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TWO POINT BOUNDARY VALUE PROBLEMS FOR THE STURM-LIOUVILLE EQUATION WITH HIGHLY DISCONTINUOUS NONLINEARITIES

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Abstract. The aim of this paper is to establish the existence of three solutions for a two-point boundary value problem with the Sturm-Liouville equation having discontinuous nonlinearities. The approach is based on the critical point theory for non-differentiable functions.

1. INTRODUCTION

Ordinary differential problems with discontinuous nonlinearities arise in several fields of theoretical and applied mathematics, and have been studied most frequently by using set-valued analysis, upper and lower solutions, or fixed point theorems (see, for instance, [5], [7], [14] and references therein). The aim of this paper is to investigate, via variational methods, the following two point boundary value problem with the Sturm-Liouville equation having discontinuous nonlinearities

$$(1.1)\qquad \begin{cases} -(\overline{p}u')' + \overline{r}u' + \overline{q}u = \lambda g(u) \ \text{ in } \ ]\mathrm{a},\mathrm{b}[\\ u(a) = u(b) = 0, \end{cases}$$

where $\overline{p}, \overline{r}, \overline{q} \in L^\infty([a,b])$, with $\operatorname{ess\,inf}_{[a,b]} \overline{p} > 0$ and $\operatorname{ess\,inf}_{[a,b]} \overline{q} \geq 0$, λ a positive real parameter, and $g : \mathbb{R} \to \mathbb{R}$ is an almost everywhere continuous function. To be precise, in the present paper, applying the critical point theory for non-differentiable functions, existence results of three solutions to Problem (1.1) are established. It is worth noticing that the set of the points of discontinuity of the nonlinear term g may also be uncountable but, notwithstanding this, the solutions of (1.1) are

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actually generalized solutions. This is different from the above cited papers, where either the nonlinearity is an increasing function (for which it has a set, at most countable, of discontinuity points), or the solutions are given in the sense of the set-valued analysis. On the other hand, because of the presence of term u' in (1.1), our results are also novel in the continuous case. In fact, for instance, it is easy to verify that Example 1.2 below and [10, Theorem 2] are mutually independent, as well as Theorems 3.3, 3.4 (where $\overline{r} \equiv 0$), and [9, Theorem 2], [1, Theorem 1]. As an example, here, we present the following particular cases of our main results (Theorems 3.5 and 3.6).

Example 1.1. Let $g : \mathbb{R} \to \mathbb{R}$ be a strictly positive, bounded, almost everywhere continuous function for which there are two positive constants c and d, with $c < d$, such that

$$\frac{\int_0^c g(\xi)d\xi}{c^2} < \frac{2}{25}\frac{\int_0^d g(\xi)d\xi}{d^2}.$$

Then, for each $\lambda \in \left]\frac{55}{4}\frac{d^2}{\int_0^d g(\xi)d\xi}, \frac{11}{10}\frac{c^2}{\int_0^c g(\xi)d\xi}\right[$ the problem

$$\begin{cases} -u'' + u' = \lambda g(u) \text{ in }]0,1[\\ u(0) = u(1) = 0 \end{cases} \tag{1.2}$$

admits at least three positive generalized solutions.

Clearly, in Example 1.1 if g is a continuous function, then the three solutions are actually classical solutions.

Example 1.2. *Let $g : [0, +\infty) \to (0, +\infty)$ be a continuous function for which there are three positive constants α, β and γ, with $\alpha < \beta < \gamma$, such that*

(a_1) $g(\xi) \le \alpha$ *for all* $\xi \in [0, \alpha]$;

(a_2) $\int_0^\beta g(\xi)d\xi \ge 22\beta^2$;

(a_3) $g(\xi) \le 2\gamma$ *for all* $\xi \in [0, 4\gamma]$.

Then, the problem

$$\begin{cases} -u'' + u' + u = g(u) \text{ in }]0,1[\\ u(0) = u(1) = 0 \end{cases} \tag{1.3}$$

admits at least three positive classical solutions u_i, $i = 1, 2, 3$, *such that*

$$\max_{t\in[0,1]} |u_i(x)| < 4\gamma, \qquad i = 1, 2, 3.$$

The note is arranged as follows. In Section 2 we recall some basic definitions and the three critical points theorems (Theorem 2.1 and Theorem 2.2) on which our approach is based, while Section 3 is devoted to Theorem 3.3 and Theorem 3.4, which deal with Problem (1.1) when $\overline{r} \equiv 0$, and our main results Theorem 3.5 and Theorem 3.6. Finally, again in Section 3, as a consequence of previous results, proofs of Example 1.1 and Example 1.2 in Introduction are presented.

2. Preliminaries

We refer to [12] and [13] for basic definitions and properties of Nonsmooth Analysis. We only recall that, given a real Banach space X, two locally Lipschitz functions $\Phi, \Psi : X \to \mathbb{R}$ and $M > 0$, the function $\Phi - \Psi$ verifies the *Palais-Smale condition at level* c*,* $c \in \mathbb{R}$*, cut off at* M (in short $(PS)_c^M$) if the function $\Phi - \Psi_M$ satisfies $(PS)_c$ condition, where

$$\Psi_M(u) = \begin{cases} \Psi(u) & \text{if } \Psi(u) \leq M; \\ M & \text{if } \Psi(u) > M. \end{cases}$$

The main tools of our paper are the following three critical points theorems for non-differentiable functions obtained from [3, Theorem 2.6] (see also [2, Theorem 3.2]) and [2, Corollary 3.1]. For an exhaustive bibliography on three critical points theorems and their applications we refer to [15] and [2]. Here, X is a reflexive Banach space, $\Phi : X \to \mathbb{R}$ is a coercive and a sequentially weakly lower semicontinuous function, $\Psi : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous function. Assume also that Φ and Ψ are locally Lipschitz functions and

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Theorem 2.1. ([3], Theorem 2.6). *Under the above assumptions on* X*,* Φ *and* Ψ*, assume that there are* $r > 0$ *and* $\overline{v} \in X$*, with* $r < \Phi(\overline{v})$*, such that:*

(i) $$\frac{\sup_{x \in \Phi^{-1}(]-\infty, r])} \Psi(x)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}.$$

Assume also that for each $\lambda \in \Lambda^{r,\overline{v}} := \left] \frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{x \in \Phi^{-1}(]-\infty, r])} \Psi(x)} \right[$ *one has*

(ii) *the function* $\Phi - \lambda\Psi$ *is bounded below and fulfils* $(PS)_c, c \in \mathbb{R}$.

Then, for each $\lambda \in \Lambda^{r,\overline{v}}$ *the function* $\Phi - \lambda\Psi$ *admits at least three distinct critical points.*

Theorem 2.2. ([2], Corollary 3.1). *Under the above assumptions on X, Φ and Ψ, assume that Φ is convex and*

(a) *for each $\lambda > 0$ and for every x_1, x_2, which are local minima for the function $\Phi - \lambda\Psi$ and such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, one has*

$$\inf_{t\in[0,1]} \Psi(tx_1 + (1-t)x_2) \geq 0.$$

Assume also that there are two positive constants r_1 and r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, such that

$$(i') \quad \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})} \quad \text{and} \quad \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})},$$

and, for each

$$\lambda \in \Lambda_{r_1,r_2,\overline{v}} := \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}; \frac{\frac{r_2}{2}}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[,$$

(ii') *the function $\Phi - \lambda\Psi$ fulfils $(PS)_c^{\frac{2\lambda}{r_2}}$, $c \in \mathbb{R}$.*

Then, for each $\lambda \in \Lambda_{r_1,r_2,\overline{v}}$ the function $\Phi - \lambda\Psi$ admits at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Now, we say that a function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ *belongs to $\mathcal{H}$* if $t \to f(t, x)$ is measurable for every $x \in \mathbb{R}$, there exists a set $I \subset [a, b]$ with $m(I) = 0$ such that the set $D_f := \bigcup_{t\in[a,b]\setminus I} \{z \in \mathbb{R} \,:\, f(t, \cdot) \,\text{is discontinuous at z}\}$ has measure zero, f is locally essentially bounded, and the functions

$$f^-(t, x) := \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|x-z|<\delta} f(t, z), \quad f^+(t, x) := \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|x-z|<\delta} f(t, z),$$

are *superpositionally measurable*, that is, $f^-(t, u(t))$ and $f^+(t, u(t))$ are measurable for all $u : [a, b] \to \mathbb{R}$ measurable.

Clearly, L^1-Carathéodory functions belong to $\mathcal{H}$.

Finally, consider the following Sturm-Liouville boundary value problem

$$(2.1) \qquad \begin{cases} -(pu')' + qu = \lambda f(x, u) \ \text{ in } \ ]\mathrm{a}, \mathrm{b}[\\ u(a) = u(b) = 0 \end{cases}$$

where λ is a positive parameter, $p, q \in L^\infty([a, b])$, with $\operatorname{ess\,inf}_{[a,b]} p > 0$ and

$\operatorname{ess\,inf}_{[a,b]} q \geq 0$, and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function.
We recall that, if f and q are continuous functions, a function $u : [a, b] \to \mathbb{R}$ is called a classical solution to (2.1) if $u \in C^1([a, b])$, $pu' \in C^1([a, b])$, $u(a) = u(b) = 0$ and $-(p(x)u'(x))' + q(x)u(x) = \lambda f(x, u(x))$ for every $x \in [a, b]$. Moreover, $u \in AC([a, b])$ is called a generalized solution of (2.1) if $pu' \in AC([a, b])$, $u(a) = u(b) = 0$ and $-(p(x)u'(x))' + q(x)u(x) = \lambda f(x, u(x))$ for almost every $x \in [a, b]$. Finally, a function $u : [a, b] \to \mathbb{R}$ is called a weak solution of (2.1) if $u \in W_0^{1,2}([a, b])$ and

$$\int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx = \lambda \int_a^b f(x, u(x))v(x)dx$$

for every $v \in W_0^{1,2}([a, b])$. Clearly, the weak solutions of (2.1) are also generalized solutions. In fact, if u is a weak solution of (2.1) one has

$$\int_a^b p(x)u'(x)v'(x)dx = -\int_a^b \Big[q(x)u(x) - \lambda f(x, u(x))\Big]v(x)dx$$

for every $v \in C_0^1([a, b])$, that is, the weak derivative of pu' is $qu - \lambda f(\cdot, u)$ which belongs to $L^2([a, b]) \subseteq L^1([a, b])$. Hence, $pu' \in AC([a, b])$ and $(p(x)u'(x))' = q(x)u(x) - \lambda f(x, u(x))$ for almost every $x \in [a, b]$.

3. Main Results

Here, and in the sequel, X is the Sobolev space $W_0^{1,2}([a, b])$ endowed with the norm

$$\|u\| = \Big(\int_a^b p|u'|^2 + \int_a^b q|u|^2\Big)^{\frac{1}{2}}$$

for all $u \in X$, which is equivalent to the usual one. Put

(3.1) $$p_0 := \operatorname*{ess\,inf}_{x\in[a,b]} p(x) > 0, \quad q_0 := \operatorname*{ess\,inf}_{x\in[a,b]} q(x) \geq 0$$

and

(3.2) $$m := \frac{2p_0}{b-a}.$$

It is known that X is compactly embedded in $C^0([a, b])$ and, moreover, one has

(3.3) $$\|u\|_\infty \leq \frac{1}{\sqrt{2m}}\|u\|.$$

Put

(3.4) $$K := \frac{6p_0}{12\|p\|_\infty + (b-a)^2\|q\|_\infty}.$$

Our first result is the following theorem.

Theorem 3.3. *Let* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be a function belonging to* $\mathcal{H}$. *Put* $F(x,\xi) := \int_0^\xi f(x,t)dt$ *for all* $(x,\xi) \in [a,b] \times \mathbb{R}$ *and assume that*

(h_1) *there exist two positive constants c,d, with* $c < d$, *such that*

$$F(x,\xi) \geq 0 \quad for \;\; all \quad (x,\xi) \in [a,b] \times [0,d]$$

and

$$\frac{\int_a^b \max_{|\xi| \leq c} F(x,\xi)dx}{c^2} < K \frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{d^2}.$$

(h_2) *there exists a positive constant s, with* $s < 2$, *and* $\mu \in L^1([a,b])$ *such that* $F(x,\xi) \leq \mu(x)(1 + |\xi|^s)$ *for almost all* $x \in [a,b]$ *and for all* $\xi \in \mathbb{R}$;

(h_3) *for almost every* $x \in [a,b]$, *for all* $z \in D_f$, *for each* $\lambda \in \Lambda_{c,d}$, *where*

$$\Lambda_{c,d} = \left] \frac{1}{K} \frac{md^2}{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}, \frac{mc^2}{\int_a^b \max_{|\xi| \leq c} F(x,\xi)dx} \right[,$$

one has that

$\lambda f^-(x,z) - q(x)z \leq 0 \leq \lambda f^+(x,z) - q(x)z$ *implies* $\lambda f(x,z) - q(x)z = 0.$

Then, for each $\lambda \in \Lambda_{c,d}$ *the problem* (2.1) *admits at least three weak solutions.*

Proof. Put

$$\Phi(u) := \frac{1}{2}\|u\|^2, \qquad \Psi(u) := \int_a^b F(x,u(x))dx$$

for all $u \in X$. Clearly, Φ, Ψ are locally Lipschitz functions in X. Fix $\lambda \in \Lambda_{c,d}$. We claim that the generalized critical points of $\Phi - \lambda\Psi$ are weak solutions for the problem (2.1). To this end, let $u_0 \in X$ be a generalized critical point of $\Phi - \lambda\Psi$, that is

$$(\Phi - \lambda\Psi)^0(u_0; v - u_0) \geq 0$$

for all $v \in X$. Therefore, $\Phi'(u_0)(w) + \lambda(-\Psi)^0(u_0; w) \geq 0$ for all $w \in X$, that is

$$-\Big(\int_a^b p(x)u_0'(x)w'(x)dx + \int_a^b q(x)u_0(x)w(x)dx\Big) \leq \lambda(-\Psi)^0(u_0; w)$$

for all $w \in X$. Setting $L(w) := -(\int_a^b p(x)u_0'(x)w'(x)dx + \int_a^b q(x)u_0(x)w(x)dx)$ for all $w \in X$, L is a continuous and linear function on X and $L \in \lambda\partial(-\Psi)(u_0)$. Now, since X is dense in $L^2([a,b])$ and Ψ is also defined and locally Lipschitz in

$L^2([a,b])$, from [6, Theorem 2.2] we obtain $\partial(-\Psi)_{|X}(u_0) \subseteq \partial(-\Psi)_{|L^2([a,b])}(u_0)$, so that L is a continuous and linear function on $L^2([a,b])$. Therefore, there exists $\overline{w} \in L^2([a,b])$ such that $L(w) = \int_a^b w(x)\overline{w}(x)dx$ for all $w \in L^2([a,b])$.

Consider now the problem

$$\begin{cases} (p(x)u(x)')' - q(x)u(x) = \overline{w}(x) \text{ in }]\mathrm{a,b}[\\ u(a) = u(b) = 0. \end{cases}$$

Therefore (see, for instance [4, Example 2, Chapter VIII.4]), there exists a unique function $\overline{u} \in W^{2,2}([a,b]) \cap X$ such that

$$-\Big(\int_a^b p(x)\overline{u}'(x)w'(x)dx + \int_a^b q(x)\overline{u}(x)w(x)dx\Big) = \int_a^b \overline{w}(x)w(x)dx = L(w)$$

for all $w \in X$. Hence, since a continuous and linear function on X is uniquely determined by a function in X (see, for instance [11, Theorem 5.9.3]), we have $\overline{u} = u_0$. For which, $u_0 \in W^{2,2}([a,b])$ and $\int_a^b (p(x)u_0'(x))'w(x)dx - \int_a^b q(x)u_0(x)w(x)dx = -\Big(\int_a^b p(x)u_0'(x)w'(x)dx + \int_a^b q(x)u_0(x)w(x)dx\Big) \leq \lambda(-\Psi)^0(u_0; w)$ for all $w \in X$. ¿From [6, Theorem 2.1] one has

$$(p(x)u_0'(x))' - q(x)u_0(x) \in [(-\lambda f)^-(x, u_0(x)), (-\lambda f)^+(x, u_0(x))]$$

for almost every $x \in [a,b]$, so that

$$-(p(x)u_0'(x))' \in [\lambda f^-(x, u_0(x)) - q(x)u_0(x), \lambda f^+(x, u_0(x)) - q(x)u_0(x)]$$

for almost every $x \in [a,b]$. It follows that, for almost every $x \in [a,b] \setminus u_0^{-1}(D_f)$, the previous condition reduces to $-(p(x)u_0'(x))' + q(x)u_0(x) = \lambda f(x, u_0(x))$, while for almost every $x \in u_0^{-1}(D_f)$, since $m(D_f) = 0$, from Lemma 1 of [8] one has $-(p(x)u_0'(x))' = 0$ and, from (h_3) one has $\lambda f(x, u_0(x)) - q(x)u_0(x) = 0$ for almost every $x \in u_0^{-1}(D_f)$, so that $-(p(x)u_0'(x))' + q(x)u_0(x) = \lambda f(x, u_0(x))$ for almost every $x \in u_0^{-1}(D_f)$. Hence, $-(p(x)u_0'(x))' + q(x)u_0(x) = \lambda f(x, u_0(x))$ for almost every $x \in [a,b]$ and our claim is proved.

Now, to obtain our assertion it is enough to apply Theorem 2.1. To this end, we observe that all assumptions on Φ and Ψ, as requested in Theorem 2.1 are verified, and moreover, as standard computation shows, condition (h_2) implies that $\Phi - \lambda\Psi$ is coercive. Hence, it is bounded below. In addition, it satisfies $(PS)_{\overline{c}}$ condition. In fact, let $\{u_n\}$ be a sequence in X such that $\Phi(u_n) - \lambda\Psi(u_n) \to \overline{c} \in \mathbb{R}$ and $(\Phi - \lambda\Psi)^0(u_n; v - u_n) \geq -\epsilon_n\|v - u_n\|$ for all $v \in X$, where $\epsilon_n \to 0^+$. Clearly, since $\Phi - \lambda\Psi$ is coercive, $\{u_n\}$ is a bounded sequence in X, then there exists a subsequence, denoted again by $\{u_n\}$, such that $u_n \rightharpoonup u \in X$ and $u_n \to u \in L^2([a,b])$. So, by rewriting the previous inequality with u instead of v, we have

$$\Phi'(u_n)(u - u_n) + \lambda(-\Psi)^0(u_n; u - u_n) \geq -\epsilon_n||u - u_n||$$

and being $\Phi'(u_n)(u-u_n) = \int_a^b p(x)u_n'(x)(u'(x)-u_n'(x))dx+\int_a^b q(x)u_n(x)(u(x)-u_n(x))dx \leq \frac{1}{2}(\int_a^b p(x){u_n'}^2(x) + q(x)u_n^2(x)dx + \int_a^b p(x)u'^2(x) + q(x)u(x)^2dx) - \|u_n\|^2 = \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|u\|^2 - \|u_n\|^2$, we have

$$-\epsilon_n\|u-u_n\| + \frac{1}{2}\|u_n\|^2 \leq \frac{1}{2}\|u\|^2 + \lambda(-\Psi)^0(u_n; u-u_n).$$

Taking into account that the function Ψ is well defined and locally Lipschitz in $L^2([a,b])$, and that one has $(-\Psi_{|X})^0(u;v) \leq (-\Psi)^0_{|X}(u;v)$ for all $u, v \in X$ (see, for instance, [6, p.111]), the upper semicontinuity of $(-\Psi)^0$ in the strong topology of $L^2([a,b]) \times L^2([a,b])$ (see, for instance, [12, Proposition 1.1]) then implies $\limsup_{n\to\infty}(-\Psi)^0(u_n; u-u_n) \leq 0$ and, therefore, the previous inequality ensures that

$$\limsup_{n\to\infty}\|u_n\| \leq ||u||.$$

Hence, since X is uniformly convex, from [4, Proposition III.30] one has $u_n \to u$ strongly in X and our claim is proved. Hence, (ii) of Theorem 2.1 is verified.

In order to prove (i) of Theorem 2.1, put $r := mc^2$ and

$$\overline{v}(x) = \begin{cases} \frac{4d}{b-a}(x-a), & x \in [a, a+\frac{1}{4}(b-a)[\ ; \\ d, & x \in [a+\frac{1}{4}(b-a), b-\frac{1}{4}(b-a)]; \\ \frac{4d}{b-a}(b-x), & x \in]b-\frac{1}{4}(b-a), b]. \end{cases}$$

Clearly, $\overline{v} \in X$ and $4d^2\Big(\frac{p_0}{b-a} + \frac{q_0}{12}(b-a)\Big) \leq \frac{1}{2}\|\overline{v}\|^2 \leq 4d^2\Big(\frac{||p||_\infty}{b-a} + \frac{||q||_\infty}{12}(b-a)\Big)$. Hence, from $c < d$ it follows that $r < \Phi(\overline{v})$. Moreover, from (3.3) one has $\max_{x\in[a,b]}|u(x)| \leq \sqrt{\frac{r}{m}} = c$ for all $u \in X$ such that $\|u\| \leq \sqrt{2r}$. Therefore, one has

$$\text{(3.5)}\qquad \frac{\sup_{x\in\Phi^{-1}(]-\infty,r])}\Psi(x)}{r} = \frac{\sup_{||u||\leq\sqrt{2r}}\int_a^b F(x,u(x))dx}{r} \leq \frac{\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx}{mc^2}.$$

On the other hand, taking into account that $F(x,\xi) \geq 0$ for all $(x,\xi) \in [a,b]\times[0,d]$, one has

$$\text{(3.6)}\qquad \begin{aligned} \frac{\Psi(\overline{v})}{\Phi(\overline{v})} &= \frac{\int_a^b F(x,\overline{v}(x))dx}{\frac{1}{2}\|\overline{v}\|^2} \geq \frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{4d^2\Big(\frac{||p||_\infty}{b-a} + \frac{||q||_\infty}{12}(b-a)\Big)} \\ &= K\frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{md^2}. \end{aligned}$$

Therefore, owing to (3.5), (3.6) and (h_1), assumption (i) of Theorem 2.1 is verified and, moreover, $\Lambda_{c,d} \subseteq \left]\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup\limits_{x\in\Phi^{-1}(]-\infty,r])}\Psi(x)}\right[$.

Hence, Theorem 2.1 ensures three distinct generalized critical points for $\Phi-\lambda\Psi$, which are, as seen before, three weak solutions for the problem (2.1) and the proof is complete. ■

Remark 3.1. If f does not depend on $x \in [a, b]$ the assumptions of Theorem 3.3 take a simpler form. To be precise, assumption (h_1) becomes

(h_1') *there exist two positive constants c,d, with $c < d$, such that*

$$F(\xi) \geq 0 \quad for \ \ all \quad \xi \in [0, d]$$

and

$$\frac{\max_{|\xi|\leq c} F(\xi)}{c^2} < \frac{K}{2}\frac{F(d)}{d^2},$$

and, in this case, the interval $\Lambda_{c,d}$ is $\left]\frac{2}{K}\frac{m}{b-a}\frac{d^2}{F(d)}, \frac{m}{b-a}\frac{c^2}{\max_{|\xi|\leq c} F(\xi)}\right[$. Clearly, if f is nonnegative in $[0, d]$ and

$$\frac{F(c)}{c^2} < \frac{K}{2}\frac{F(d)}{d^2},$$

then (h_1') is verified.

Obviously, in the autonomous case, the following condition

$$\lim_{t\to+\infty}\frac{f(t)}{t^r} = 0,$$

with $0 < r < 1$, implies (h_2).

Finally, if $f : \mathbb{R} \to \mathbb{R}$ is a locally essentially bounded and almost everywhere continuous function, then it belongs to $\mathcal{H}$. In addition, if, for instance, for each $\overline{z} \in D_f$ there exists a neighbourhood V of $\overline{z}$ such that

$$\inf_V f > \frac{p_0 K}{m^2}\frac{F(d)}{d^2}\max\left\{\|q\|_\infty\overline{z}; q_o\overline{z}\right\},$$

then (h_3) is verified. In fact, if $\overline{z} > 0$ one has $f(z) - \frac{p_0K}{m^2}\frac{F(d)}{d^2}\|q\|_\infty\overline{z} \geq \inf_V f - \frac{p_0K}{m^2}\frac{F(d)}{d^2}\|q\|_\infty\overline{z} := \overline{l} > 0$, $f(z) - \frac{(b-a)K}{2m}\frac{F(d)}{d^2}\|q\|_\infty\overline{z} \geq \overline{l}$, $\frac{2m}{(b-a)K}\frac{d^2}{F(d)}f(z) - \|q\|_\infty\overline{z} \geq \overline{l}\frac{2m}{(b-a)K}\frac{d^2}{F(d)}$, $\lambda f(z) - q(x)\overline{z} \geq \overline{l}\frac{2m}{(b-a)K}\frac{d^2}{F(d)}$ for all $z \in V$, $\lambda > \frac{2m}{(b-a)K}\frac{d^2}{F(d)}$, a.e. $x \in [a, b]$; hence, $\lambda f^-(\overline{z}) - q(x)\overline{z} > 0$ for all $\lambda > \frac{2m}{(b-a)K}\frac{d^2}{F(d)}$, a.e. $x \in [a, b]$. If $\overline{z} \leq 0$, arguing as before with q_0 instead of $\|q\|_\infty$, the same conclusion is obtained.

In particular, when $q \equiv 0$ and f is strictly positive, that is $\inf\limits_{\mathbb{R}} f > 0$, then (h_3) is true.

The other result is the following theorem.

Theorem 3.4. *Let* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be a function belonging to* $\mathcal{H}$. *Put* $F(x,\xi) := \int_0^\xi f(x,t)dt$ *for all* $(x,\xi) \in [a,b] \times \mathbb{R}$ *and assume that*

(k_1) $f(x,u) \geq 0$ *for almost every* $x \in [a,b]$ *and for all* $u \geq 0$;

(k_2) *there exist three positive constants* c_1, d, c_2, *with* $c_1 < d < \sqrt{\frac{K}{2}}c_2$, *such that*

$$\frac{\int_a^b F(x,c_1)dx}{c_1^2} < \frac{2}{3}K\frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{d^2}.$$

and

$$\frac{\int_a^b F(x,c_2)dx}{c_2^2} < \frac{1}{3}K\frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{d^2}.$$

(k_3) *for almost every* $x \in [a,b]$, *for all* $z \in D_f$, *for each* $\lambda \in \Lambda_{c_1,c_2,d}$, *where*

$$\Lambda_{c_1,c_2,d} := \left]\frac{3}{2}\frac{1}{K}\frac{md^2}{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx},\right. \left.\frac{m}{2}\min\left\{\frac{2c_1^2}{\int_a^b F(x,c_1)dx}, \frac{c_2^2}{\int_a^b F(x,c_2)dx}\right\}\right[,$$

one has that

$\lambda f^-(x,z) - q(x)z \leq 0 \leq \lambda f^+(x,z) - q(x)z$ *implies* $\lambda f(x,z) - q(x)z = 0$.

Then, for each $\lambda \in \Lambda_{c_1,c_2,d}$ *the problem* (2.1) *admits at least three nonnegative weak solutions* u_i, $i = 1,2,3$, *such that*

$$\max_{x\in[a,b]} |u_i(x)| < c_2, \qquad i = 1,2,3.$$

Proof. The proof is similar to that of Theorem 3.3 and we give only an outline. Let Φ and Ψ be as in the proof of Theorem 3.3, and fix $\lambda \in \Lambda_{c_1,c_2,d}$. As seen before, generalized critical points of $\Phi - \lambda\Psi$ are weak solutions of the Problem (2.1). Our aim is to apply Theorem 2.2. First, we observe that, owing to the maximum principle and (k_1), assumption (a) is easily verified and, moreover, for all $M > 0$ the function $\Phi - \lambda\Psi_M$, arguing as in the proof of Theorem 3.3

and taking into account that it is coercive since Φ is corcive, satisfies the $(PS)_c$-condition, $c \in \mathbb{R}$. Next, we set $r_1 := mc_1^2$, $r_2 := mc_2^2$, and $\overline{v} \in X$ as in the proof of Theorem 3.3. From $c_1 < d < \sqrt{\frac{K}{2}}c_2$ one has $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$. Moreover (see (3.5) and (3.6)), one has

$$\frac{\sup\limits_{x\in\Phi^{-1}(]-\infty,r_1])}\Psi(x)}{r_1} \le \frac{\int_a^b F(x,c_1)dx}{mc_1^2}, \quad \frac{\sup\limits_{x\in\Phi^{-1}(]-\infty,r_2])}\Psi(x)}{r_2} \le \frac{\int_a^b F(x,c_2)dx}{mc_2^2},$$

$\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})} \ge K\dfrac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{md^2}$. Hence, owing to (k_2) the assumption (i') of Theorem 2.2 is verified and the conclusion is obtained. ■

Remark 3.2. When $f(\cdot,0) \not\equiv 0$ the three solutions in the conclusion of Theorem 3.4 are positive. Otherwise, Theorem 3.4 ensures at least two positive solutions.

Remark 3.3. If f does not depend on $x \in [a,b]$, as seen in Remark 3.1, the assumptions of Theorem 3.4 take a simpler form. In particular, assumption (k_2) becomes

(k_2') there exist three positive constants c_1, d, c_2, with $c_1 < d < \sqrt{\frac{K}{2}}c_2$, such that

$$\frac{F(c_1)}{c_1^2} < \frac{K}{3}\frac{F(d)}{d^2},$$

and

$$\frac{F(c_2)}{c_2^2} < \frac{K}{6}\frac{F(d)}{d^2},$$

and, in this case, the interval $\Lambda_{c_1,c_2,d}$ is $\left]\frac{3}{K}\frac{m}{b-a}\frac{d^2}{F(d)}, \frac{1}{2}\frac{m}{b-a}\min\left\{\frac{2c_1^2}{F(c_1)}, \frac{c_2^2}{F(c_2)}\right\}\right[$.

Remark 3.4. The conclusion of Theorem 3.4 is more precise than that of Theorem 3.3. In addition, in Theorem 3.3 asymptotic conditions on the nonlinear term (see (h_2) in Theorem 3.4) are not requested. On the other hand, the sign assumption on the nonliner term in Theorem 3.4 is stronger than the one in Theorem 3.3. We also note that the first inequality in (k_2) is more restrictive than the one in (h_1) since one has $\frac{2}{3}K$ instead of K.

Now, as an application of previous theorems, we present multiplicity results for Problem (1.1). To this end, taking into account that $\frac{\overline{r}}{\overline{p}}$ is (Lebesgue) integrable in $[a,b]$, denote by R the function such that

$$R' = \frac{\overline{r}}{\overline{p}}$$

almost everywhere in $[a,b]$, and put

$$C := \int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} e^{-R(x)}dx, \quad D := \int_a^b e^{-R(x)}dx, \quad m_1 := \frac{2}{b-a}\operatorname*{ess\,inf}_{[a,b]}(e^{-R}\overline{p}),$$

$$\overline{K} := \frac{6\,\mathrm{ess\,inf}_{[a,b]}(e^{-R}\overline{p})}{12||e^{-R}\overline{p}||_\infty + (b-a)^2||e^{-R}\overline{q}||_\infty}, \quad K_1 := \frac{C}{D}\overline{K}.$$

Theorem 3.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function. Put $G(\xi) := \int_0^\xi g(x)dx$ for every $\xi \in \mathbb{R}$ and assume that*

(j_1) *there exist two positive constants c,d, with $c < d$, such that*

$$G(\xi) \geq 0 \quad \textit{for all} \quad \xi \in [0,d]$$

and

$$\frac{\max_{|\xi|\leq c} G(\xi)}{c^2} < K_1 \frac{G(d)}{d^2},$$

(j_2) *there exist two positive constants a and s, with $s < 2$, such that*

$G(\xi) \leq a(1+|\xi|^s)$ *for all $\xi \in \mathbb{R}$;*

(j_3) *for almost every $x \in [a,b]$, for all $z \in \mathcal{D}_g$, for each $\lambda \in \Lambda_{c,d}$, where*

$$\Lambda_{c,d} = \left]\frac{m_1}{K_1 D}\frac{d^2}{G(d)}, \frac{m_1}{D}\frac{c^2}{\max_{|\xi|\leq c} G(\xi)}\right[,$$

one has that

$\lambda g^-(z) - \overline{q}(x)z \leq 0 \leq \lambda g^+(z) - \overline{q}(x)z$ *implies* $\lambda g(z) - q(x)z = 0$.

Then, for each $\lambda \in \Lambda_{c,d}$ problem (1.1) admits at least three generalized solutions.

Proof. Consider the following problem

$$(3.1) \quad \begin{cases} -(e^{-R(x)}\overline{p}(x)u'(x))' + e^{-R(x)}\overline{q}(x)u(x) = \lambda e^{-R(x)}g(u(x)) \text{ in }]\mathrm{a},\mathrm{b}[\\ u(a) = u(b) = 0. \end{cases}$$

It is easy to verify that all the solutions of problem (3.1) are also solutions of (1.1). Hence, setting $p(x) = e^{-R(x)}\overline{p}(x)$, $q(x) = e^{-R(x)}\overline{q}(x)$ and $f(x,u(x)) = e^{-R(x)}g(u(x))$, from Theorem 3.3 the conclusion follows. ■

Theorem 3.6. *Let $g : [0,+\infty) \to [0,+\infty)$ be a locally essentially bounded and almost everywhere continuous function. Put $G(\xi) := \int_0^\xi g(x)dx$ for every $\xi \in \mathbb{R}$ and assume that*

(i_1) *there exist three positive constants* c_1, d, c_2, *with* $c_1 < d < \sqrt{\frac{K}{2}}c_2$, *such that*

$$\frac{G(c_1)}{c_1^2} < \frac{2}{3}K_1\frac{G(d)}{d^2},$$

and

$$\frac{G(c_2)}{c_2^2} < \frac{1}{3}K_1\frac{G(d)}{d^2}.$$

(i_2) *for almost every* $x \in [a,b]$, *for all* $z \in \mathcal{D}_g$, *for each* $\lambda \in \Lambda_{c_1,c_2,d}$, *where*

$$\Lambda_{c_1,c_2,d} := \left]\frac{3}{2}\frac{m_1}{DK_1}\frac{d^2}{G(d)}, \frac{m_1}{2D}\min\left\{\frac{2c_1^2}{G(c_1)}, \frac{c_2^2}{G(c_2)}\right\}\right[,$$

one has that

$\lambda g^-(z) - \overline{q}(x)z \le 0 \le \lambda g^+(z) - \overline{q}(x)z$ *implies* $\lambda g(z) - q(x)z = 0$.

Then, for each $\lambda \in \Lambda_{c_1,c_2,d}$ *the problem* (1.1) *admits at least three nonnegative generalized solutions* u_i, $i = 1, 2, 3$, *such that*

$$\max_{t\in[a,b]} |u_i(x)| < c_2, \qquad i = 1, 2, 3.$$

Proof. Arguing as in the proof of Theorem 3.5, from Theorem 3.4 the conclusion follows. ■

Proof of Example 1.1. Since $K_1 = \frac{e^{3/4}-e^{1/4}}{2e(e-1)} > \frac{2}{25}$, $\frac{m_1}{K_1 D} = \frac{4e}{e^{3/4}-e^{1/4}} < \frac{55}{4}$, $\frac{m_1}{D} = \frac{2}{e-1} > \frac{11}{10}$, the conclusion is an immediate consequence of Theorem 3.5, taking Remark 3.1 into account.

Proof of Example 1.2. Since $\sqrt{\frac{K}{2}} = \sqrt{\frac{3}{13e}} > \frac{1}{4}$, $\frac{m_1}{D} = \frac{2}{e-1} > 1$, $\frac{3}{2}\frac{m_1}{K_1 D} = \frac{13}{2}\frac{e}{e^{3/4}-e^{1/4}} < 22$, by choosing $c_1 = \alpha$, $d = \beta$ and $c_2 = 4\gamma$, one has $c_1 < d < \sqrt{\frac{K}{2}}c_2$. Moreover, from a_1), a_2) and a_3) one has $\frac{G(c_1)}{c_1^2} \le 1$, $\frac{G(d)}{d^2} \ge 22$ and $\frac{G(c_2)}{c_2^2} \le \frac{1}{2}$. Hence, one has $\frac{3}{2}\frac{m_1}{DK_1}\frac{d^2}{G(d)} < 1 < \min\{\frac{m_1}{D}\frac{c_1^2}{G(c_1)}, \frac{m_1}{D}\frac{c_2^2/2}{G(c_2)}\}$ and the conclusion is an immediate consequence of Theorem 3.6.

References

1. G. A. Afrouzi and S. Heidarkhani, Three solutions for a quasilinear boundary value problem, *Nonlinear Anal.*, **69** (2008), 3330-3336.

2. G. Bonanno and P. Candito, Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, *J. Differential Equations*, **244** (2008), 3031-3059.
3. G. Bonanno and S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, in press.
4. H. Brézis, *Analyse Fonctionelle- Théorie et Applications*, Masson, Paris, 1983.
5. A. Cabada and S. Heikkilä, Extremality results for discontinuous explicit and implicit diffusion problems, *Appl. Math. Comput.*, **143** (2002), 69-80.
6. K.-C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.*, **80** (1981), 102-129.
7. J. A. Cid and R. L. Pouso, On first-order ordinary differential equations with non-negative right-hand sides, *Nonlinear Anal.*, **52** (2003), 1961-1977.
8. E. De Giorgi, G. Buttazzo and G. Dal Maso, On the lower semicontinuity of certain integral functionals, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8) Mat. Appl.*, **74** (1983), 274-282.
9. X. He and W. Ge, Existence of three solutions for a quasilinear two-point boundary value problem, *Comput. Math. Appl.*, **45** (2003), 765-769.
10. J. Henderson and H. B. Thompson, Existence of multiple solutions for second order boundary value problems, *J. Differential Equations*, **166** (2000), 443-454.
11. A. Kufner, O. John and S. Fučik, *Function spaces*, Kluwer, Leyden, 1977.
12. S. A. Marano and D. Motreanu, On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems, *Nonlinear Anal.*, **48** (2002), 37-52.
13. D. Motreanu and V. Radulescu, *Variational and Non-variational Methods in Nonlinear Analysis and Boundary Value Problems*, Nonconvex Optimization and Applications, Kluwer Academic Publishers, 2003.
14. N. S. Papageorgiou and V. Staicu, The method of upper-lower solutions for nonlinear second order differential inclusions, *Nonlinear Anal.*, **67** (2007), 708-726.
15. B. Ricceri, A three critical points theorem revisited, *Nonlinear Anal.*, **70** (2009), 3084-3089.

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