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FIXED POINT THEOREMS FOR THE GENERALIZED Ψ-SET CONTRACTION MAPPING ON AN ABSTRACT CONVEX SPACE

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Abstract. In this paper, we establish some fixed point theorems for the generalized Ψ-set contraction mapping on an abstract convex space, which need not to be a compact map.

1. Introduction and Preliminaries

In 1929, Knaster, Kuratowski and Mazurkiewicz [8] had proved the well-known KKM theorem on n-simplex. Besides, in 1961, Ky Fan [7] had generalized the KKM theorem in the infinite dimensional topological vector space. Later, Chang and Yen [4] introduced the generalized KKM property on a convex subset of a Haudorff topological vector space and they establish some fixed point theorems on this class. Recently, Amini et al. [1] had showed that each compact closed multifunction $F \in S\text{-}KKM_{\mathcal{C}}(X, X, X)$ has a fixed point in an abstract convex space X. In this paper, we establish some fixed point theorems for the generalized Ψ-set contraction mapping on an abstract convex space $(X, \mathcal{C})$, which need not to be a compact map.

Let X and Y be two sets, and let $T : X \to 2^Y$ be a set-valued mapping. We shall use the following notations in the sequel.

(i) $T(x) = \{y \in Y : y \in T(x)\}$,

(ii) $T(A) = \cup_{x \in A} T(x)$,

(iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$,

(iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$, and

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(v) if D is a nonempty subset of X, then $\langle D \rangle$ denotes the class of all nonempty finite subset of D.

For the case that X and Y are two topological spaces, a set-valued map $T : X \to 2^Y$ is said to be closed if its graph $\mathcal{G}_T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed. T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y.

Definition 1. [1]. An abstract convex space $(X, \mathcal{C})$ consists of a nonempty topological space X and a family $\mathcal{C}$ of subsets of X such that X and ϕ belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection.

Suppose A is a nonempty subset of an abstract convex space $(X, \mathcal{C})$. Then

(i) the $\mathcal{C}$-admissible hull of A is defined by

$$ad_{\mathcal{C}}(A) = \cap\{B \in \mathcal{C} : A \subset B\},$$

(ii) a subset A is called $\mathcal{C}$-admissible if $A = ad_{\mathcal{C}}(A)$, and

(iii) A is called $\mathcal{C}$-subadmissible if for each $D \in \langle A \rangle$, $ad_{\mathcal{C}}(D) \subset A$.

Remark 1. It is clear that if A_i is $\mathcal{C}$-subadmissible for each $i \in I$, then $\cap_{i \in I} A_i$ is $\mathcal{C}$-subadmissible.

The following is a main example of an abstract convex space.

Example 1. Let (M, d) be a bounded metric space, and A be a subset of M. Then

(i) $ad(A) = \cap\{B \subset M : B$ is a closed ball in M such that $A \subset B\}$.

(ii) a subset A is called admissible if $A = ad(A)$.

(iii) A is called subadmissible if for each $D \in \langle A \rangle$, $ad(D) \subset A$.

Let A be a nonempty subset of an abstract convex space $(X, \mathcal{C})$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$. Then A is called $\mathcal{C}$-almost subadmissible if for any $K = \{x_1, x_2, ..., x_n\} \in \langle A \rangle$ and for any $V \in \mathcal{N}$, there exists a mappping $h_{K,V} : K \to A$ such that $h_{K,V}(x_i) \in V[x_i]$ for all $i \in \{1, 2, ..., n\}$ and $ad_{\mathcal{C}}(h_{K,V}(K)) \subset A$. Moreover, we call the mapping $h_{K,V} : K \to A$ a $\mathcal{C}$-subadmissible-inducing mapping.

Remark 2. It is clear that every $\mathcal{C}$-subadmissible set must be $\mathcal{C}$-almost subadmissible, but the converse is not true.

Proposition 1. *Let $(X,\mathcal{C})$ be an abstract convex space which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$. If A is a nonempty $\mathcal{C}$-almost subadmissible subset of X and B is a nonempty open $\mathcal{C}$-subadmissible subset of X, then $A \cap B$ is $\mathcal{C}$-almost subadmissible.*

Proof. Let $K = \{x_1, x_2, ..., x_n\} \in \langle A \cap B \rangle$. Since B is open, there exists a $U \in \mathcal{N}$ such that $U[K] \subset B$. For any $V \in \mathcal{N}$ with $V \circ V \subset U$, there exists a $\mathcal{C}$-subadmissible-inducing mapping $h_{K,V} : K \to A$ such that $h_{K,V}(x) \in V[x]$ for all $x \in K$ and $ad_{\mathcal{C}}(h_{K,V}(K)) \subset A$, since A is $\mathcal{C}$-almost subadmissible. Since $h_{K,V}(K) \subset V[K] \subset B$ and B is $\mathcal{C}$-subadmissible, $ad_{\mathcal{C}}(h_{K,V}(K)) \subset B$. Thus $ad_{\mathcal{C}}(h_{K,V}(K)) \subset A \cap B$. ■

Remark 3. Let us note that the open condition of the above Proposition 1 is really needed. For instance, if we consider the metric space (M,d), $M = R^2$ and $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $x = (x_1, x_2), y = (y_1, y_2) \in M$, let $X = N(0,1) \cup \{(1,1),(1,-1),(-1,1),(-1,-1)\}$ and $Y = B(-2,1)$ be two nonempty subsets of M, then X is $\mathcal{C}$-almost subadmissible, Y is $\mathcal{C}$-subadmissible, but $X \cap Y = \{(-1,1),(-1,-1)\}$ is not $\mathcal{C}$-almost subadmissible.

Recently, Amini et al.[1] introduced the class of multifunctions with the KKM and $S - KKM$ properties in abstract convex spaces.

Definition 2. [1]. Let Z be a nonempty set, $(X,\mathcal{C})$ an abstract convex space, and Y a topological space. If $S : Z \to 2^X$, $T : X \to 2^Y$ and $F : Z \to 2^Y$ are three multifunctions satisfying

$$T(ad_{\mathcal{C}}(S(A))) \subset \cup_{x \in A} F(x), \text{ for each } A \in \langle Z \rangle,$$

then F is called a $\mathcal{C}$-S-KKM mapping with respect to T. If the multifunction $T : X \to 2^Y$ satisfies the requirement that for any $\mathcal{C}$-S-KKM mapping F with respect to T, the family $\{clF(x) : x \in Z\}$ has the finite intersection property, then T is said to have the S-KKM property with respect to $\mathcal{C}$. We define

$$S - KKM_{\mathcal{C}}(Z,X,Y) := \{T : X \to 2^Y | T \text{ has the } S - KKM \text{ property with respect to } \mathcal{C}\}$$

Remark 4. It is clear that if S is the identity mapping I, then S-$KKM_{\mathcal{C}}(X,X,Y)$ = $KKM_{\mathcal{C}}(X,Y)$.

Moreover, $KKM_{\mathcal{C}}(X,Y)$ is contained in S-$KKM_{\mathcal{C}}(Z,X,Y)$ for any $S : Z \to 2^X$.

Definition 3. Let X be a nonempty $\mathcal{C}$-almost subadmissible subset of an abstract convex space $(E, \mathcal{C})$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$, and Y a topological space. If $T : X \to 2^Y$ and $F : X \to 2^Y$ are two multifunctions such that for each $A \in \langle X \rangle$ and for each $V \in \mathcal{N}$, there exists a $\mathcal{C}$-subadmissible-inducing mapping $h_{A,V} : A \to X$ satisfying

$$T(ad_{\mathcal{C}}(h_{A,V}(A))) \subset F(A), \text{ for each } A \in \langle X \rangle,$$

then F is called a $\mathcal{C}$-KKM^* mapping with respect to T. If the multifunction $T : X \to 2^Y$ satisfies the requirement that for any generalized $\mathcal{C}$-KKM^* mapping F with respect to T, the family $\{clF(x) : x \in X\}$ has the finite intersection property, then T is said to have the $\mathcal{C}$-KKM^* property with respect to $\mathcal{C}$. We define

$$KKM^*_{\mathcal{C}}(X, Y) := \{T : X \to 2^Y | T \text{ has the } KKM^* \text{ property with respect to } \mathcal{C}\}$$

The Φ-mapping and the Φ-spaces, in an abstract convex space setting, were also introduced by Amini et al.[1].

Definition 4. [1]. Let $(X, \mathcal{C})$ be an abstract convex space, and Y a topological space. A map $T : Y \to 2^X$ is called a Φ-mapping if there exists a multifunction $F : Y \to 2^X$ such that

(i) for each $y \in Y$, $A \in \langle F(y) \rangle$ implies $ad_{\mathcal{C}}(A) \subset T(y)$, and

(ii) $Y = \cup_{x \in X} intF^{-1}(x)$.

The mapping F is called a companion mapping of T.

Furthermore, if the abstract convex space $(X, \mathcal{C})$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$, then X is called a Φ-space if for each entourage $V \in \mathcal{N}$, there exists a Φ-mapping $T : X \to 2^X$ such that $\mathcal{G}_T \subset V$.

Remark 5.

(i) If $T : Y \to 2^X$ is a Φ-mapping, then for each nonempty subset Y_1 of Y, $T|_{Y_1} : Y_1 \to X$ is also a Φ-mapping.

(ii) It is easy to see that if $X_1 \subset X$ and $\mathcal{C}_1 = \{C \cap X_1 : C \in \mathcal{C}\}$, then $(X_1, \mathcal{C}_1)$ is also a Φ-space.

Definition 5. An abstract convex space $(X, \mathcal{C})$ is said to be a locally abstract convex space if X is a uniform topological space with uniformity $\mathcal{U}$ which has an open basis $\mathcal{N} = \{V_i : i \in I\}$of symmetric encourages such that for each $V \in \mathcal{N}$, the set $V[x]$ is an $\mathcal{C}$-subadmissible subset of X.

The measure of noncompactness of topological vector spaces were introduced in [2]. In the following, we extend the definition to the abstract convex spaces.

Definition 6. Let $(X, \mathcal{C})$ be an abstract convex space and $\alpha : 2^X \to \Re^+$, where $\Re^+$ denote the set of all nonegative real nunbers. α is called a measure of noncompactness with respect to $\mathcal{C}$ provided that the following conditions hold.

(i) $\alpha(ad_{\mathcal{C}}(A)) = \alpha(A)$ for each $A \in 2^X$,

(ii) $\alpha(A) = 0$ if and only if A is precompact, and

(iii) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$, for each $A, B \in 2^X$.

Remark 6. It is clear that if $A \subset B$, then $\alpha(A) \leq \alpha(B)$.

In the sequel, we let $\Psi = \{\psi : \Re^+ \to \Re^+ : \psi$ is upper semicontinuous with $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0\ \}$. The following proposition have showed by Chen [6], and it plays an important role for this paper.

Proposition 2. *If $\psi \in \Psi$, then there exists a strictly increasing, continuous function $\alpha : \Re^+ \to \Re^+$ such that $\psi(t) \leq \alpha(t) < t$ for all $t > 0$.*

Remark 7. In above Proposition 2, the function α is invertible. If for each $t > 0$, we denote $\alpha^0(t) = 0$ and $\alpha^{-n}(t) = \alpha^{-1}(\alpha^{-n+1}(t))$ for each $n \in N$, then we have $\lim_{n\to\infty} \alpha^{-n}(t) = \infty$, that is; $\lim_{n\to\infty} \alpha^n(t) = 0$. Moreover, we also conclude that $\lim_{n\to\infty} \alpha^n(t) = 0$.

Proof. Let $t > 0$. Suppose that $\lim_{n\to\infty} \psi^{-n}(t) = \eta$ for some positive real number η. Then

$$\eta = \lim_{n\to\infty} \alpha^{-n}(t) = \alpha^{-1}(\lim_{n\to\infty} \alpha^{-n+1}(t)) = \alpha^{-1}(\eta) > \eta,$$

which is a contradiction. ■

Definition 7. Let $(X, \mathcal{C})$ be an abstract convex space. A mapping $T : X \to 2^X$ is said to be a generalized Ψ-set contraction mapping with respect to $\mathcal{C}$, if, there exists an $\psi \in \Psi$ such that for each $A \subset X$ with A bounded, $T(A)$ is bounded and $\alpha(T(A)) \leq \psi(\alpha(A))$.

2. Main Results

The following theorem that due to Amini et al. [1], will help us to get two fixed point theorems for the generalized Ψ-set contraction mapping.

Theorem 1. [1]. *Let $(X, \mathcal{C})$ be a Φ-space and $s : X \to X$ be a surjective function. Suppose that $T \in s - KKM_{\mathcal{C}}(X, X, X)$ is compact and closed. Then T has a fixed point.*

We now establish the main fixed point theorem for this paper, as follows:

Theorem 2. *Let $(X, \mathcal{C})$ be a bounded abstract convex space. If $T : X \to 2^X$ is a generalized Ψ-set contraction mapping with respect to $\mathcal{C}$, then there exists a nonempty precompact $\mathcal{C}$-subadmissible subset K of X such that $T(K) \subset K$.*

Proof. Since T is a generalized Ψ-set contraction mapping, there exists an $\psi \in \Psi$ such that $\alpha(T(A)) \leq \psi(\alpha(A))$ for each $A \subset X$. Take $x_0 \in X$, and we let

$$X_0 = X, \quad X_1 = ad_{\mathcal{C}}(T(X_0) \cup \{x_0\}), \text{ and}$$

$$X_{n+1} = ad_{\mathcal{C}}(T(X_n) \cup \{x_0\}), \text{ for each } n \in N.$$

Then

(1) $X_{n+1} \subset X_n$, for each $n \in N$,
(2) $T(X_n) \subset X_{n+1}$, for each $n \in N$, and
(3) X_n is $\mathcal{C}$-subadmissible, for each $n \in N$.

We claim that $\alpha(X_{n+1}) \leq \psi^{n+1}(\alpha(X_0))$

Since

$$\alpha(X_{n+1}) \leq \alpha(ad_{\mathcal{C}}(T(X_n) \cup \{x_0\}) \leq \alpha(T(X_n)), \text{ and}$$

$$\alpha(T(X_n)) \leq \psi(\alpha(X_n)), \text{ for each } n \in N,$$

we have

$$\alpha(X_{n+1}) \leq \psi^{n+1}(\alpha(X_0)).$$

Thus $\alpha(X_n) \to 0$, as $n \to \infty$. Let $X_\infty = \cap_{n \geq 1} X_n$. Then X_∞ is a nonempty precompact $\mathcal{C}$-subadmissible subset of X. Moreover, by (i) and (ii), we also have that $T(X_\infty) = T(\cap_{n \geq 1} X_n) \subset X_\infty$.This completes the proof. ∎

Remark 8. In the process of the proof of Theorem 2 , we call the set X_∞ the precompact-inducing $\mathcal{C}$-subadmissible subset of X, and in the sequel, we always denote X_∞ be this set.

By Theorem 1 and Theorem 2, we can conclude the following fixed point theorem.

Theorem 3. *Let $(X, \mathcal{C})$ be a bounded Φ-space and let $s : X \to X$ be a single-valued function with $s(X_\infty) = X_\infty$. If $T \in s - KKM_\mathcal{C}(X, X, X)$ is generalized Ψ-set contraction with respect to $\mathcal{C}$ and closed , then T has a fixed point in X.*

Proof. By Theorem 2, we get a precompact-inducing $\mathcal{C}$-subadmissible subset X_∞ of X with $T(X_\infty) \subset X_\infty$, and we can conclude that $\alpha(T(X_n)) \to 0$, as $n \to \infty$, hence $T(X_\infty)$ is a precompact subset of X_∞. By the definition of the function s, we have that $s(X_\infty) = X_\infty$ and $T|_{X_\infty} \in s - KKM_\mathcal{C}(X_\infty, X_\infty, X_\infty)$, since $T \in s - KKM_\mathcal{C}(X, X, X)$ and $s(X_\infty) = X_\infty$.

By Remark 5, we let $\mathcal{C}_1 = \{C \cap X_\infty : C \in \mathcal{C}\}$, then $(X_\infty, \mathcal{C}_1)$ is also a Φ-space. Let $\mathcal{N}$ be a basis of the uniform structure of X_∞, and let $V \in \mathcal{N}$. Then there exists a Φ-mapping $F : X_\infty \to 2^{X_\infty}$ such that $\mathcal{G}_F \subset V$. Since F is a Φ-mapping, there exists a companion mapping $G : X_\infty \to 2^{X_\infty}$ such that $X_\infty = \cup_{x \in X_\infty} intG^{-1}(x)$. Let $K = \overline{T(X_\infty)}$. Then there exists a finite subset A of X_∞ such that $K \subset \cup_{x \in A} intG^{-1}(x)$. Since $s(X_\infty) = X_\infty$, there exists a finite subset B of X_∞ such that $K \subset \cup_{z \in B} intG^{-1}(s(z))$. Now, we define $P : X_\infty \to 2^{X_\infty}$ by

$$P(z) = K \backslash intG^{-1}(s(z)), \text{ for each } z \in X_\infty.$$

By the definition of P, we obtain that P is not a $\mathcal{C}$-s-KKM mapping with respect to $T|_{X_\infty}$. Hence, there exists $N = \{z_1, z_2, ..., z_k\} \subset X_\infty$ such that $T(ad_\mathcal{C} s(N)) \nsubseteq \cup_{i=1}^k P(z_i)$. So, there exist $x \in ad_\mathcal{C} s(N)$ and $y \in T(x)$ such that $y \notin \cup_{i=1}^k P(z_i)$. Consequently, $y \in \cap_{i=1}^k intG^{-1}(z_i)$, and so $s(z_i) \in G(y)$ for all $i = 1, 2, .., k$. Since F is a Φ-mapping, we have $ad_\mathcal{C} s(N) \subset F(y)$, and so $x \in F(y)$, ie $(x, y) \in \mathcal{G}_F \subset V$. Therefore, $y \in V[x] \cap T(x)$, and we are easy to prove that T has a fixed point in X. ∎

Corollary 1. *Let $(X, \mathcal{C})$ be a bounded Φ-space, and let $T \in KKM_\mathcal{C}(X, X)$ be generalized Ψ-set contraction with respect to $\mathcal{C}$ and closed. Then T has a fixed point in X.*

Lemma 1. *Let $(X, \mathcal{C})$ be an abstract convex space, and Y a topological space. Then $T|_D \in KKM_\mathcal{C}(D, Y)$ whenever $T \in KKM_\mathcal{C}(X, Y)$ and D is a $\mathcal{C}$-subadmissible subset of X.*

Proof. The proof is similar to one given by Chang and Yen [4]. ∎

Lemma 2. *Let Z be a nonempty set, $(X, \mathcal{C})$ an abstract convex space, and Y, W a topological space. If $T \in S - KKM_\mathcal{C}(Z, X, Y)$, then $fT \in S - KKM_\mathcal{C}(Z, X, W)$ for each $f \in C(Y, W)$.*

Proof. The proof is similar to one given by Chang et al.[3]. ∎

Theorem 4. *Let X be a nonempty $\mathcal{C}$-subadmissible subset of a locally abstract convex space $(E, \mathcal{C})$, and let $s : X \to X$ be a single-valued mapping. If $T \in s - KKM_{\mathcal{C}}(X, X, X)$ is compact and closed with $\overline{T(X)} \subset s(X)$, then T has a fixed point in X.*

Proof. Since E is a locally abstract convex space, there exists a uniform structure $\mathcal{U}$. Let $\mathcal{N}$ be an open symmetric base family for the uniform structure $\mathcal{U}$ such that for any $U \in \mathcal{N}$, the set $U[x] = \{y \in X : (x, y) \in U\}$ is an open $\mathcal{C}$-subadmissible subset of E for each $x \in X$.

We now claim that for any $V \in \mathcal{N}$, there exists $x_V \in X$ such that $V[x_V] \cap T(x_V) \neq \phi$. Suppose it is not the case, then there is an $V \in \mathcal{N}$ such that $V[x_V] \cap T(x_V) = \phi$, for all $x_V \in X$. Since T is compact, hence $K = \overline{T(X)}$ is a compact subset of X. Define $F : X \to 2^X$ by

$$F(x) = K \backslash V[s(x)] \quad \text{for each } x \in X.$$

We will show that

(i) $F(x)$ is nonempty and closed for each $x \in X$, and

(ii) F is a $\mathcal{C}$-s-KKM generalized mapping with respect to T.

(1) is obviuos. To prove (2), we use the contradiction. Suppose, there exists $A = \{x_1, x_2, ..., x_n\} \in \langle X \rangle$ such that $T(ad_{\mathcal{C}}(s(A))) \nsubseteq F(A)$. Then there exists $y \in ad_{\mathcal{C}}(s(A))$, $z \in T(y)$, and $z \notin F(A)$. Since $z \notin F(A)$, $z \notin \cup_{i=1}^{n}(K \backslash V[s(x_i)])$, and so $z \in V[s(x_i)]$ for each $i \in \{1, 2, ..., n\}$, that is; $(s(x_i), z) \in V$ for each $i \in \{1, 2, ..., n\}$. Since V is symmetric, we conclude that $(z, s(x_i)) \in V$ and $s(x_i) \in V[z]$ for each $i \in \{1, 2, ..., n\}$. Furthermore, $ad_{\mathcal{C}}(\{s(x_1), s(x_2, ..., s(x_n))\}) \subset V[z]$, since $V[z]$ is $\mathcal{C}$-subadmissible. Hence, $y \in V[z]$, $z \in V[y]$, and so we have $z \in T(y) \cap V[y]$. This contradicts with $T(y) \cap V[y] = \phi$ for each $y \in X$.

Since $T \in s - KKM_{\mathcal{C}}(X, X, X)$ and K is compact, so $\cap_{x \in X} F(x) \neq \phi$. Let $\eta \in \cap_{x \in X} F(x) \subset K = \overline{T(X)} \subset s(X)$, then there exists $\xi \in X$ such that $s(\xi) = \eta$. So we have $\eta \in F(\xi) = K \backslash V[s(\xi)] = K \backslash V[\eta]$, that is; $(\eta, \eta) \notin V$. So we get a contradiction. Therefore, we have proved that for each $V_i \in \mathcal{N}$, there exists $x_{V_i} \in X$ such that $V_i[x_{V_i}] \cap T(x_{V_i}) \neq \phi$. Let $y_{V_i} \in V_i[x_{V_i}] \cap T(x_{V_i})$. Then $(x_{V_i}, y_{V_i}) \in V$ and $(x_{V_i}, y_{V_i}) \in \mathcal{G}_T$. Since T is compact, we may assume that $\{y_{V_i}\}_{i \in I}$ converges to y_0 in X. Now, for $W \in \mathcal{N}$, take $U \in \mathcal{N}$ such that $U \circ U \subset W$. Since $y_{V_i} \to y_0$, there exists $U_0 \in \mathcal{N}$ with $U_0 \subset U$ such that $y_{V_i} \in U[y_0]$ for $V_i \in \mathcal{N}$ with $V_i \subset U_0$, that is; $(y_{V_i}, y_0) \in U$ for $V_i \in \mathcal{N}$ with $V_i \subset U_0$. So we have $(x_{V_i}, y_0) = (x_{V_i}, y_{V_i}) \circ (y_{V_i}, y_0) \in U \circ U \subset W$, that is; $x_{V_i} \in W[y_0]$ for $V_i \in \mathcal{N}$ with $V_i \subset U_0$. This shows that $x_{V_i} \to y_0$. Since T is closed, we have $(y_0, y_0) \in \mathcal{G}_T$, so $y_0 \in T(y_0)$. We complete the proof. ∎

By Theorem 4, we also conclude the following fixed point theorem for the generalized Ψ-set contraction mapping.

Theorem 5. *Let X be a nonempty bounded $\mathcal{C}$-subadmissible subset of a locally abstract convex space $(E, \mathcal{C})$, and let $s : X \to X$ be a single-valued mapping with $s(X_\infty) = X_\infty$. If $T \in s - KKM_{\mathcal{C}}(X, X, X)$ is generalized Ψ-set contraction with respect to $\mathcal{C}$ and closed with $\overline{T(X_\infty)} \subset s(X_\infty)$, then T has a fixed point in X.*

Proof. By Theorem 2, we get $T(X_\infty) \subset X_\infty$, and we can conclude that $\alpha(T(X_n)) \to 0$, as $n \to \infty$, hence $T(X_\infty)$ is a precompact subset of X_∞. By the definition of the function s, we have that $s(X_\infty) = X_\infty$ and $T|_{X_\infty} \in s - KKM_{\mathcal{C}}(X_\infty, X_\infty, X_\infty)$, since $T \in s - KKM_{\mathcal{C}}(X, X, X)$ and $s(X_\infty) = X_\infty$.

Let $K = \overline{T(X_\infty)}$, and we define $F : X_\infty \to 2^{X_\infty}$ by

$$F(x) = K \backslash V[s(x)] \qquad \text{for each } x \in X_\infty.$$

The remainder proof is similar to Theorem 4, we omit it. ■

Next, we use the other proof's skill to get a precompact-inducing $\mathcal{C}$-almost subadmissible subset X_∞ of an abstract convex space X, and then, we establish the fixed point theorems for the generalized Ψ-set contraction mapping having the $\mathcal{C}$-$KKM^*_{\mathcal{C}}(X, Y)$ property on this $\mathcal{C}$-almost subadmissible set.

Theorem 6. *Let X be a nonempty $\mathcal{C}$-almost subadmissible subset of a locally abstract convex space $(E, \mathcal{C})$. If $T \in KKM^*_{\mathcal{C}}(X, X)$ is compact and closed, then T has a fixed point in X.*

Proof. The proof is analogous to the proof of Theorem 2.5 of Chen et al.[5], we omit it. ■

By Theorem 6, we also conclude the following fixed point theorem.

Theorem 7. *Let X be a nonempty bounded $\mathcal{C}$-almost subadmissible subset of a locally abstract convex space $(E, \mathcal{C})$. If $T \in KKM^*_{\mathcal{C}}(X, X)$ is generalized Ψ-set contraction with respect to $\mathcal{C}$ and closed with $\overline{T(X)} \subset X$ and $intT(x) \neq \phi$ for each $x \in X$, then T has a fixed point in X.*

Proof. Since T is a generalized Ψ-set contraction mapping, there exists an $\psi \in \Psi$ such that $\alpha(T(A)) \leq \psi(\alpha(A))$ for each $A \subset X$. Take $x_0 \in X$, and we let

$$X_0 = X, \;\; X_1 = intad_{\mathcal{C}}(T(X_0 \cup \{x_0\})) \cap X, \text{ and}$$
$$X_{n+1} = intad_{\mathcal{C}}(T(X_n \cup \{x_0\})) \cap X, \text{ for each } n \in N.$$

Then

(i) $X_{n+1} \subset X_n$, for each $n \in N$, and

(ii) by Proposition 1, X_n is $\mathcal{C}$-almost subadmissible, for each $n \in N$.

We now claim that $\alpha(X_{n+1}) \leq \psi^{n+1}(\alpha(X_0))$. Since

$$\alpha(T(X_n)) \leq \psi(\alpha(X_n)), \text{ for each } n \in N, \text{ and}$$
$$\alpha(X_{n+1}) \leq \alpha(intad_{\mathcal{C}}(T(X_n \cup \{x_0\})) \leq \alpha(ad_{\mathcal{C}}(T(X_n \cup \{x_0\})) \leq \alpha(T(X_n)),$$

we have

$$\alpha(X_{n+1}) \leq \psi^{n+1}(\alpha(X_0)).$$

Thus $\alpha(X_n) \to 0$, as $n \to \infty$. Let $X_\infty = \cap_{n \geq 1} X_n$. Then X_∞ is a nonempty precompact $\mathcal{C}$-almost subadmissible subset of X. Moreover, we also conclude that $\alpha(T(X_n)) \to 0$, as $n \to \infty$, and so $\overline{T(X_\infty)}$ is a compact subset of X. The remainder conclusion follows from Theorem 6. ■

Corollary 2. *Let X be a nonempty bounded $\mathcal{C}$-subadmissible subset of a locally abstract convex space $(E, \mathcal{C})$. If $T \in KKM_{\mathcal{C}}(X, X)$ is generalized Ψ-set contraction with respect to $\mathcal{C}$ and closed with $\overline{T(X)} \subset X$, then T has a fixed point in X.*

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