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EXTENDED JACOBSON DENSITY THEOREM FOR GRADED RINGS WITH DERIVATIONS AND AUTOMORPHISMS

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Abstract. We introduce and study M-outer derivations and automorphisms of graded rings. We prove a version of the Chevalley-Jacobson density theorem for graded rings with such derivations and automorphisms.

1. Introduction

Chevalley-Jacobson density theorem is one of the important tools in ring theory. This celebrated theorem has been generalized in various directions. For example, Liu, Beattie and Fang proved the density theorem for primitive rings graded by groups in [14], and Beidar and Brešar considered the density theorem for rings with derivations and automorphisms in [3]. The purpose of this paper is to generalize some results in [3] to graded rings. We shall prove a version of the density theorem for graded rings with M-outer derivations and automorphisms.

In the next section, we set the basic terminology and prove the density theorem for gr-local modules. In Section 3 and 4, we study the density theorems concerning automorphisms and derivations, respectively. In Section 5, we prove a version of the Chevalley-Jacobson density theorem for graded rings with derivations and automorphisms. In Section 6, some applications of the density theorems concerning derivations in graded rings are presented.

The method used here is very similar to that in [3]. In fact, most of the technical details has already been presented in [3].

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2. Density and gr-Local Modules

Let G be a group with identity e. A ring A called a *G-graded ring* if $A = \bigoplus_{g\in G} A_g$, where each A_g is an additive subgroup of A, and $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$. A left A-module M is said to be *graded* if $M = \bigoplus_{g\in G} M_g$, where each M_g is an additive subgroup of M, and $A_gM_h \subseteq M_{gh}$ for all $g, h \in G$. We denote by $h(A)$ the set of homogeneous elements of the graded ring A and $h(M)$ the set of homogeneous elements of the graded module M, i.e. $h(A) = \bigcup_{g\in G} A_g$ and $h(M) = \bigcup_{g\in G} M_g$. A submodule N of M is called a *graded submodule* if $N = \bigoplus_{g\in G} \left(N \bigcap M_g\right)$. A graded left A-module M is said to be *gr-simple* if $AM \neq 0$ and M has no nonzero proper graded submodules.

Consider graded A-modules M and N. An A-linear map $\phi : M \to N$ is said to be a *graded morphism of degree* g, $g \in G$, if $(M_h)\phi \subseteq N_{hg}$ for all $h \in G$. The graded morphisms of degree g form an additive subgroup HOM$({}_AM, {}_AN)_g$ of Hom$({}_AM, {}_AN)$. The group $\bigoplus_{g\in G}$ HOM$({}_AM, {}_AN)_g$ will be denoted by HOM$({}_AM, {}_AN)$. Further, we set END$({}_AM)$ = HOM$({}_AM, {}_AM)$, and note that END$({}_AM)$ is a graded ring with unity under the pointwise addition and composition of maps. It is well known that if M is finitely generated or G is a finite group, then HOM$({}_AM, {}_AN)$ = Hom$({}_AM, {}_AN)$ [15, Corollary A.I.2.11.].

A graded ring D is a *gr-division ring* if D has a unity 1 in D_e and every nonzero element of $h(D)$ is invertible. If M is a gr-simple graded A-module, then $D =$ END$({}_AM)$ is a gr-division ring and M becomes a graded right D-module.

We recall that a set $\{m_1, m_2, \cdots, m_n\} \subseteq h(M)$ is not D-independent if there exist $d_i \in h(D)$, not all zero, such that $\sum_{i=1}^n m_id_i = 0$. By [14, Corollary 1.4] we know that every homogeneous basis of M has the same cardinality, and thus the notion of dimension, denoted by dim(M_D), of the graded module M over the gr-division ring D makes sense. It is easy to see that a set $\{m_1, m_2, \cdots, m_n\} \subseteq M_g$ is D-independent if and only if the set is D_e-independent. For if $\sum_{i=1}^n m_id_i = 0$, then we may assume that $d_i \in D_h$ for some $h \in G$. Without loss of generality, we may assume $d_1 \neq 0$. Then $\sum_{i=1}^n m_id_id_1^{-1} = 0$ with each $d_id_1^{-1} \in D_e$.

Throughout the paper, A will be a G-graded ring with a gr-simple module M and $D =$ END$({}_AM)$. Since we are considering density theorem in graded cases, throughout this paper we shall omit the adjective 'graded' for graded modules.

In [14], Liu et al. proved the density theorem for primitive rings graded by groups. The following result is a corollary of [14, Theorem 2.5].

Theorem 2.1. *Let M be a gr-simple A-module and $D =$ END$({}_AM)$. Let x_1, x_2, $\ldots$, $x_k \in h(M)$ be D-independent, and let $y_1, y_2, \ldots, y_k \in M$. Then there* *exists an element* $a \in A$ *such that* $ax_i = y_i$ *for all* $i = 1, 2, \ldots, k$.

First we generalize Theorem 2.1.

Theorem 2.2. *Let* M_i $(i = 1, 2, \ldots, n)$ *be pairwise nonisomorphic gr-simple* A*-modules and* $D_i = END({}_AM_i)$. *Further, for positive integers* $k_1, k_2, \ldots, k_n$ *and* $i = 1, 2, \ldots, n$, *let* $x_{i1}, x_{i2}, \ldots, x_{ik_i} \in h(M_i)$ *be* D_i*-independent, and* $y_{i1}, y_{i2}, \ldots, y_{ik_i} \in M_i$. *Then there exists an element* $a \in A$ *such that for every* i, $ax_{ij} = y_{ij}$ *for all* $j = 1, 2, \ldots, k_i$.

Proof. We proceed by induction on n. The case $n = 1$ is exactly Theorem 2.1. Assume $n > 1$. We fix an element $u \in A$ such that $ux_{nj} = y_{nj}$ for $j = 1, 2, \ldots, k_n$. Next we set $B = \{a \in A \mid ax_{ni} = 0 \text{ for all } i = 1, 2, \ldots, k_n\}$. Clearly B is a left ideal of A.

Now we define a module homomorphism $\rho : A \to M_n^{k_n}$ by

$$a^\rho = (ax_{n1}, ax_{n2}, \ldots, ax_{nk_n}) \quad \text{for all } a \in A.$$

Since $x_{n1}, x_{n2}, \ldots, x_{nk_i} \in h(M_n)$ are D_n-independent, it follows from Theorem 2.1 that ρ is surjective. Given $g \in G$, we set

$$(M_n^{k_n})_g = \{(ax_{n1}, ax_{n2}, \ldots, ax_{nk_n}) \mid a \in A_g\} = (A_gx_{n1}, A_gx_{n2}, \ldots, A_gx_{nk_n}).$$

It is clear that $M_n^{k_n} = \bigoplus_{g \in G}(M_n^{k_n})_g$ is an A-module.

Let $1 \leq i < n$ and $x \in h(M_i) \setminus \{0\}$. We claim that $Bx \neq 0$. Assume the contrary. Then the map $\tau : A \to M_i$, given by the rule of $a^\tau = ax$, is an epimorphism of modules with $B \subseteq \text{Ker}(\tau)$. Since $\text{Ker}(\rho) = B \subseteq \text{Ker}(\tau)$, there exists an epimorphism $\sigma : M_n^{k_n} \to M_i$ of modules with $\tau = \rho\sigma$.

Let $N_j = (0, \ldots, 0, Ax_{nj}, 0, \ldots, 0)$ where $1 \leq j \leq k_n$. Then N_j is a submodule of $M_n^{k_n}$ for all j, $1 \leq j \leq k_n$. Since σ is an epimorphism of nonzero modules $M_n^{k_n}$ and M_i, there exists a j, $1 \leq j \leq k_n$, such that $N_j\sigma \neq 0$. Without loss of generality, we may assume $j = 1$. Let σ' be the restriction of σ to N_1. Then $N_1\sigma' = N_1\sigma$ is a nonzero submodule of M_i. Since M_i is gr-simple, we have $\text{Ker}(\sigma') = 0$ and $\text{Im}(\sigma') = N_1\sigma' = M_i$. That is, $M_i \cong N_1 = (Ax_{n1}, 0, \ldots, 0) \cong M_n$, a contradiction.

Therefore $Bx \neq 0$ for all $x \in h(M_i) \setminus \{0\}$, $1 \leq i < n$. Since B is a left ideal of A, Bx is a submodule of M_i. As each M_i is gr-simple, we have $Bx = M_i$ for all $x \in h(M_i) \setminus \{0\}$ and can be regarded as a gr-simple B-module. Clearly

END$({}_AM_i) \subseteq$ END$({}_BM_i)$. Also, we have END$({}_AM_i) \supseteq$ END$({}_BM_i)$. Indeed, let $\phi \in$ END$({}_BM_i)$, $a \in A$ and $m \in M_i$. Then $m = bx$ for some $b \in B$ and

$$(am)^\phi = (abx)^\phi = ab(x)^\phi = a(bx)^\phi = a(m)^\phi.$$

Hence $\phi \in$ END$({}_AM_i)$ and END$({}_AM_i) =$ END$({}_BM_i)$.

Now by the inductive assumption, there exists an element $v \in B$ with $vx_{ij} = y_{ij} - ux_{ij}$ for all $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, k_i$. Hence $a = u + v$ is the desired element and the proof is completed. ■

The goal of this section is to prove the density theorem for gr-local modules. Recall that a submodule L of M is *gr-maximal* if and only if M/L is a gr-simple module. A left A-module M is said to be *gr-local*, if it contains a unique gr-maximal submodule L and every proper submodule of M is contained in L.

We also recall that the *gr-Jacobson radical* of a module M of A, denoted by $J(M)$, is the intersection of all gr-maximal submodules K of M such that M/K is a gr-simple A-module. If M has no such submodules, set $J(M) = M$.

The next lemma gives some results on $J(M)$ of a gr-local module M.

Lemma 2.3. *Let M, N and N_i, $1 \leq i \leq n$, be left A-modules, and L a submodule of M.*

- (a) *If $\alpha \in$ HOM$({}_AM, {}_AN)$ is an epimorphism, then $J(M)\alpha \subseteq J(N)$.*
- (b) *$J(\bigoplus_{i=1}^n N_i) = \bigoplus_{i=1}^n J(N_i)$.*
- (c) *If M has a generating set $\{x_1, x_2, \ldots, x_n\} \subseteq h(M)$ such that $x_i \in Ax_i$ for all i, and L is a submodule of M such that $L + J(M) = M$, then $L = M$.*
- (d) *M is gr-local with gr-maximal submodule L if and only if $Ax = M$ for all $x \in h(M)\backslash h(L)$. Moreover, if M is a gr-local module with gr-maximal submodule L, then $J(M) = L$.*

Proof.

- (a) Let $\alpha \in$ HOM$({}_AM, {}_AN) = \bigoplus_{g\in G}$ HOM$({}_AM, {}_AN)_g$. Then $\alpha = \sum_{g\in G} \alpha_g$ for $\alpha_g \in$ HOM$({}_AM, {}_AN)_g$ and so $(M_h)\alpha_g \subseteq N_{hg}$ for all $h \in G$. Let K be a gr-maximal submodule of N. Then $K\alpha_g^{-1}$ is a gr-maximal submodule of M. Thus we have $J(M) \subseteq K\alpha_g^{-1}$, and so $J(M)\alpha_g \subseteq K$, for all $g \in G$. Hence $J(M)\alpha = J(M)(\sum_{g\in G} \alpha_g) = \sum_{g\in G}(J(M)\alpha_g) \subseteq K$. This shows that $J(M)\alpha \subseteq J(N)$.

(b) Applying (a) to the canonical projections $\pi_j : \bigoplus_{i=1}^n N_i \to N_j$, we have

$$J\left(\bigoplus_{i=1}^n N_i\right)\pi_j \subseteq J(N_j).$$

Thus $\bigoplus_{j=1}^n \big(J\big(\bigoplus_{i=1}^n N_i\big)\pi_j\big) \subseteq \bigoplus_{j=1}^n J(N_j)$ and so $J\big(\bigoplus_{i=1}^n N_i\big) \subseteq \bigoplus_{j=1}^n J(N_j)$. On the other hand, let $\gamma \in \mathrm{HOM}(\bigoplus_{i=1}^n N_i, K)_g$ is an epimorphism onto a gr-simple A-module K for some $g \in G$, and let γ_i be the restriction of γ to N_i. Since K is gr-simple, either $(N_i)\gamma_i = 0$ or $(N_i)\gamma_i = K$. If $(N_i)\gamma_i = 0$, then $J(N_i) \subseteq N_i \subseteq \mathrm{Ker}(\gamma_i) \subseteq \mathrm{Ker}(\gamma)$. And if $(N_i)\gamma_i = K$, then $N_i/\mathrm{Ker}(\gamma_i) \cong K$ and so $\mathrm{Ker}(\gamma_i)$ is a gr-maximal submodule of N_i. Thus $J(N_i) \subseteq \mathrm{Ker}(\gamma_i) \subseteq \mathrm{Ker}(\gamma)$ for all i. Therefore $\bigoplus_{i=1}^n J(N_i) \subseteq \mathrm{Ker}(\gamma)$ and by [15, Lemma A.I.7.4.] we have $\bigoplus_{i=1}^n J(N_i) \subseteq J(\bigoplus_{i=1}^n N_i)$.

(c) Assume that $L \neq M$. Since M is finitely generated, by Zorn's Lemma, there exists a gr-maximal submodule N of M containing L. Clearly $x_i \notin N$ for some i and so $Ax_i \nsubseteq N$. It follows that M/N is a gr-simple module. But then $J(M) \subseteq N$, contradicting $L + J(M) = M$.

(d) Suppose that M is a gr-local with a gr-maximal submodule L and $x \in h(M)\setminus h(L)$. Since M/L is a gr-simple A-module, $Ax \nsubseteq L$ and so $Ax = M$. On the other hand, if $Ax = M$ for all $x \in h(M)\setminus h(L)$, then M/L is a gr-simple A-module and every proper submodule of M is contained in L. Hence M is a gr-local A-module. In particular $J(M) = L$. ■

We now have the main theorem of this section.

Theorem 2.4. *Let A be a graded ring. For $i = 1, \ldots, n$, let M_i be a gr-local left A-module with a gr-maximal submodule L_i, and let $N_i = M_i/L_i$ and $D_i = END({}_AN_i)$. Further, let $n, m_1, m_2, \ldots, m_n$ be positive integers, and for each i, $1 \leq i \leq n$, let $x_{i1}, x_{i2}, \ldots, x_{im_i}$ be homogeneous elements of M_i linearly independent over D_i modulo L_i. Suppose that $N_i \ncong N_j$ if $i \neq j$. Then for any $y_{i1}, y_{i2}, \ldots, y_{im_i} \in M_i$, $1 \leq i \leq n$, there exists $a \in A$ such that $ax_{ij} = y_{ij}$ for all i and j.*

Proof. Set $M = \bigoplus_{i=1}^n (M_i^{m_i})$ where $M_i^{m_i}$ is the direct sum of m_i copies of M_i. Let

$$\overline{x} = (x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n}) \quad \text{and}$$
$$\overline{y} = (y_{11}, \ldots, y_{1m_1}, y_{21}, \ldots, y_{2m_2}, \ldots, y_{n1}, \ldots, y_{nm_n}).$$

According to Lemma 2.3 (b), $J(M) = J\big(\bigoplus_{i=1}^n M_i^{m_i}\big) = \bigoplus_{i=1}^n J(M_i^{m_i})$ and so $M/J(M) = \bigoplus_{i=1}^n \big[M_i/J(M_i)\big]^{m_i}$. It follows from Theorem 2.2 that $A(\overline{x} + J(M)) = M/J(M)$ and so $A\overline{x} + J(M) = M$. Now by Lemma 2.3 (c), we have $A\overline{x} = M$. In particular there exists $a \in A$ with $a\overline{x} = \overline{y}$. Thus $ax_{ij} = y_{ij}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m_i$. ∎

3. Density and M-Outer Automorphisms

Let A be a graded ring with a gr-simple left A-module M and $D = \text{END}({}_AM)$. An automorphism α of A is called a *graded automorphism* if $(A_g)\alpha = A_g$ for all $g \in G$. With an automorphism α of A, we define an A-module M_α as follows. As an abelian group, $M_\alpha = M$. Next $(M_\alpha)_g = M_g$ for all $g \in G$. Given $a \in A$ and $x \in M_\alpha$ we set $a *_\alpha x = a^\alpha x$. Clearly M_α is a gr-simple left A-module, and

$$(a *_\alpha x)u = (a^\alpha x)u = a^\alpha(xu) = a *_\alpha (xu)$$

for all $u \in D$, $a \in A$ and $x \in M_\alpha$. Therefore there exists a monomorphism of graded rings $D \to \text{END}({}_AM_\alpha)$. Since ${}_AM \cong {}_A(M_\alpha)_{\alpha^{-1}}$, we conclude that $D \cong \text{END}({}_AM_\alpha)$. Henceforward we shall identify them, i.e.,

$$D = \text{END}({}_AM_\alpha). \tag{1}$$

Let $g \in G$. Consider M as an additive group and denote by $\text{END}(M)_g$ the set of all endomorphisms $\psi : M \to M$ such that $\psi(M_h) \subseteq M_{hg}$ for all $h \in G$. Set $\text{END}(M) = \bigoplus_{g\in G} \text{END}(M)_g$ and consider $\text{END}(M_D)$ as a subring of $\text{END}(M)$. Given $a \in A$, we define a map $L_a : M \to M$ by $L_a x = ax$ for all $x \in M$. Clearly, if $a \in h(A)$, then $L_a \in \text{END}(M_D)$.

Now we introduce the concepts of M-inner and M-outer automorphisms.

Definition 3.1. An automorphism α of a graded ring A is called M*-inner* if there exists an invertible homogeneous element $T \in \text{END}(M)$ such that

$$TL_aT^{-1} = L_{a^\alpha} \quad \text{for all } a \in A;$$

otherwise it is called M*-outer*.

We shall say that two automorphisms α and β of A are M*-independent* if the automorphism $\alpha^{-1}\beta$ (and hence $\beta^{-1}\alpha$) is M-outer; otherwise they are called M*-dependent*.

Proposition 3.2. *Let A be a graded ring with a gr-simple left A-module M, and let α and β be automorphisms of A. Then α and β are M-dependent if and only if the left A-modules M_α and M_β are isomorphic.*

Proof. Assume that α and β are M-dependent, then there exists an invertible homogeneous element $T \in \mathrm{END}(M)$ such that $L_{a^{\alpha^{-1}\beta}} = TL_aT^{-1}$ for all $a \in A$. Therefore $L_{a^\beta} = TL_{a^\alpha}T^{-1}$ and so

$$L_{a^\beta}T = TL_{a^\alpha} \quad \text{for all } a \in A. \tag{2}$$

Consider T as a bijective additive map $M_\alpha \to M_\beta$. Then for $a \in A$ and $x \in M_\alpha$, we have

$$T(a *_\alpha x) = T(L_{a^\alpha}x) = L_{a^\beta}(Tx) = a *_\beta (Tx).$$

Therefore, T is an isomorphism of A-modules M_α and M_β.

Conversely, let $T : M_\alpha \to M_\beta$ be an isomorphism of left A-modules. Then

$$T(L_{a^\alpha}x) = T(a *_\alpha x) = a *_\beta (Tx) = L_{a^\beta}(Tx) \quad \text{for all } a \in A, x \in M_\alpha.$$

That is, $TL_{a^\alpha} = L_{a^\beta}T$. Substituting $a^{\alpha^{-1}}$ for a, we obtain $TL_a = L_{a^{\alpha^{-1}\beta}}T$ and so $TL_aT^{-1} = L_{a^{\alpha^{-1}\beta}}$ for all $a \in A$. This shows that α and β are M-dependent. ∎

The main result of this section is

Theorem 3.3. *Let A be a graded ring with a gr-simple left A-module M, $D = END({}_AM)$ and α_1, α_2, ..., α_n automorphisms of A. Then the following conditions are equivalent:*

(a) *α_i and α_j are M-independent for all $i \neq j$;*

(b) *given any $x_1, x_2, \ldots, x_k \in h(M)$ linearly independent over D, and any $y_{ij} \in M$, $1 \leq i \leq n, 1 \leq j \leq k$, there exists an element $a \in A$ such that $a^{\alpha_i}x_j = y_{ij}$ for all i and j.*

Proof. Suppose that α_i and α_j are M-independent for all $i \neq j$. According to Proposition 3.2, M_{α_1}, M_{α_2}, ..., M_{α_n} are pairwise nonisomorphic. By equation (1), we get $D = \mathrm{END}({}_AM_{\alpha_1}) = \mathrm{END}({}_AM_{\alpha_2}) = \cdots = \mathrm{END}({}_AM_{\alpha_n})$. It follows from Theorem 2.2 that there exists an element $a \in A$ such that $a^{\alpha_i}x_j = a *_{\alpha_i} x_j = y_{ij}$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k$.

To prove the converse assume that two automorphisms $\alpha = \alpha_i$ and $\beta = \alpha_j$, $i \neq j$, are M-dependent. Let $T \in \mathrm{END}(M)$ be invertible and such that the equation (2) holds.

Now, pick $x \in h(M) \setminus \{0\}$. If x and Tx are linearly independent over D, then by the assumption there exists an element $a \in A$ with $a^\beta x = 0 = a^\beta Tx$ and $a^\alpha x = x$. On the other hand, if x and Tx are linearly dependent, we choose $a \in A$ such that $a^\beta x = 0$ and $a^\alpha x = x$. Since x and Tx are linearly dependent, we also have $a^\beta Tx = 0$.

With such an a, we have

$$Tx = T(a^\alpha x) = T(L_{a^\alpha})x = L_{a^\beta} Tx = a^\beta Tx = 0.$$

This is a contradiction since T is invertible. Hence α and β are linearly independent. This completes the proof. ∎

4. Density and M-Outer Derivations

Let A be a graded ring with a gr-simple left A-module M. Put $D = \text{END}({}_A M)$ and let F be the prime subfield of the gr-division ring D. Clearly M is a vector space over F. By a ***derivation*** of A we mean an additive map $d : A \to \text{END}(M_D)$ satisfying $(A_g)d \subseteq \text{END}(M_D)_g$ for all $g \in G$ and $(xy)^d = L_x y^d + x^d L_y$ for all $x, y \in h(A)$. This extends the usual concept of a derivation of a graded ring into itself since any derivation $d : A \to A$ gives rise to a derivation $\overline{d} : A \to \text{END}(M_D)$ given by $a^{\overline{d}} = L_{a^d}$.

Definition 4.1. A derivation $d : A \to \text{END}(M_D)$ is called M-***inner*** if there exists a homogeneous element T of $\text{END}(M)_e$ such that

$$[T, L_a] = a^d \quad \text{for all } a \in A; \tag{3}$$

otherwise it is called M-***outer***.

Of course, if d is a derivation of A into itself, then we shall say that d is M-inner if $\overline{d} : A \to \text{END}(M_D)$ is M-inner.

Recall that an additive mapping $\varphi : M \to M$ is called a ***differential transformation*** if there exists a map $\gamma : D \to D$ such that $\varphi(xu) - (\varphi x)u = xu^\gamma$ for all $x \in M, u \in D$.

Let $d : A \to \text{END}(M_D)$ be an M-inner derivation, and $T \in \text{END}(M)_e$ such that $[T, L_a] = a^d$ for all $a \in A$. We claim that T is a differential transformation of M. Given $u \in D$, we define a map $R_u : M \to M$ by $R_u x = xu$ for all $x \in M$. Clearly $D^\circ = \{R_u : u \in D\}$ is a gr-subdivision ring of $\text{END}(M)$ anti-isomorphic to D. Note that $[R_u, L_a] = 0$ for all $u \in D$ and $a \in A$. Clearly

$$D^\circ = \{\Lambda \in \text{END}(M) : [\Lambda, L_a] = 0 \quad \text{for all } a \in A\}.$$

Note that (3) yields

$$0 = [T, [R_u, L_a]] = [R_u, [T, L_a]] + [[T, R_u], L_a] = [L_a, [R_u, T]]$$

for all $u \in D$ and $a \in A$ because $[R_u, L_a] = 0 = [R_u, a^d]$. Then $[R_u, T] \subseteq D^\circ$. Therefore $[R_u, T] = R_{u'}$ for some $u' \in D$ we see that $T(xu) - (Tx)u = xu'$ for all $x \in M$. That is to say, T is a differential transformation of M.

Let V be a vector space over F with basis $\{d, e\}$ and $d : A \to \text{END}(M_D)$ a derivation. We can define a left A-module structure on the vector space $M_d = V \otimes_F M$ by the following action:

$$a(d \otimes x + e \otimes y) = d \otimes ax + e \otimes (a^d x + ay) \quad \text{for all } x, y \in h(M),\ a \in A.$$

Also, let $L(d) = e \otimes M$. Then $L(d)$ is a submodule of M_d since

$$\begin{aligned} e \otimes M = e \otimes \left(\bigoplus_{g \in G} M_g \right) &= \bigoplus_{g \in G} (e \otimes M_g) \\ &= \bigoplus_{g \in G} ((e \otimes M_g) \bigcap (M_d)_g) = \bigoplus_{g \in G} (L(d) \bigcap (M_d)_g). \end{aligned}$$

Moreover, $M_d/L(d)$ is isomorphic to M via the map $d \otimes x + L(d) \mapsto x$, for all $x \in h(M)$. Finally, we identify D with END$({}_A(M_d/L(d)))$:

$$(d \otimes x + L(d))\lambda = d \otimes (x\lambda) + L(d), \text{ for all } \lambda \in D \text{ and } x \in h(M)\ .$$

Proposition 4.2. *Let A be a graded ring with a gr-simple left module M and $d : A \to$ END(M_D) be an M-outer derivation. Then M_d is a gr-local left A-module with gr-maximal submodule $L(d)$.*

Proof. If M_d is gr-local, then $L(d)$ is a gr-maximal submodule since $L(d)$ is a proper submodule of M_d and $M_d/L(d) \cong M$ is gr-simple. Suppose that M_d is not gr-local. Then there exists a proper A-submodule N of M_d which is not contained in $L(d)$. Since $L(d)$ is a gr-simple A-module, either $N \supset L(d)$ or $N \cap L(d) = \{0\}$. If $N \supset L(d)$, then $N/L(d)$ must be a nonzero proper submodule of the gr-simple A-module $M_d/L(d) \cong M$, which is impossible. Therefore $N \cap L(d) = \{0\}$. It follows that N is isomorphic to the gr-simple left A-module $M_d/L(d) \cong M$.

Let $0 \neq e \otimes y + d \otimes x \in N$. Then $x \neq 0$ since $N \cap L(d) = \{0\}$. It is clear that y is uniquely determined by x because $N \cap L(d) = \{0\}$. Further, for $a \in A$, we have $a(e \otimes y + d \otimes x) = e \otimes (ay + a^d x) + d \otimes ax$. As $x \neq 0$, $Ax = M$.

Thus, for every $u \in M_g$ and $g \in G$, there exists a uniquely determined $v \in M_g$ with $e \otimes v + d \otimes u \in N$. Let $T : M \to M$ be given via setting $Tu = v$. Clearly T is well-defined and $T \in \mathrm{END}(M)_e$. Since $a(e \otimes v + d \otimes u) = e \otimes (av + a^d u) + d \otimes au$, we see that $T(au) = av + a^d u = aTu + a^d u$ and so $[T, L_a]u = a^d u$ for all $u \in h(M)$. That is $[T, L_a] = a^d$ for all $a \in A$, contradicting that d is M-outer. ∎

Here is the density theorem concerning one derivation.

Theorem 4.3. *Let A be a graded ring with a gr-simple left module M, let $D = \mathrm{END}({}_A M)$ and let $d : A \to \mathrm{END}(M_D)$ be a derivation. Then the following conditions are equivalent:*

- (a) *d is M-outer;*
- (b) *given any elements $x_1, x_2, \ldots, x_n \in h(M)$ linearly independent over D and arbitrary elements $y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in M$, there exists an element $a \in A$ such that $ax_i = y_i$ and $a^d x_i = z_i$ for $i = 1, 2, \ldots, n$.*

Proof. Suppose that d is M-outer. Then M_d is a gr-local A-module by Proposition 4.2. Set $\overline{x}_i = d \otimes x_i$ and $\overline{y}_i = d \otimes y_i + e \otimes z_i$, $i = 1, 2, \ldots, n$. By Theorem 2.4, there exists an $a \in A$ such that $a\overline{x}_i = \overline{y}_i$ for $i = 1, 2, \ldots, n$. That is to say, $ax_i = y_i$ and $a^d x_i = z_i$ for $i = 1, 2, \ldots, n$.

Conversely, suppose that (b) is fulfilled and there exists $T \in \mathrm{END}(M)_e$ with $a^d = [T, L_a]$ for all $a \in A$. Let $x \in h(M) \setminus \{0\}$. If x and Tx are linearly independent over D, then by the assumption there exists an element $a \in A$ with $ax = 0 = aTx$ and $a^d x = x$. On the other hand, if x and Tx are linearly dependent, we choose $a \in A$ such that $ax = 0$ and $a^d x = x$. Since x and Tx are linearly dependent, we also have $aTx = 0$. Now, we get

$$x = a^d x = [T, L_a]x = Tax - aTx = 0,$$

a contradiction. Thus no any $T \in \mathrm{END}(M)_e$ can satisfy $a^d = [T, L_a]$ for all $a \in A$. Therefore, d is M-outer. ∎

We shall now prove an analogue of Theorem 4.3 with several derivations.

Consider derivations $d_i : A \to \mathrm{END}(M_D)$, $i = 1, 2, \ldots, n$. Denoting by $D_M(A)$ the set of all M-inner derivations. We say that $d_1, d_2, \ldots, d_n$ are *dependent over D modulo $D_M(A)$*, if there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_n \in h(D)$, not all zero, and $T \in \mathrm{END}(M)_e$ such that $\sum_{i=1}^{n} a^{d_i} x \lambda_i + [T, L_a]x = 0$ for all $a \in A$ and $x \in M$; otherwise $d_1, d_2, \ldots, d_n$ are called *independent over D modulo $D_M(A)$*.

Let V be a vector space over F with basis $\{\overline{d}_n, e_1, e_2, \ldots, e_n\}$. We set $M_{\overline{d}_n} = V \otimes M$. Clearly $M_{\overline{d}_n}$ is a left A-module under the multiplication

$$a\left(\overline{d}_n \otimes x + \sum_{i=1}^{n} e_i \otimes x_i\right) = \overline{d}_n \otimes ax + \sum_{i=1}^{n} e_i \otimes (a^{d_i}x + ax_i)$$

for all $x, x_1, x_2, \ldots, x_n \in h(M)$, $a \in A$. We set $L(\overline{d}_n) = \sum_{i=1}^{n} e_i \otimes M$. Clearly $L(\overline{d}_n)$ is a submodule of $M_{\overline{d}_n}$ isomorphic to the direct sum of n copies of M and $M_{\overline{d}_n}/L(\overline{d}_n) \cong M$. In proving the density theorem for derivations $d_1, d_2, \ldots, d_n$ (Theorem 4.5 below), the modules $M_{\overline{d}_n}$ and $L(\overline{d}_n)$ will play the same role as that M_d and $L(d)$ play in the proof of Theorem 4.3.

We first generalize Proposition 4.2.

Proposition 4.4. *Suppose that d_1, d_2, …, d_n are independent over D modulo $D_M(A)$. Then $M_{\overline{d}_n}$ is a gr-local module with gr-maximal submodule $L(\overline{d}_n)$.*

Proof. We proceed by induction on n. The case $n = 1$ follows from Proposition 4.2. In the inductive case we assume that $M_{\overline{d}_{n-1}}$ is a gr-local module with gr-maximal submodule $L(\overline{d}_{n-1}) = \sum_{i=1}^{n-1} e_i \otimes M$.

Let $g \in G$ with $M_g \neq 0$. Pick $x, x_1, x_2, \ldots, x_n \in M_g$ with $x \neq 0$ and set

$$\overline{x} = \overline{d}_{n-1} \otimes x + \sum_{i=1}^{n-1} e_i \otimes x_i \in \left(M_{\overline{d}_{n-1}}\right)_g \text{ and } \quad \overline{y} = d_n \otimes x + e \otimes x_n \in (M_{d_n})_g.$$

Suppose that for all $a \in A$, $a\overline{x} = 0$ implies $a\overline{y} = 0$. Note that $A\overline{x} = M_{\overline{d}_{n-1}}$ and $A\overline{y} = M_{d_n}$ by Lemma 2.3(d). It follows that exists an epimorphism of modules $\beta : M_{\overline{d}_{n-1}} \to M_{d_n}$. By Lemma 2.3(a), β maps the gr-Jacobson radical $L(\overline{d}_{n-1})$ of $M_{\overline{d}_{n-1}}$ into the gr-Jacobson radical $L(d_n) = e \otimes M$ of M_{d_n}. Since each $e_i \otimes M \cong M$, there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_n \in h(D)$ such that

$$\left(\sum_{i=1}^{n-1} e_i \otimes z_i\right)\beta = \sum_{i=1}^{n-1} e \otimes z_i\lambda_i \quad \text{for all } z_1, z_2, \ldots, z_{n-1} \in M_g.$$

If $(L(\overline{d}_{n-1}))\beta = 0$, then β induces an epimorphism from the gr-simple module $M \cong M_{\overline{d}_{n-1}}/L(\overline{d}_{n-1})$ onto the gr-local module M_{d_n}, which is impossible. Therefore $L(\overline{d}_{n-1})\beta \neq 0$, and so not all λ_i's are equal to 0.

Clearly β induces a homomorphism of modules

$$M \cong M_{\overline{d}_{n-1}}/L(\overline{d}_{n-1}) \to M_{d_n}/L(d_n) \cong M$$

so there exist elements $\lambda \in h(D)$ and $T \in \mathrm{END}(M)_e$ such that

$$(\overline{d}_{n-1} \otimes z)\beta = d_n \otimes z\lambda + e \otimes Tz \quad \text{for all } z \in h(M).$$

Given $a \in A$ and $z \in h(M)$, we have

$$\begin{aligned} & d_n \otimes az\lambda + e \otimes a^{d_n} z\lambda + e \otimes aTz \\ &= a(d_n \otimes z\lambda + e \otimes Tz) = a[(\overline{d}_{n-1} \otimes z)\beta] \\ &= [a(\overline{d}_{n-1} \otimes z)]\beta = \left(\overline{d}_{n-1} \otimes az + \sum_{i=1}^{n-1} e_i \otimes a^{d_i} z\right)\beta \\ &= d_n \otimes az\lambda + e \otimes T(az) + \sum_{i=1}^{n-1} e \otimes a^{d_i} z\lambda_i. \end{aligned}$$

But then, for all $a \in A$ and $z \in h(M)$,

$$\sum_{i=1}^{n-1} a^{d_i} z\lambda_i - a^{d_n} z\lambda + [T, L_a]z = 0,$$

a contradiction. This shows that there exists an element $a \in A$ such that $a\overline{x} = 0$ and $a\overline{y} \neq 0$. That is

(4) $$ax = a^{d_1} + ax_1 = \cdots = a^{d_{n-1}}x + ax_{n-1} = 0 \quad \text{and} \quad a^{d_n}x + ax_n \neq 0.$$

Let $\overline{z} \in h\big(M_{\overline{d}_n}\big)\backslash h\big(L(\overline{d}_n)\big)$. According to Lemma 2.3(d), it is enough to show that $A\overline{z} = M_{\overline{d}_n}$. To this end, we set $\overline{z} = \overline{d}_n \otimes x + \sum_{i=1}^{n} e_i \otimes x_i$ where $x, x_i \in M_g$ for some $g \in G$. Clearly $x \neq 0$. Therefore, by what we have just shown, there exists some $a \in A$ such that (4) is fulfilled. Hence $a\overline{z} = e_n \otimes (a^{d_n}x + ax_n) \neq 0$, and so $e_n \otimes M \subseteq A\overline{z}$. Since $M_{\overline{d}_n}\big/(e_n \otimes M) \cong M_{\overline{d}_{n-1}}$ which is gr-local by the induction assumption. We thus conclude that $A\overline{z} = A\overline{z} + e_n \otimes M = M_{\overline{d}_n}$. ■

We are now in a position to prove the main result of this section.

Theorem 4.5. *Let A be a graded ring with a gr-simple left module M with $D = END({}_AM)$. Let $d_j : A \to END(M_D)$, $j = 1, 2, \ldots, m$, be derivations. Then the following conditions are equivalent:*

- (a) *$d_1, d_2, \ldots, d_m$ are independent over D modulo $D_M(A)$;*
- (b) *given any elements $x_1, x_2, \ldots, x_n \in h(M)$ linearly independent over D and arbitrary elements y_i, $z_{ij} \in M$ ($1 \leq i \leq n$ and $1 \leq j \leq m$), there exists an element $a \in A$ such that*

$$ax_i = y_i,\ a^{d_j}x_i = z_{ij},\ \text{for all } i \text{ and } j.$$

Proof. Suppose that (a) is fulfilled. Then $M_{\overline{d}_m}$ is a gr-local A-module by Proposition 4.4. For $i = 1, 2, \ldots, n$, set $\overline{x}_i = \overline{d}_m \otimes x_i$ and $\overline{y}_i = \overline{d}_m \otimes y_i + \sum_{j=1}^{m} e_j \otimes z_{ij}$. By Theorem 2.4 there exists an $a \in A$ such that $a\overline{x}_i = \overline{y}_i$ for all $i = 1, 2, \ldots, n$. Clearly a is the desired element.

Now suppose that (b) is true but not (a). That is, there exist elements $\lambda_1, \ldots, \lambda_m \in h(D)$, not all zero, and $T \in \mathrm{END}(M)_e$ such that $\sum_{j=1}^{m} a^{d_j} x \lambda_j + [T, L_a]x = 0$ for all $a \in A$ and $x \in M$. We may assume that $\lambda_1 \neq 0$. Arguing similarly as in the proof of Theorem 4.3, we see that given any nonzero $x \in h(M)$, there is some $a \in A$ such that $ax = aTx = a^{d_2}x = \cdots = a^{d_m}x = 0$ and $a^{d_1}x = x$, which yields $x\lambda_1 = 0$, a contradiction. This completes the proof. ■

5. The Main Theorem

Let A be a graded ring with a gr-simple left A-module M. Put $D = \mathrm{END}({}_AM)$ and let F be the prime subfield of the gr-division ring D. Let $D(A)$ be the additive group of derivations from A into A. Let n, m_1, $m_2, \ldots, m_n$ be positive integers and let d_1, $d_2, \ldots, d_n \in D(A)$. We set

$$\begin{aligned} \overline{m} &= (m_1, m_2, \ldots, m_n), \\ \Omega(\overline{m}) &= \{\overline{s} = (s_1, s_2, \ldots, s_n) \mid 0 \leq s_i \leq m_i, i = 1, 2, \ldots, n\}, \\ \Delta_{\overline{s}} &= d_1^{s_1} d_2^{s_2} \ldots d_n^{s_n},\ \overline{s} \in \Omega(\overline{m}) \quad \text{and} \\ \overline{0} &= (0, 0, \ldots, 0). \end{aligned}$$

It is understood that $\Delta_{\overline{0}} = e$, the identity map on A. Given $\overline{s}, \overline{r} \in \Omega(\overline{m})$, we shall write $\overline{s} \geq \overline{r}$ provides that $s_i \geq r_i$ for all $i = 1, 2, \ldots, n$.

Let $a, b \in A$ and $\overline{s} \in \Omega(\overline{m})$. According to Leibnitz Formula [5, Remark 1.1.1],

$$(ab)^{\Delta_{\overline{s}}} = \sum_{\overline{r} \in \Omega(\overline{m}),\ \overline{r} \leq \overline{s}} \binom{\overline{s}}{\overline{r}} a^{\Delta_{\overline{s}-\overline{r}}} b^{\Delta_{\overline{r}}} \tag{5}$$

where $\binom{\overline{s}}{\overline{r}} = \prod_{i=1}^{n} \binom{s_i}{r_i}$.

Let V be the vector space over F with basis $\{\Delta_{\overline{s}} \mid \overline{s} \in \Omega(\overline{m})\}$. We set $M_{\Omega(\overline{m})} = V \otimes_F M$. Then

$$M_{\Omega(\overline{m})} = \bigoplus_{g \in G} \big(V \otimes_F M_g\big) = \bigoplus_{g \in G} \big(M_{\Omega(\overline{m})}\big)_g$$

is a graded A-module. When the context is clear, we shall simply write M_Ω for $M_{\Omega(\overline{m})}$ and Ω for $\Omega(\overline{m})$.

It follows from (5) that M_Ω is a left A-module under the operation

$$a(\Delta_{\overline{s}} \otimes x) = \sum_{\overline{r}\in\Omega,\ \overline{r}\leq\overline{s}} \binom{\overline{s}}{\overline{r}} \Delta_{\overline{s}-\overline{r}} \otimes a^{\Delta_{\overline{r}}} x \tag{6}$$

for all $a \in A$, $x \in h(M), \overline{s} \in \Omega$.

Next, we set

$$L(\Omega) = \sum_{\overline{s}\in\Omega,\ \overline{s}<\overline{m}} \Delta_{\overline{s}} \otimes M. \tag{7}$$

Then

$$L(\Omega) = \sum_{\overline{s}\in\Omega,\ \overline{s}<\overline{m}} \Delta_{\overline{s}} \otimes \left(\bigoplus_{g\in G} M_g\right) = \bigoplus_{g\in G} \big(L(\Omega) \cap (M_\Omega)_g\big)$$

is a graded submodule of M_Ω such that, as graded A-modules,

$$M_\Omega / L(\Omega) \cong M. \tag{8}$$

Given $\overline{s} \in \Omega$, note that

$$M_{\Omega(\overline{s})} = \sum_{\overline{r}\in\Omega(\overline{s})} \Delta_{\overline{r}} \otimes M \subseteq M_\Omega \tag{9}$$

for all $\overline{s} \in \Omega$. Finally, if $\overline{s} = (s_1, s_2, \ldots, s_n) \in \Omega$, we set $|\overline{s}| = \sum_{i=1}^n s_i$. The reader will see that the modules M_Ω and $L(\Omega)$ are main tools in the proof of our main result.

Proposition 5.1. *Suppose that the following conditions are fulfilled:*

(i) $d_1, d_2, \ldots, d_n \in D(A)$ *are independent over* D *modulo* $D_M(A)$*;*

(ii) *either* $char(D) = 0$ *or* $char(D) = p > 0$ *and* $m_i < p$ *for all* i*.*

Then M_Ω *is a gr-local* A*-module with gr-maximal submodule* $L(\Omega)$*.*

Proof. Given $\overline{x} \in h(M_\Omega) \setminus h(L(\Omega))$, in view of Lemma 2.3(d), it is enough to show that $A\overline{x} = M_\Omega$. In order to prove the equality, we shall make use of the triple induction. The first induction is on $|\overline{m}|$. On the inductive step on $|\overline{m}|$ we shall represent $L(\Omega)$ as an union of certain A-submodules N_k, $-1 \leq k \leq |\overline{m}| - 1$ (with $N_{-1} = 0$) and proceed by induction on k to show that $N_k \subseteq A\overline{x}$ for all k (and so $L(\Omega) \subset A\overline{x}$ forcing $A\overline{x} = M_\Omega$ in view of (8)). Making the induction step on k, we shall introduce the concept of the height $\hbar(z)$ of the element $z \in h(A\overline{x}) \setminus h(N_{k-1})$ and proceed by the induction on $\hbar(z)$.

We now proceed by induction on $|\overline{m}|$. If $|\overline{m}| = 1$, then $n = 1 = m_1$. In this case $M_\Omega = M_{d_1}$, $L(\Omega) = L(d_1)$ and the result follows from Proposition 4.2.

In the inductive case we may assume that each $M_{\Omega(\overline{s})}$ is gr-local for all $\overline{s} \in \Omega$ with $|\overline{s}| < |\overline{m}|$. Let

$$\overline{x} = \sum_{\overline{s} \in \Omega} \Delta_{\overline{s}} \otimes x_{\overline{s}} \in h(M_\Omega)$$

where $x_{\overline{s}} \in M_g$ for some $g \in G$ with $x_{\overline{m}} \neq 0$. By Lemma 2.3(d) it is enough to show that $A\overline{x} = M_\Omega$. It follows from (6) that

$$a\overline{x} - \Delta_{\overline{m}} \otimes ax_{\overline{m}} \in L(\Omega)$$

and so $A\overline{x} + L(\Omega) = \Delta_{\overline{m}} \otimes Ax_{\overline{m}} + L(\Omega) = M_\Omega$. Therefore it is enough to show that $A\overline{x} \supseteq L(\Omega)$.

Let k be a nonnegative integer. We set $N_{-1} = 0$ and

$$N_k = \sum_{\overline{s} \in \Omega,\ |\overline{s}| \leq k} \Delta_{\overline{s}} \otimes M \subseteq M_\Omega.$$

Clearly $L(\Omega) = N_{|\overline{m}|-1}$. We proceed by induction on k to prove that $N_k \subseteq A\overline{x}$. The case $k = -1$ is clear. In the inductive case we assume that $N_{k-1} \subseteq A\overline{x}$. If $k = |\overline{m}|$, then $N_{k-1} = L(\Omega)$ and there is nothing to prove. Therefore we may assume that $k < |\overline{m}|$. Pick any $\overline{s} \in \Omega$ with $|\overline{s}| = k$. Since

$$N_k = \sum_{\overline{r} \in \Omega,\ |\overline{r}| = k} \Delta_{\overline{r}} \otimes M + N_{k-1},$$

it suffices to show that $\Delta_{\overline{s}} \otimes M \subseteq A\overline{x}$. To this end, we introduce the following concept. A nonzero element $\overline{y} = \sum_{\overline{s} \in \Omega} \Delta_{\overline{s}} \otimes y_{\overline{s}} \in h(M_\Omega)$ is said to have a *height* provided that there exists an $\overline{r} \in \Omega$ such that

$$\overline{y} = \Delta_{\overline{r}} \otimes y_{\overline{r}} + \sum_{\overline{p} \in \Omega,\ |\overline{p}| < |\overline{r}|} \Delta_{\overline{p}} \otimes y_{\overline{p}} \quad \text{and} \quad y_{\overline{r}} \neq 0.$$

In this case we shall write the height $\hbar(\overline{y}) = \overline{r}$. For example, $\hbar(\overline{x}) = \overline{m}$. In the set

$$\{\overline{y} \in h(A\overline{x}) \setminus h(N_{k-1}) \mid \hbar(\overline{y}) \geq \overline{s}\}$$

we choose an element $\overline{z}$ with minimal possible height (with respect to the partial order on Ω). Suppose that $|\hbar(\overline{z})| = |\overline{s}| = k$. Then $\overline{z} = \Delta_{\overline{s}} \otimes z_{\overline{s}} + \overline{u}$ for some

$\overline{u} \in h(N_{k-1})$. Because $M_{\Omega(\overline{s})}$ is gr-local by the induction assumption (recall that $k < |\overline{m}|$), we have

$$A\overline{x} \supseteq A\overline{z} + N_{k-1} \supseteq A(\Delta_{\overline{s}} \otimes z_{\overline{s}}) = M_{\Omega(\overline{s})}$$

and so $\Delta_{\overline{s}} \otimes M \subseteq A\overline{x}$. Therefore we may assume that $|\hbar(\overline{z})| > k = |\overline{s}|$. Let $\overline{r} = \hbar(\overline{z})$. We have $\overline{r} > \overline{s}$, and whence there exists an index i such that $r_i > s_i$. Say, $i = 1$. By Theorem 4.5 there exists an $a \in A$ such that

$$\begin{aligned} az_{\overline{p}} &= 0, \quad \overline{p} \in \Omega \\ a^{d_i} z_{\overline{r}} &= 0, \quad i = 2, 3, \ldots, n, \quad and \\ a^{d_1} z_{\overline{r}} &\neq 0. \end{aligned}$$

Therefore

$$a\overline{z} = r_1 \Delta_{\overline{q}} \otimes a^{d_1} z_{\overline{r}} + \sum_{\substack{\overline{p} \in \Omega, \\ |\overline{p}| < |\overline{r}| - 1}} \Delta_{\overline{p}} \otimes w_{\overline{p}}$$

for suitable $w_{\overline{p}} \in h(M)$, where $\overline{q} = (r_1 - 1, r_2, \ldots, r_n)$. By our assumption r_1 is a nonzero element of F and whence the element $a\overline{z}$ has a height and $\overline{r} > (a\overline{z}) = \overline{q} \geq \overline{s}$, a contradiction. ■

Theorem 5.2. *Let A be a graded ring, M_1, M_2, …, M_k gr-simple left A-modules ($k \geq 1$), and $m_1, \ldots, m_n$ positive integers. Further, for positive integers l, n, let $d_1, d_2, \ldots, d_n \in D(A)$, and let $x_{i1}, x_{i2}, \ldots, x_{il} \in h(M_i)$ be linearly independent over D_i = END$({}_AM_i)$. Also for $1 \leq i \leq k$, $1 \leq j \leq l$, and $\overline{s} \in \Omega = \Omega(\overline{m}) \setminus \{\overline{0}\}$, let $y_{ij}, z_{ij\overline{s}} \in M_i$, where $\overline{m} = (m_1, m_2, \ldots, m_n)$. Suppose that the following conditions are fulfilled:*

(*i*) *$d_1, d_2, \ldots, d_n$ are independent over D_i modulo $D_{M_i}(A)$ for every i,*

(*ii*) *for all i, either $\mathrm{char}(D_i) = 0$ or $\mathrm{char}(D_i) = p_i > 0$ and each $m_j < p_i$, and*

(*iii*) *${}_AM_i \ncong_A M_j$ for all $i \neq j$.*

Then there exists an $a \in A$ such that

$$ax_{ij} = y_{ij} \quad \textit{and} \quad a^{\Delta_{\overline{s}}} x_{ij} = z_{ij\overline{s}}$$

for all i, j, and $\overline{s} \in \Omega \setminus \{\overline{0}\}$.

Proof. By Proposition 5.1, let $N_i = (M_i)_\Omega$ is a gr-local A-module with gr-maximal proper submodule L_i such that $N_i/L_i \cong M_i$. It follows from (iii) that $N_i/L_i \ncong N_j/L_j$ if $i \neq j$. For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$, set

$$\overline{x}_{ij} = \Delta_{\overline{m}} \otimes x_{ij} \quad \text{and} \quad \overline{y}_{ij} = \Delta_{\overline{m}} \otimes y_{ij} + \sum_{\overline{s}\in\Omega,\ \overline{s}\neq\overline{0}} \binom{\overline{m}}{\overline{s}} \Delta_{\overline{m}-\overline{s}} \otimes z_{ij\overline{s}}.$$

Note that each $\binom{\overline{m}}{\overline{s}}$ is a nonzero element of every D_i. By Theorem 2.4 there exists some $a \in A$ such that $a\overline{x}_{ij} = \overline{y}_{ij}$ for all i and j. The result now follows from (6). ■

Finally, we have arrived at the main result of the present article.

Theorem 5.3. *Let A be a graded ring with a gr-simple left A-module M, let k, l, n, m_1, $m_2, \ldots, m_n$ be positive integers, let d_1, $d_2, \ldots, d_n \in D(A)$, and let α_1, α_2, ..., α_l be automorphisms of A. Suppose that the following conditions are fulfilled:*

(*i*) $d_1, d_2, \ldots, d_n$ *are independent over* $D = \mathit{END}({}_AM)$ *modulo* $D_M(A)$*;*

(*ii*) *either* $\operatorname{char}(D) = 0$ *or* $\operatorname{char}(D) = p > 0$ *and each* $m_i < p$*;*

(*iii*) α_i *and* α_j *are M-independent for all* $i \neq j$.

Then for any elements $x_1, x_2, \ldots, x_k \in h(M)$ linearly independent over D, and for any $z_{i\overline{s}j} \in M$, $i = 1, 2, \ldots, k$, $\overline{s} \in \Omega = \Omega(\overline{m})$, $j = 1, 2, \ldots, l$, there exists an $a \in A$ such that

$$a^{\Delta_{\overline{s}}\alpha_j} x_i = z_{i\overline{s}j}$$

for all i, j, and $\overline{s} \in \Omega$.

Proof. Set $N = M_\Omega$ and $L = L(\Omega)$. By Proposition 5.1, N is a gr-local module with gr-maximal submodule L and $N/L \cong M$. Set $N_j = N_{\alpha_j}$ and $L_j = L_{\alpha_j} \subseteq N_j$. It is easy to see that N_j is a gr-local module with gr-maximal submodule L_j and $N_j/L_j \cong M_{\alpha_j}$. By Proposition 3.2, $N_i/L_i \cong N_j/L_j$ if and only if $i = j$. According to (iii), END$({}_AM_{\alpha_j}) = D$. In particular,

$$\Delta_{\overline{m}} \otimes x_1,\ \Delta_{\overline{m}} \otimes x_2,\ \ldots,\ \Delta_{\overline{m}} \otimes x_k$$

are independent elements of $h(N_j)$ over D modulo L_j $(1 \leq j \leq l)$. The result now follows from Theorem 5.2. ■

6. Applications

We state with a lemma.

Lemma 6.1. *Let M be a right vector space over a gr-division ring D. Let $g \in G$ be such that dim$(M_g)_{D_e} \geq 2$. If T is an additive homogeneous endomorphism of M such that x and Tx are linearly dependent for every $x \in M_g$. Then there exists a $\lambda_g \in h(D)$ such that $Tx = x\lambda_g$ for all $x \in M_g$.*

Proof. By assumption, for every $x \in M_g$, there is a $\lambda_x \in h(D)$ such that $Tx = x\lambda_x$. Fix a $y \in M_g \setminus \{0\}$. Since dim$(M_g)_{D_e} \geq 2$, there is $x \in M_g$ such that x and y are linearly independent over D_e. From $(x+y)\lambda_{x+y} = T(x+y) = Tx + Ty = x\lambda_x + y\lambda_y$, we get $x(\lambda_{x+y} - \lambda_x) + y(\lambda_{x+y} - \lambda_y) = 0$. Since x and y are linearly independent, we see that $\lambda_x = \lambda_{x+y} = \lambda_y$.

Now let z be any nonzero element in M_g. If y and z are linearly independent, then $\lambda_z = \lambda_y$ as we have just seen. If y and z are linearly dependent, then x and z are linearly independent and so $\lambda_z = \lambda_x = \lambda_y$. Thus $\lambda_z = \lambda_y$ for all $z \in M_g \setminus \{0\}$. This completes the proof. ■

For a positive integer n, let $J_n(A)$ be the ideal of A consisting of those elements $a \in A$ such that $aM = 0$ for all gr-simple left A-module M with the property that if $M_g \neq \{0\}$, $g \in G$, then dim$(M_g)_{D_e} \geq n$. Here, $D =$ END$(_AM)$. Of course, $J_1(A) = J(A)$ is the gr-Jacobson radical of A.

We shall consider derivations satisfying certain conditions which might appear special at first. However, it is actually more general than various conditions considered by other authors.

Derivations d of (ungraded) prime rings R such that

$$(a^d)^n = 0 \text{ for all } \ a \in R$$

was studied in [10, 11, 12, 16]. Further, Felzenszwalb and Lanski [9] investigated derivations d with $(a^d)^n = 0$, $a \in R$, where $n = n(a)$ is not fixed but depends on a. In a theorem [3, Theorem 7.2], Beidar and Brešar treated derivations with all the above conditions mentioned.

Lemma 6.2. *Let A be a graded ring with a gr-simple module M. Let $D =$ END$(_AM)$ and let d be a derivation of A. Suppose that for every $a \in A$ there exist positive integers $n = n(a)$ and $m = m(a)$ such that $(a^d)^nA \subseteq A(a^d)^m + J(A)$. If dim$M_D > 1$, then d is M-inner.*

Proof. Assume that d is M-outer. Since dim$M_D > 1$, there are $x, y \in h(M)$ linearly independent over D. By Theorem 4.3, there is an $a \in A$ such that $a^d x = 0$, $a^d y = y$, and $ax = y$. But then $(a^d)^n ax = y$ while $A(a^d)^m x = 0$ for all positive integers n and m, contradicting the assumption. Thus d is M-inner. ∎

Theorem 6.3. *Let A be a graded ring and d be a derivation of A. Suppose that for every $a \in A$ there exist positive integers $n = n(a)$, $m = m(a)$ such that $(a^d)^n A \subseteq A(a^d)^m + J(A)$. Then $A^d \subseteq J_3(A)$.*

Proof. Pick any gr-simple left A-module M such that dim$(M_g)_{D_e} \geq 3$ for all $g \in G' = \{g \in G \mid M_g \neq 0\}$, and let $D = \text{END}({}_A M)$. The goal is to show that $A^d M = 0$. By Lemma 6.2, we can assume that d is M-inner, and let $T : M \to M$ be a differential transformation such that $[T, L_a] = L_{a^d}$ for $a \in A$.

Suppose for all $g \in G'$ and for every $x \in M_g$, Tx and x are D-dependent. By Lemma 6.1 for every $g \in G'$, there is a $\lambda_g \in h(D)$ such that $Tx = x\lambda_g$ for all $x \in M_g$. But then $a^d x = T(ax) - a(Tx) = (ax)\lambda_g - a(x\lambda_g) = 0$ for any $a \in A$ and $x \in M_g$, and so $A^d M = A^d(\oplus_{g \in G} M_g) = 0$, a contradiction. Therefore, there is some $g \in G'$ with an $x \in M_g$ such that $y = Tx$ and x are D-independent.

We claim that there is a nonzero $z \in h(M)$ such that $y \notin (Tz)D + zD + xD$. Indeed, pick any $z_0 \notin yD + xD$ (such z_0 exists for dim$M_D \geq$ dim$(M_g)_{D_e} \geq 3$ by assumption). With no loss of generality we may assume that $y \in (Tz_0)D + z_0 D + xD$, say, $y = (Tz_0)\lambda + z_0\mu + x\nu$ for some $\lambda, \mu, \nu \in h(D)$. Clearly, $\lambda \neq 0$. Now set $z = z_0 - x\lambda^{-1}$. Of course, $z \neq 0$. Since T is a differential transformation, we have $T(x\lambda^{-1}) = (Tx)\lambda^{-1} + x(\lambda^{-1})^\gamma$ where $\gamma : D \to D$ is a map. Whence

$$\begin{aligned} Tz &= Tz_0 - T(x\lambda^{-1}) \\ &= y\lambda^{-1} - z_0\mu\lambda^{-1} - x\nu\lambda^{-1} - y\lambda^{-1} - x(\lambda^{-1})^\gamma \\ &\in z_0 D + xD = zD + xD. \end{aligned}$$

As $y \notin z_0 D + xD$, we get $y \notin (Tz)D + zD + xD$, and our claim is proved.

By Theorem 2.2, there exist $a, b \in A$ such that $aTz = az = ax = 0$, $ay = -x$, and $bz = x$. Then $(a^d)^n bz = x$ while $(A(a^d)^m + J(A))z = 0$ for all positive integers n and m. This clearly contradicts our initial assumption. ∎

The condition treated in the next theorem also unifies several conditions treated in the literature (see [3, Section 7] for detail).

A well-known theorem of Posner from 1957 [17] considers derivations d such that $[a^d, a]$ is central for every a in a (ungraded) ring R. This theorem was generalized in various ways, in particular, derivations satisfying certain Engel type conditions were studied. Lanski [13] considered the following general condition:

$$[[...[[(a^{n_0})^d, a^{n_1}], a^{n_2}], ..], a^{n_k}] = 0$$

where the n_i's are fixed positive integers. Also, Chuang [7] another condition

$$[(a^{n(a)})^d, a^{n(a)}]_k = 0$$

(here, $[x, y]_k$ is defined as $[x, y]_0 = x$ and for $k > 1$, $[x, y]_k = [x, y]_{k-1}y - y[x, y]_{k-1}$. Beidar and Brešar made a unified treatment for these conditions just mentioned in [3, Theorem 7.3]. Now we shall prove a similar result in graded rings.

Theorem 6.4. *Let A be a graded ring and d a derivation of A. Suppose that for each $a \in A$ there is a nonnegative integer $n = n(a)$ such that $a^n a^d \in Aa + J(A)$. Then $A^d \subseteq J_2(A)$.*

Proof. Pick any gr-simple left A-module M such that $\dim(M_g)_{D_e} \geq 2$ for all $g \in G' = \{g \in G \mid M_g \neq 0\}$ and put $D = \text{END}({}_A M)$. We will show that $A^d M = 0$.

We claim that d is M-inner, but assume on the contrary that d is M-outer. Choose any D-independent elements $x, y \in h(M)$. By Theorem 4.3 there is an $a \in A$ such that $ax = 0$, $ay = y$ and $a^d x = y$. Then $a^n a^d x = y$ for every positive integer n, while $(Aa + J(A))x = 0$, a contradiction. Therefore, d is indeed M-inner.

Let T be a differential transformation T such that $[T, L_a] = L_{a^d}$ for all $a \in A$. Suppose that for some $g \in G'$, there is an $x \in M_g$ such that x and Tx are D-independent. By Theorem 2.2, there is an $a \in A$ such that $ax = 0$ and $a(Tx) = Tx$. But then $a^n a^d x = -Tx$ for any positive integer n, yet since $(Aa + J(A))x = 0$, this is impossible. Therefore x and Tx are D-dependent for all $x \in M_g$ with $g \in G'$. Using Lemma 6.1 and arguing as in the proof of Theorem 6.3, we would arrive at the desired conclusion. ■

As discussed in [3, Section 7], we cannot claim that $A^d \subseteq J(A)$ in general. After all, A could be commutative and so the condition of Theorem 6.4 would be trivially satisfied; but it is not true that every derivation of a commutative graded ring has the range in the gr-Jacobson radical.

However, in graded Banach algebras this is true indeed (by a graded Banach algebra we shall always mean a complex Banach graded algebra). In particular, many conditions under which a derivation of a noncommutative graded Banach algebra has the range in the gr-Jacobson radical have been found. As a corollary to Theorem 6.4 we will now obtain a result of such type.

Corollary 6.5. *Let A be a graded Banach algebra and d be a continuous derivation of A. Suppose that for each $a \in A$ there is a nonnegative integer $n = n(a)$ such that $a^n a^d \in Aa + J(A)$. Then $A^d \subseteq J(A)$.*

Proof. It is enough to show is that $A^d M = 0$ for any gr-simple left A-module M which is one-dimensional over $D = \text{END}({}_A M)$, a subfield of the complex field [6, Corollary 5, p. 128].

Clearly, $\text{ann}(M) = \{a \in A \mid ax = 0 \text{ for all } x \in M\}$ is an ideal of A and $(\text{ann}(M))^d \subseteq \text{ann}(M)$ by Sinclair's theorem [18]. Therefore, d induces a derivation on the algebra $A/\text{ann}(M)$ which is given by $a + \text{ann}(M) \mapsto a^d + \text{ann}(M)$. However, $A/\text{ann}(M)$ is isomorphic to the complex field and so the induced derivation is trivial. That is, $A^d \subseteq \text{ann}(M)$ and the proof is completed. ■

Finally, let us point out a special case of Corollary 6.5 that concern the Engel condition.

Corollary 6.6. *Let A be a graded Banach algebra and d a continuous derivation of A. Suppose that for each $a \in A$ there is a nonnegative integer $n = n(a)$ such that $[a^d, a]_n \in J(A)$. Then $A^d \subseteq J(A)$.*

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