can be avoided altogether. The literature, however, focuses almost exclusively on the description of $\mathbb{M}^{r}_{d,n,1}$, and so we provide (simpler) proofs of the main results based on $\mathbb{M}^{o}_{d,n,1}$ in this section.

As a set, $\mathbb{M}_{d,n,1}$ can be identified with $\mathbb{R}^{d^2+dn+d}$, and so we equip $\mathbb{M}_{d,n,1}$ with the usual topology of $\mathbb{R}^{d^2+dn+d}$. The subsets $\mathbb{M}^{r}_{d,n,1}$, $\mathbb{M}^{o}_{d,n,1}$, and $\mathbb{M}^{r,o}_{d,n,1}$ are given induced topologies so that the inclusion maps $\mathbb{M}^{w}_{d,n,1} \hookrightarrow \mathbb{M}_{d,n,1}$ are continuous for $w \in \{r, o, \{r, o\}\}$. Let $\mathcal{M}_{d,n,1} = \mathbb{M}_{d,n,1}/\mathrm{GL}_d$ be the set of $\mathrm{GL}_d$-orbits in $\mathbb{M}_{d,n,1}$, and let $\pi_{d,n,1} \colon \mathbb{M}_{d,n,1} \to \mathcal{M}_{d,n,1}$ be the projection map. We equip $\mathcal{M}_{d,n,1}$ with the quotient topology so that $\pi_{d,n,1}$ is continuous with respect to this topology. The subsets $\mathcal{M}^{w}_{d,n,1} = \pi_{d,n,1}(\mathbb{M}^{w}_{d,n,1}) \subset \mathcal{M}_{d,n,1}$ are given the topology induced from $\mathcal{M}_{d,n,1}$.

**Proposition 4.1.** *The subsets* $\mathbb{M}^r_{d,n,1}$, $\mathbb{M}^o_{d,n,1}$, *and* $\mathbb{M}^{r,o}_{d,n,1}$ *are open in* $\mathbb{M}_{d,n,1}$.

*Proof.* Let $\Phi_{d,n,1}, \Psi_{d,n,1} \colon \mathbb{M}_{d,n,1} \to \mathbb{R}$ be defined by $\Phi_{d,n,1}(\Sigma) = \det[R(\Sigma)\, R(\Sigma)^T]$ and $\Psi_{d,n,1}(\Sigma) = \det Q(\Sigma, d)$. Then $\Phi_{d,n,1}$ and $\Psi_{d,n,1}$ are polynomial maps and are hence continuous. Since $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$ and

$$\mathbb{M}^r_{d,n,1} = \{\Sigma \in \mathbb{M}_{d,n,1} \mid \det[R(\Sigma)R(\Sigma)^T] \neq 0\} = \Phi^{-1}_{d,n,1}(\mathbb{R} \setminus \{0\}), \tag{4.1}$$

$$\mathbb{M}^o_{d,n,1} = \{\Sigma \in \mathbb{M}_{d,n,1} \mid \det Q(\Sigma, d) \neq 0\} = \Psi^{-1}_{d,n,1}(\mathbb{R} \setminus \{0\}), \tag{4.2}$$

it follows from continuity of $\Phi_{d,n,1}$ and $\Psi_{d,n,1}$ that $\mathbb{M}^r_{d,n,1}$ and $\mathbb{M}^o_{d,n,1}$ are open. Finally, $\mathbb{M}^{r,o}_{d,n,1}$ is open since $\mathbb{M}^{r,o}_{d,n,1} = \mathbb{M}^r_{d,n,1} \cap \mathbb{M}^o_{d,n,1}$. ■

Note that $\mathbb{M}_{d,n,1}$ may be equipped with the Zariski topology of $\mathbb{R}^{d^2+dn+d}$. Then since $\Phi_{d,n,1}$ and $\Psi_{d,n,1}$ are polynomial (algebraic) maps, $\mathbb{M}^o_{d,n,1}$, $\mathbb{M}^o_{d,n,1}$, and $\mathbb{M}^{r,o}_{d,n,1}$ are Zariski open in $\mathbb{M}_{d,n,1}$ and we may consider the description of the moduli space with respect to the Zariski topology. However, we do not pursue this in this paper.

For any $(A, C) \in \mathbb{M}_{d\times d} \times \mathbb{M}_{1\times d}$, define $Q(A, C, d) = (C; CA; \cdots; CA^{d-1})$ and $Q(A, C)_i$ as for $Q(\Sigma)$ above.

**Lemma 4.2.** *For any* $v \in \mathbb{R}^d$, *there exists unique* $(A, C) \in \mathbb{M}_{d\times d} \times \mathbb{M}_{1\times d}$ *such that* $(A, C)$ *is observable,* $Q(A, C, d) = I_d$, *and* $Q(A, C)_{d+1} = v$, *where* $I_d$ *is the* $d \times d$ *identity matrix.*

*Proof.* The proof is by construction. Let $e_i$ be the $i$-th standard basis element of $\mathbb{R}^d$ and let $A_i$ be the $i$-th row of $A$. Firstly, since $Q(A, C, d) = I_d$, we must have $C = Q(A, C)_1 = e_1$. But then $e_2 = Q(A, C)_2 = CA = e_1A = A_1$, and by induction $A_i = CA^i = e_{i+1}$ for $1 \le i \le d - 1$. For $A_d$, note that $v = CA^d = (CA^{d-1})A = e_dA$ and so $A_d = v$. The existence and uniqueness is clear from the construction. ■

If a group $G$ acts on sets $X$ and $Y$, then a map $\psi \colon X \to Y$ is said to be *G-equivariant* if $\psi(gx) = g\psi(x)$ for all $x \in X$ and $g \in G$. That is, if $\psi$ commutes with group actions on $X$ and $Y$.

For any $\Sigma \in \mathbb{M}^o_{d,n,1}$, let $g_\Sigma = Q(\Sigma, d)$. Then $g_\Sigma \in \mathrm{GL}_d$ by Lemma 2.2, and so we can define $\phi_{d,n,1} \colon \mathbb{M}^o_{d,n,1} \to \mathrm{GL}_d \times \mathbb{R}^{dn} \times \mathbb{R}^d$ by

$$\phi_{d,n,1}(\Sigma) = (g_\Sigma^{-1}, g_\Sigma B, Q(g_\Sigma \Sigma)_{d+1}), \tag{4.3}$$

and $\psi_{d,n,1} \colon \mathrm{GL}_d \times \mathbb{R}^{dn} \times \mathbb{R}^d \to \mathbb{M}^o_{d,n,1}$ by

$$\psi_{d,n,1}(g, B, v) = g(A, B, C), \tag{4.4}$$

where $(A, C)$ is the unique observable pair with $Q(A, C, d) = I_d$ and $Q(A, C)_{d+1} = v$ as given by Lemma 4.2. We equip $\mathrm{GL}_d \times \mathbb{R}^{dn} \times \mathbb{R}^d$ with the topology induced from $\mathbb{R}^{d^2} \times \mathbb{R}^{dn} \times \mathbb{R}^d$, and let $\mathrm{GL}_d$ act on this set by left multiplication on the first component.

**Proposition 4.3.** *$\phi_{d,n,1}$ is a $GL_d$-equivariant homeomorphism and $\psi_{d,n,1} = \phi_{d,n,1}^{-1}$. In particular, $\mathbb{M}^o_{d,n,1} \cong GL_d \times \mathbb{R}^{dn} \times \mathbb{R}^d$.*

*Proof.* Since $\phi_{n,d,1}$ is rational, it is continuous. Next, let $\Sigma = (A, B, C) \in \mathbb{M}^o_{d,n,1}$. Then $\phi_{d,n,1}(\Sigma) = (g_\Sigma^{-1}, g_\Sigma B, Q(g_\Sigma \Sigma)_{d+1})$ and so

$$(\psi_{d,n,1} \circ \phi)(\Sigma) = \psi_{d,n,1}(g_\Sigma^{-1}, g_\Sigma B, Q(g_\Sigma \Sigma)_{d+1}) = g_\Sigma^{-1}(A', g_\Sigma B, C'),$$

where $(A', C')$ is the unique observable pair with the property $Q(A', C', d) = I_d$ and $Q(A', C')_{d+1} = Q(g_\Sigma \Sigma)_{d+1}$. We show that $(g_\Sigma A g_\Sigma^{-1}, C g_\Sigma^{-1})$ has these properties, whence it will follow by uniqueness that $(A', C') = (g_\Sigma A g_\Sigma^{-1}, C g_\Sigma^{-1})$. But $Q(g_\Sigma A g_\Sigma^{-1}, C g_\Sigma^{-1}, d) = Q(\Sigma, d\, g_\Sigma^{-1}) = I_d$ by definition of $g_\Sigma$, and similarly we have $Q(g_\Sigma A g_\Sigma^{-1}, C g_\Sigma^{-1})_{d+1} = Q(g_\Sigma \Sigma)_{d+1}$. It follows that

$$(\psi_{d,n,1} \circ \phi_{d,n,1})(\Sigma) = g_\Sigma^{-1}(g_\Sigma A g_\Sigma^{-1}, g_\Sigma B, C g_\Sigma^{-1}) = \Sigma$$

whence $\psi_{d,n,1} \circ \phi_{d,n,1} = \mathrm{id}$. Similar computation gives $\phi_{d,n,1} \circ \psi_{d,n,1} = \mathrm{id}$ which implies $\phi_{d,n,1}$ is a homeomorphism. It remains to show that $\phi_{d,n,1}$ is $\mathrm{GL}_d$-equivariant. For this, note that for any $h \in \mathrm{GL}_d$ we have $g_{h\Sigma} = g_\Sigma h^{-1}$ and so

$$\phi_{d,n,1}(h\Sigma) = ((g_\Sigma h^{-1})^{-1}, g_\Sigma h^{-1}(hB), Q(g_\Sigma h^{-1}(h\Sigma))_{d+1}) = h\phi_{d,n,1}(\Sigma).$$

This completes the proof. ■

It follows from the previous proposition that in the language of fibre bundles ([15]), $\mathbb{M}^o_{d,n,1}$ is the trivial $\mathrm{GL}_d$-bundle over $\mathcal{M}^o_{d,n,1}$, and similarly $\mathbb{M}^{r,o}_{d,n,1}$ is the trivial $\mathrm{GL}_d$-bundle over $\mathcal{M}^{r,o}_{d,n,1}$. Now, let $\tilde{\pi}_{d,n,1} : \mathrm{GL}_d \times \mathbb{R}^{dn} \times \mathbb{R}^d \to \mathbb{R}^{dn} \times \mathbb{R}^d$ be the projection map given by $\tilde{\pi}_{d,n,1}(g, v, w) = (v, w)$.

**Corollary 4.4.** *We have $\mathcal{M}^o_{d,n,1} \cong \mathbb{R}^{dn} \times \mathbb{R}^d$ and $\mathcal{M}^{r,o}_{d,n,1}$ is homeomorphic to the open subset, $\tilde{\pi}_{d,n,1} \circ \phi_{d,n,1}(\mathbb{M}^{r,o}_{d,n,1})$, of $\mathbb{R}^{dn} \times \mathbb{R}^d$.*

The set of Gaussian HJM models with fdr corresponding to $\mathbb{M}^{r,o}_{d,n,1}$ can hence be identified with an open subset of $\mathbb{R}^{dn} \times \mathbb{R}^d$.

**Corollary 4.5.** *Let $\mathcal{M}$ be the set of all Gaussian HJM models with* fdr. *Then $\mathcal{M} = \coprod_{(d,n) \in \mathbb{N}^2_+} \mathcal{M}^{r,o}_{d,n,1}$, where each $\mathcal{M}^{r,o}_{d,n,1}$ is homeomorphic to an open subset of $\mathbb{R}^{dn} \times \mathbb{R}^d$.*

We now consider the canonical representatives of each $\mathrm{GL}_d$-orbit in $\mathbb{M}^o_{d,n,1}$, and a method for transforming a given matrix triple to its canonical representative, or form. Let $c^\star\colon \Sigma \in \mathbb{M}^o_{d,n,1} \to \mathbb{M}^o_{d,n,1}$ be defined by $c^\star(\Sigma) = g_\Sigma \Sigma$, where $g_\Sigma = Q(\Sigma, d)$ as defined above. Then $c^*$ is *continuous*, and $c^\star(\Sigma) \sim \Sigma$ by definition. Hence, $c^\star$ defines a continuous *canonical form* on $\mathbb{M}^o_{d,n,1}$ in the following sense.

**Lemma 4.6.** *We have $c^\star \circ c^\star = c^\star$. Moreover, for $\Sigma_1, \Sigma_2 \in \mathbb{M}^o_{d,n,1}$ we have $\Sigma_1 \sim \Sigma_2$ if and only if $c^*(\Sigma_1) = c^\star(\Sigma_2)$.*

*Proof.* The first statement follows easily from the fact that $g_{c^\star(\Sigma)} = I_d$ for all $\Sigma \in \mathbb{M}^o_{d,n,1}$. For the second statement, suppose firstly that $\Sigma_1 \sim \Sigma_2$. Then there exists $h \in \mathrm{GL}_d$ such that $\Sigma_2 = h\Sigma_1$ and so

$$c^\star(\Sigma_2) = c^\star(h\Sigma_1) = g_{h\Sigma_1}(h\Sigma_1) = (g_{\Sigma_1}h^{-1})(h\Sigma) = g_{\Sigma_1}\Sigma_1 = c^\star(\Sigma_1).$$

Conversely, if $c^\star(\Sigma_1) = c^\star(\Sigma_2)$, then $g_{\Sigma_1}\Sigma_1 = g_{\Sigma_2}\Sigma_2$ and it follows that $\Sigma_2 = g\Sigma_1$ with $g = g^{-1}_{\Sigma_2} g_{\Sigma_1} \in \mathrm{GL}_d$. ■

**Corollary 4.7.** *Term structure models with* fdr *corresponding to $\Sigma_1, \Sigma_2 \in \mathbb{M}^{r,o}_{d,n,1}$ are equivalent if and only if $c^\star(\Sigma_1) = c^\star(\Sigma_2)$.*

Since the computation of $c^\star(\Sigma)$ is purely mechanical, it gives an easy way of checking whether or not two elements of $\mathbb{M}^o_{d,n,1}$, and hence the corresponding term structure models, are equivalent.

From definitions, we have $\tilde{\pi}_{d,n,1} \circ \phi_{d,n,1}(\Sigma) = \tilde{\pi}_{d,n,1} \circ \phi_{d,n,1} \circ c^\star(\Sigma)$, and since $g_{c^\star(\Sigma)} = I_d$ we have

$$\tilde{\pi}_{d,n,1} \circ \phi_{d,n,1}(\Sigma) = (Q(\Sigma, d)B, CA^d Q(\Sigma, d)^{-1}). \tag{4.5}$$

Given this description, we can define a metric, $\rho$, on $\mathcal{M}^o_{d,n,1}$, and hence on $\mathcal{M}^{r,o}_{d,n,1}$, by setting

$$\begin{aligned} \rho(\Sigma_1, \Sigma_2) &= |(Q(\Sigma_1, d)B_1, C_1A_1^d Q(\Sigma_1, d)^{-1}) \\ &\quad -(Q(\Sigma_2, d)B_2, C_2A_2^d Q(\Sigma_2, d)^{-1})|, \end{aligned} \tag{4.6}$$

where $\Sigma_i = (A_i, B_i, C_i)$ and $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^{dn} \times \mathbb{R}^d$. The metric, $\rho$, is compatible with the given topology on $\mathcal{M}^o_{d,n,1}$ which, in turn, corresponds to the usual topology on $\mathbb{R}^{dn} \times \mathbb{R}^d$ under the homeomorphism established in Corollary 4.4. Hence, we have a natural notion of *distance* between two term structure models in $\mathcal{M}^{r,o}_{d,n,1}$, and the corresponding notion of term structure models being *close* to a given one.

It should be noted that there are alternative metrics that can be placed on $\mathcal{M}^o_{d,n,1}$. For example, it follows from [14] that

$$\tilde{\rho}(\Sigma_1, \Sigma_2) = \sum_{j=1}^{n} \max_k \frac{2^{-k} p_k(\varsigma_{\Sigma_1,j} - \varsigma_{\Sigma_2,j})}{1 + p_k(\varsigma_{\Sigma_1,j} - \varsigma_{\Sigma_2,j})},$$

where $\varsigma_{\Sigma_i,j}(x) = (C_i e^{A_i x} B_i)_j$ and $p_k(f) = \sup_{x\in[0,k]} |f(x)|$, is another metric on $\mathcal{M}^o_{d,n,1}$ compatible with the topology on $\mathcal{C}(\mathbb{R}_+)^n$ defined by the seminorms $\{p_k\}$. Although, being defined in terms of forward rate volatilities, $\tilde{\rho}$ may be economically more meaningful, $\rho$ is much easier to compute and hence practically more useful. Moreover, it is unclear whether or not $\tilde{\rho}$ is compatible with the given topology of $\mathcal{M}^o_{d,n,1}$.

## 5. Conclusion

In this paper, we obtained an explicit parametrization of Gaussian HJM models with fdr considered in [2] by establishing that they decompose into components, $\mathcal{M}^{r,o}_{d,n,1}$, that are homeomorphic to open subsets of $\mathbb{R}^{dn} \times \mathbb{R}^d$. Using this homeomorphism, we defined a notion of distance between term structure models in $\mathcal{M}^{r,o}_{d,n,1}$, along with the associated topology. In particular, this leads to the notion of open sets about a given term structure model and hence to the calculus of functions defined on term structure models.

We also determined the structure of Lie algebras for the Gaussian HJM models, along with the necessary and sufficient conditions under which these Lie algebras are isomorphic. It was shown that the isomorphism class of Lie algebras is insufficient to distinguish term structure models in general, and hence unable to parameterize these models.

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