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ALGORITHMS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS APPROACH TO MINIMIZATION PROBLEMS

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Abstract. In this paper, we introduce two algorithms for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Furthermore, we prove that the proposed algorithms converge strongly to a solution of the minimization problem of finding $x^* \in \Gamma$ such that $\|x^*\| = \min_{x\in\Gamma} \|x\|$ where Γ stands for the intersection set of the solution set of the equilibrium problem and the fixed points set of a nonexpansive mapping.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

It is clear that any α-inverse-strongly monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. Let $f : C \to H$ be a ρ-contraction; that is, there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$. A mapping $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$. Denote the set of fixed points of S by $Fix(S)$.

Let $A : C \to H$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. Now we concern the following equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.1}$$

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The solution set of (1.1) is denoted by EP. If $A = 0$, then (1.1) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \forall y \in C. \tag{1.2}$$

If $F = 0$, then (1.1) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Az, y - z \rangle \geq 0, \forall y \in C. \tag{1.3}$$

The equilibrium problem (1.2) and the variational inequality problem (1.3) have been investigated by many authors. Please see [6]-[18], [21-29] and the references therein. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others. See, e.g., [1], [3], [4], [5].

For solving equilibrium problem (1.1), Moudafi [5] introduced an iterative algorithm and proved a weak convergence theorem. Further, Takahashi and Takahashi [3] introduced another iterative algorithm for finding an element of $EP \cap Fix(S)$ and they obtained a strong convergence result.

It is our purpose in this paper that we introduce two algorithms for finding an element of $EP \cap Fix(S)$ in a real Hilbert space. Furthermore, we prove that the proposed algorithms converge strongly to a solution of the minimization problem of finding $x^* \in EP \cap Fix(S)$ such that $\|x^*\| = \min_{x \in EP \cap Fix(S)} \|x\|$.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x, x) = 0$ for all $x \in C$;

(H2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(H3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2, \forall x,y\in H. \tag{2.1}$$

Moreover, P_C is characterized by the following properties:

$$\langle x-P_Cx, y-P_Cx\rangle \leq 0, \tag{2.2}$$

and

$$\|x-y\|^2 \geq \|x-P_Cx\|^2+\|y-P_Cx\|^2, \tag{2.3}$$

for all $x\in H$ and $y\in C$.

We need the following lemmas for proving our main results.

Lemma 2.1. ([2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F: C\times C\to R$ be a bifunction which satisfies conditions (H1)-(H4). Let $r>0$ and $x\in C$. Then, there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y\in C.$$

Further, if $T_r(x)=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0, \forall y\in C\}$, then the following hold:

- (i) *T_r is single-valued and T_r is firmly nonexpansive, i.e., for any $x,y\in H$, $\|T_rx-T_ry\|^2\leq\langle T_rx-T_ry, x-y\rangle$;*
- (ii) *EP is closed and convex and $EP=Fix(T_r)$.*

Lemma 2.2. ([8]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A: C\to H$ be α-inverse strongly monotone and $r>0$ be a constant. Then, we have*

$$\|(I-rA)x-(I-rA)y\|^2\leq\|x-y\|^2+r(r-2\alpha)\|Ax-Ay\|^2, \forall x,y\in C.$$

In particular, if $0\leq r\leq 2\alpha$, then $I-rA$ is nonexpansive.

Lemma 2.3. ([19]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$. Suppose $x_{n+1}=(1-\beta_n)y_n+\beta_nx_n$ for all $n\geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0$. Then, $\lim_{n\to\infty}\|y_n-x_n\|=0$.*

Lemma 2.4. ([10]). *Let C be a closed convex subset of a real Hilbert space H and let $S: C\to C$ be a nonexpansive mapping. Then, the mapping $I-S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n\to x^*$ weakly and $(I-S)x_n\to y$ strongly, then $(I-S)x^*=y$.*

Lemma 2.5. ([20]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. MAIN RESULTS

In this section we will introduce two algorithms for finding the minimum norm element x^* of $\Gamma := Ep \cap Fix(S)$. Namely, we want to find the unique point x^* which solves the following minimization problem:

$$x^* = \arg\min_{x\in\Gamma}\|x\|. \tag{3.1}$$

Let $S : C \to C$ be a nonexpansive mapping and $A : C \to H$ be an α-inverse strongly monotone mapping. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)-(H4). In order to find a solution of the minimization problem (3.1), we construct the following implicit algorithm

$$x_t = SP_C[(1-t)T_r(x_t - rAx_t)], \forall t \in (0,1), \tag{3.2}$$

where T_r is defined as Lemma 2.1. We will show that the net $\{x_t\}$ defined by (3.2) converges to a solution of the minimization problem (3.1). As matter of fact, in this paper, we will study the following general algorithm.

Let $f : C \to H$ be a ρ-contraction. For each $t \in (0,1)$, we consider the following mapping W_t given by

$$W_t x = SP_C[tf(x) + (1-t)T_r(I - rA)x], \forall x \in C.$$

Since the mappings S, P_C, T_r and $I - rA$ are nonexpansive, then we can check easily that $\|W_t x - W_t y\| \leq [1-(1-\rho)t]\|x-y\|$ which implies that W_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of W_t in C, i.e.,

$$x_t = SP_C[tf(x_t) + (1-t)T_r(x_t - rAx_t)]. \tag{3.3}$$

In this point, we would like to point out that algorithm (3.3) includes algorithm (3.2) as a special case due to the contraction f is a possible nonself mapping.

Now we show our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping and $A : C \to H$ be an α-inverse strongly monotone mapping. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)-(H4). Let $f : C \to H$ be a ρ-contraction and $r > 0$ be a constant with $r < 2\alpha$. Suppose $\Gamma \neq \emptyset$. Then the net $\{x_t\}$ generated by the implicit method (3.3) converges in norm, as $t \to 0$, to the unique solution x^* of the following variational inequality*

(3.4) $$x^* \in \Gamma, \quad \langle (I-f)x^*, x - x^* \rangle \geq 0, \quad x \in \Gamma.$$

In particular, if we take $f = 0$, then the net $\{x_t\}$ defined by (3.2) converges in norm, as $t \to 0$, to a solution of the minimization problem (3.1).

Proof. First, we prove that $\{x_t\}$ is bounded. Set $u_t = T_r(x_t - rAx_t)$ for all $t \in (0,1)$. Take $z \in \Gamma$. It is clear that $z = T_r(z - rAz)$. Since T_r is nonexpansive and A is α-inverse-strongly monotone, we have from Lemma 2.2 that

(3.5) $$\begin{aligned} \|u_t - z\|^2 &\leq \|x_t - rAx_t - (z - rAz)\|^2 \\ &\leq \|x_t - z\|^2 + r(r - 2\alpha)\|Ax_t - Az\|^2 \\ &\leq \|x_t - z\|^2. \end{aligned}$$

So, we have that

$$\|u_t - z\| \leq \|x_t - z\|.$$

It follows from (3.3) that

$$\begin{aligned} \|x_t - z\| &= \|SP_C[tf(x_t) + (1-t)u_t] - SP_C z\| \\ &\leq \|t(f(x_t) - z) + (1-t)(u_t - z)\| \\ &\leq t\|f(x_t) - f(z)\| + t\|f(z) - z\| + (1-t)\|u_t - z\| \\ &\leq t\rho\|x_t - z\| + t\|f(z) - z\| + (1-t)\|x_t - z\| \\ &= [1 - (1-\rho)t]\|x_t - z\| + t\|f(z) - z\|, \end{aligned}$$

that is,

$$\|x_t - z\| \leq \frac{\|f(z) - z\|}{1 - \rho}.$$

So, $\{x_t\}$ is bounded. Hence $\{u_t\}$ and $\{f(x_t)\}$ are also bounded. Now we can choose a constant $M > 0$ such that

$$\sup_t \left\{ 2\|f(x_t) - z\|\|u_t - z\| + \|f(x_t) - z\|^2, 2r\|x_t - u_t\|, \|u_t - f(x_t)\|^2 \right\} \leq M.$$

From (3.3) and (3.5), we have

$$\begin{aligned}
&\|x_t - z\|^2 \\
&\leq \|(1-t)(u_t - z) + t(f(x_t) - z)\|^2 \\
&= (1-t)^2\|u_t - z\|^2 + 2t(1-t)\langle f(x_t) - z, u_t - z\rangle + t^2\|f(x_t) - z\|^2 \\
&\leq \|u_t - z\|^2 + tM \\
&\leq \|x_t - z\|^2 + r(r - 2\alpha)\|Ax_t - Az\|^2 + tM
\end{aligned} \tag{3.6}$$

that is,

$$r(2\alpha - r)\|Ax_t - Az\|^2 \leq tM \to 0.$$

Since $r(2\alpha - r) > 0$, we derive

$$\lim_{t\to 0}\|Ax_t - Az\| = 0. \tag{3.7}$$

From Lemma 2.1, Lemma 2.2 and (3.3), we obtain

$$\begin{aligned}
\|u_t - z\|^2 &= \|T_r(x_t - rAx_t) - T_r(z - rAz)\|^2 \\
&\leq \langle (x_t - rAx_t) - (z - rAz), u_t - z\rangle \\
&= \frac{1}{2}\Big(\|(x_t - rAx_t) - (z - rAz)\|^2 + \|u_t - z\|^2 \\
&\quad -\|(x_t - z) - r(Ax_t - Az) - (u_t - z)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|x_t - z\|^2 + \|u_t - z\|^2 - \|(x_t - u_t) - r(Ax_t - Az)\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_t - z\|^2 + \|u_t - z\|^2 - \|x_t - u_t\|^2 \\
&\quad +2r\langle x_t - u_t, Ax_t - Az\rangle - r^2\|Ax_t - Az\|^2\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|u_t - z\|^2 \\
&\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2r\langle x_t - u_t, Ax_t - Az\rangle - r^2\|Ax_t - Az\|^2 \\
&\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2r\|x_t - u_t\|\|Ax_t - Az\| \\
&\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + M\|Ax_t - Az\|.
\end{aligned} \tag{3.8}$$

By (3.6) and (3.8), we have

$$\begin{aligned}
\|x_t - z\|^2 &\leq \|u_t - z\|^2 + tM \\
&\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + (\|Ax_t - Az\| + t)M.
\end{aligned}$$

It follows that

$$\|x_t - u_t\|^2 \leq (\|Ax_t - Az\| + t)M.$$

This together with (3.7) imply that

$$\lim_{t\to 0} \|x_t - u_t\| = 0.$$

It follows that

$$\begin{aligned} \|x_t - Sx_t\| &= \|SP_C[tf(x_t) + (1-t)u_t] - SP_Cx_t\| \\ &\leq t\|f(x_t) - x_t\| + (1-t)\|u_t - x_t\| \to 0. \end{aligned} \tag{3.9}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $u_n := u_{t_n}$. From (3.9), we get

$$\|x_n - Sx_n\| \to 0. \tag{3.10}$$

By (3.3), we deduce

$$\begin{aligned} \|x_t - z\|^2 &= \|SP_C[tf(x_t) + (1-t)u_t] - SP_Cz\|^2 \\ &\leq \|u_t - z - tu_t + tf(x_t)\|^2 \\ &= \|u_t - z\|^2 - 2t\langle u_t, u_t - z\rangle + 2t\langle f(x_t), u_t - z\rangle + t^2\|u_t - f(x_t)\|^2 \\ &= \|u_t - z\|^2 - 2t\langle u_t - z, u_t - z\rangle - 2t\langle z, u_t - z\rangle \\ &\quad + 2t\langle f(x_t) - f(z), u_t - z\rangle + 2t\langle f(z), u_t - z\rangle + t^2\|u_t - f(x_t)\|^2 \\ &\leq [1 - 2(1-\rho)t]\|x_t - z\|^2 + 2t\langle f(z) - z, u_t - z\rangle \\ &\quad + t^2\|u_t - f(x_t)\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_t - z\|^2 &\leq \frac{1}{1-\rho}\langle z - f(z), z - u_t\rangle + \frac{t}{2(1-\rho)}\|u_t - f(x_t)\|^2 \\ &\leq \frac{1}{1-\rho}\langle z - f(z), z - u_t\rangle + \frac{t}{2(1-\rho)}M. \end{aligned}$$

In particular,

$$\|x_n - z\|^2 \leq \frac{1}{1-\rho}\langle z - f(z), z - u_n\rangle + \frac{t_n}{2(1-\rho)}M, \quad z \in \Gamma. \tag{3.11}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$. Also $u_n \to x^*$ weakly. Noticing (3.10) we can use Lemma 2.4 to get $x^* \in Fix(S)$.

Now we show $x^* \in EP$. Since $u_n = T_r(x_n - rAx_n)$, for any $y \in C$ we have

$$F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq 0.$$

From the monotonicity of F, we have

$$\frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq F(y, u_n), \forall y \in C.$$

Hence,

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\rangle \geq F(y, u_{n_i}), \forall y \in C. \tag{3.12}$$

Put $z_t = ty + (1-t)x^*$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.12) we have

$$\begin{aligned}\langle z_t - u_{n_i}, Az_t\rangle \geq\ & \langle z_t - u_{n_i}, Az_t\rangle - \langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\rangle \\ & + F(z_t, u_{n_i}) \\ =\ & \langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle + \langle z_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\ & - \langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\rangle + F(z_t, u_{n_i}).\end{aligned} \tag{3.13}$$

Note that $\|Au_{n_i} - Ax_{n_i}\| \leq \frac{1}{\alpha}\|u_{n_i} - x_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle \geq 0$. Letting $i \to \infty$ in (3.13), we have

$$\langle z_t - x^*, Az_t\rangle \geq F(z_t, x^*). \tag{3.14}$$

From (H1), (H4) and (3.14), we also have

$$\begin{aligned}0 =\ & F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, x^*) \\ \leq\ & tF(z_t, y) + (1-t)\langle z_t - x^*, Az_t\rangle \\ =\ & tF(z_t, y) + (1-t)t\langle y - x^*, Az_t\rangle\end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle Az_t, y - x^*\rangle. \tag{3.15}$$

Letting $t \to 0$ in (3.15), we have, for each $y \in C$,

$$0 \leq F(x^*, y) + \langle y - x^*, Ax^*\rangle.$$

This implies that $x^* \in EP$. Therefore, $x^* \in \Gamma$.

We substitute x^* for z in (3.11) to get

$$\|x_n - x^*\|^2 \leq \frac{1}{1-\rho}\langle x^* - f(x^*), x^* - u_n\rangle + \frac{t_n}{2(1-\rho)}M.$$

Hence, the weak convergence of $\{u_n\}$ to x^* implies that $x_n \to x^*$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

Now we return to (3.11) and take the limit as $n \to \infty$ to get

$$\|x^* - z\|^2 \leq \frac{1}{1-\rho}\langle z - f(z), z - x^*\rangle, \quad z \in \Gamma. \tag{3.16}$$

In particular, x^* solves the following variational inequality

$$x^* \in \Gamma, \quad \langle (I-f)z, z - x^*\rangle \geq 0, \quad z \in \Gamma$$

or the equivalent dual variational inequality

$$x^* \in \Gamma,\ \langle (I-f)x^*, z - x^*\rangle \geq 0,\ z \in \Gamma.$$

Therefore, $x^* = (P_\Gamma f)x^*$. That is, x^* is the unique fixed point in Γ of the contraction $P_\Gamma f$. Clearly this is sufficient to conclude that the entire net $\{x_t\}$ converges in norm to x^* as $t \to 0$.

Finally, if we take $f = 0$, then (3.16) is reduced to

$$\|x^* - z\|^2 \leq \langle z, z - x^*\rangle, \quad z \in \Gamma.$$

Equivalently,

$$\|x^*\|^2 \leq \langle x^*, z\rangle, \quad z \in \Gamma.$$

This clearly implies that

$$\|x^*\| \leq \|z\|, \quad z \in \Gamma.$$

Therefore, x^* is a solution of minimization problem (3.1). This completes the proof. ■

Next we introduce an explicit algorithm for finding a solution of minimization problem (3.1). This scheme is obtained by discretizing the implicit scheme (3.3). We will show the strong convergence of this algorithm.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping and $A : C \to H$ be an α-inverse strongly monotone mapping. Let $F : C \times C \to R$ be a bifunction which* *satisfies conditions* (*H1*)*-*(*H4*)*. Let* $f : C \to H$ *be a* ρ*-contraction and* $r > 0$ *be a constant with* $r < 2\alpha$*. Suppose* $\Gamma \neq \emptyset$*. For given* $x_0 \in C$ *arbitrarily, let the sequence* $\{x_n\}$ *be generated iteratively by*

(3.17) $$x_{n+1} = \beta_n x_n + (1-\beta_n) SP_C[\alpha_n f(x_n) + (1-\alpha_n) T_r(x_n - rAx_n)], n \geq 0,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $[0, 1]$ *satisfying the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *converges strongly to* x^* *which is the unique solution of variational inequality (3.4). In particular, if* $f = 0$*, then the sequence* $\{x_n\}$ *generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) SP_C[(1-\alpha_n) T_r(x_n - rAx_n)], n \geq 0,$$

converges strongly to a solution of the minimization problem (*3.1*)*.*

Proof. First, we prove that the sequence $\{x_n\}$ is bounded.

Let $z = P_C(z - rAz)$. Set $u_n = T_r(x_n - rAx_n)$ for all $n \geq 0$. From (3.17), we get

$$\begin{aligned} \|u_n - z\| &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\| \\ &\leq \|x_n - z\|, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(SP_C[\alpha_n f(x_n) + (1-\alpha_n)u_n] - z)\| \\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)\|\alpha_n(f(x_n) - z) + (1-\alpha_n)(u_n - z)\| \\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)[\alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| \\ &\quad +(1-\alpha_n)\|u_n - z\|] \\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)[\alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\quad +(1-\alpha_n)\|x_n - z\|] \\ &= [1 - (1-\rho)(1-\beta_n)\alpha_n]\|x_n - z\| + \alpha_n(1-\beta_n)\|f(z) - z\| \\ &\leq \max\{\|x_n - z\|, \frac{\|f(z) - z\|}{1-\rho}\}. \end{aligned}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{\|f(z) - z\|}{1-\rho}\right\}.$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$, $\{f(x_n)\}$ and $\{Ax_n\}$ are all bounded. Let $M > 0$ be a constant such that

$$\sup_n \Big\{ \|u_n\| + \|f(x_n)\|, 2\|u_n - f(x_n)\|\|u_n - z\| + \|u_n - f(x_n)\|^2, \\ (\|x_n - z\| + \|x_{n+1} - z\|), 2r\|x_n - u_n\| \Big\} \leq M.$$

Next we show $\lim_{n\to\infty} \|u_n - Su_n\| = 0$.

Define $x_{n+1} = \beta_n x_n + (1-\beta_n) v_n$ for all $n \geq 0$. It follows from (3.17) that

$$\begin{aligned} \|v_{n+1} - v_n\| &= \|SP_C[\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1}) u_{n+1}] \\ &\quad - SP_C[\alpha_n f(x_n) + (1-\alpha_n) u_n]\| \\ &\leq \|\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1}) u_{n+1} - \alpha_n f(x_n) - (1-\alpha_n) u_n\| \\ &\leq \|u_{n+1} - u_n\| + \alpha_{n+1}(\|u_{n+1}\| + \|f(x_{n+1})\|) \\ &\quad + \alpha_n(\|u_n\| + \|f(x_n)\|) \\ &\leq \|T_r(x_{n+1} - rAx_{n+1}) - T_r(x_n - rAx_n)\| + M(\alpha_{n+1} + \alpha_n) \\ &\leq \|x_{n+1} - x_n\| + M(\alpha_{n+1} + \alpha_n). \end{aligned}$$

This together with (i) imply that

$$\limsup_{n\to\infty} \Big(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\| \Big) \leq 0.$$

Hence by Lemma 2.3, we get $\lim_{n\to\infty} \|v_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|v_n - x_n\| = 0.$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &= \|\beta_n(x_n - z) + (1-\beta_n)(v_n - z)\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|v_n - z\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|u_n - z - \alpha_n(u_n - f(x_n)\|^2 \qquad (3.18) \\ &= \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|u_n - z\|^2 - 2\alpha_n\langle u_n - f(x_n), u_n - z\rangle \\ &\quad + \alpha_n^2\|u_n - f(x_n)\|^2] \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 + \alpha_n M. \end{aligned}$$

From Lemma 2.2, we get

$$\begin{aligned} \|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\ &\leq \|(x_n - rAx_n) - (z - rAz)\|^2 \\ &\leq \|x_n - z\|^2 + r(r - 2\alpha)\|Ax_n - Az\|^2. \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.18), we have

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|x_n - z\|^2 + r(r-2\alpha)\|Ax_n - Ax\|^2] + \alpha_n M \\ &= \|x_n - z\|^2 + (1-\beta_n)r(r-2\alpha)\|Ax_n - Az\|^2 + \alpha_n M. \end{aligned}$$

Therefore,

$$\begin{aligned} &(1-\beta_n)r(2\alpha - r)\|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M \\ &\leq (\|x_n - z\| + \|x_{n+1} - z\|)\|x_n - x_{n+1}\| + \alpha_n M \\ &\leq (\|x_n - x_{n+1}\| + \alpha_n)M. \end{aligned}$$

Since $\liminf_{n\to\infty}(1-\beta_n)r(2\alpha - r) > 0$, $\|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we derive

$$\lim_{n\to\infty} \|Ax_n - Az\| = 0.$$

From Lemma 2.1 and (3.17), we obtain

$$\begin{aligned} \|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\ &\leq \langle (x_n - rAx_n) - (z - rAz), u_n - z\rangle \\ &= \frac{1}{2}\Big(\|(x_n - rAx_n) - (z - rAz)\|^2 + \|u_n - z\|^2 \\ &\quad - \|(x_n - z) - r(Ax_n - Az) - (u_n - z)\|^2\Big) \\ &\leq \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 \\ &\quad - \|(x_n - u_n) - r(Ax_n - Az)\|^2\Big) \\ &= \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r\langle x_n - u_n, Ax_n - Az\rangle - r^2\|Ax_n - Az\|^2\Big). \end{aligned}$$

Thus, we deduce

$$
\begin{aligned}
\|u_n - z\|^2 &\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\| \\
&\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + M\|Ax_n - Az\|.
\end{aligned} \tag{3.20}
$$

By (3.18) and (3.20), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 + \alpha_n M \\
&\le \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|x_n - z\|^2 - \|x_n - u_n\|^2 \\
&\quad +M\|Ax_n - Az\|] + \alpha_n M \\
&\le \|x_n - z\|^2 - (1-\beta_n)\|x_n - u_n\|^2 + (\|Ax_n - Az\| + \alpha_n)M.
\end{aligned}
$$

It follows that

$$
(1-\beta_n)\|x_n - u_n\|^2 \le (\|x_{n+1} - x_n\| + \|Ax_n - Az\| + \alpha_n)M.
$$

Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Ax_n - Az\| \to 0$, we derive that

$$
\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.21}
$$

Hence,

$$
\begin{aligned}
\|Su_n - u_n\| &= \|Su_n - v_n\| + \|v_n - x_n\| + \|x_n - u_n\| \\
&= \|SP_C u_n - SP_C[\alpha_n f(x_n) + (1-\alpha_n)u_n]\| \\
&\quad +\|v_n - x_n\| + \|x_n - u_n\| \\
&\le \alpha_n\|f(x_n) - u_n\| + \|v_n - x_n\| + \|x_n - u_n\| \to 0.
\end{aligned}
$$

Next we prove

$$
\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - u_n\rangle \le 0
$$

where $x^* = P_\Gamma f(x^*)$.

Indeed, we can choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$
\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - u_n\rangle = \lim_{i\to\infty}\langle x^* - f(x^*), x^* - u_{n_i}\rangle.
$$

Without loss of generality, we may further assume that $u_{n_i} \to \tilde{x}$ weakly. By the same argument as that of Theorem 3.1, we can deduce that $\tilde{x} \in \Gamma$. Therefore,

$$
\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - u_n\rangle = \langle x^* - f(x^*), x^* - \tilde{x}\rangle \le 0.
$$

From (3.17), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(u_n-x^*)\|^2 \\
&\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)[(1-\alpha_n)^2\|u_n-x^*\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x_n)-x^*, u_n-x^*\rangle+\alpha_n^2\|f(x_n)-x^*\|^2] \\
&= \beta_n\|x_n-x^*\|^2+(1-\beta_n)[(1-\alpha_n)^2\|u_n-x^*\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x_n)-f(x)^*, u_n-x^*\rangle \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x)^*-x^*, u_n-x^*\rangle+\alpha_n^2\|f(x_n)-x^*\|^2] \\
&\leq [1-2(1-\rho)(1-\beta_n)\alpha_n]\|x_n-x^*\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)(1-\beta_n)\langle f(x)^*-x^*, u_n-x^*\rangle+(1-\beta_n)\alpha_n^2 M \\
&= (1-\gamma_n))\|x_n-x^*\|^2+\delta_n\gamma_n,
\end{aligned}
$$

where $\gamma_n = 2(1-\rho)(1-\beta_n)\alpha_n$ and $\delta_n/ = \frac{(1-\alpha_n)}{1-\rho}\langle f(x)^*-x^*, u_n-x^*\rangle+\frac{\alpha_n M}{2(1-\rho)}$. It is clear that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta \leq 0$. Hence, all conditions of Lemma 2.5 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$.

Finally, if we take $f = 0$, by the similar argument as that Theorem 3.1, we deduce immediately that x^* is a solution of minimization problem (3.1). This completes the proof. ■

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