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HYPERSURFACES IN SPACE FORMS SATISFYING THE CONDITION $L_k x = Ax + b$

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Abstract. We study hypersurfaces either in the sphere $\mathbb{S}^{n+1}$ or in the hyperbolic space $\mathbb{H}^{n+1}$ whose position vector x satisfies the condition $L_k x = Ax + b$, where L_k is the linearized operator of the $(k+1)$-th mean curvature of the hypersurface for a fixed $k = 0, \ldots, n-1$, $A \in \mathbb{R}^{(n+2)\times(n+2)}$ is a constant matrix and $b \in \mathbb{R}^{n+2}$ is a constant vector. For every k, we prove that when A is self-adjoint and $b = 0$, the only hypersurfaces satisfying that condition are hypersurfaces with zero $(k+1)$-th mean curvature and constant k-th mean curvature, and open pieces of standard Riemannian products of the form $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{S}^{n+1}$, with $0 < r < 1$, and $\mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{H}^{n+1}$, with $r > 0$. If H_k is constant, we also obtain a classification result for the case where $b \neq 0$.

1. Introduction

In [4] and inspired by Garay's extension of Takahashi theorem [18, 6, 7] and its subsequent generalizations and extensions [8, 11, 10, 12, 2, 3], the first author jointly with Gürbuz started the study of hypersurfaces in the Euclidean space satisfying the general condition $L_k x = Ax + b$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant matrix and $b \in \mathbb{R}^{n+1}$ is a constant vector (we refer the reader to the Introduction of [4] for further details). In particular, the following classification result was given in [4, Theorem 1].

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Theorem 1.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface immersed into the Euclidean space and let L_k be the linearized operator of the $(k+1)$-th mean curvature of M, for some fixed $k = 0, \ldots, n-1$. Then the immersion satisfies the condition $L_k x = Ax + b$ for some constant matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and some constant vector $b \in \mathbb{R}^{n+1}$ if and only if it is one of the following hypersurfaces in $\mathbb{R}^{n+1}$:*

(1) *a hypersurface with zero $(k+1)$-th mean curvature,*
(2) *an open piece of a round hypersphere $\mathbb{S}^n(r)$,*
(3) *an open piece of a generalized right spherical cylinder $\mathbb{S}^m(r) \times \mathbb{R}^{n-m}$, with $k+1 \leq m \leq n-1$.*

In this paper, and as a natural continuation of the study started in [4], we consider the study of hypersurfaces M^n immersed either into the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ or into the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ whose position vector x satisfies the condition $L_k x = Ax + b$. Here and for a fixed integer $k = 0, \ldots, n-1$, L_k stands for the linearized operator of the $(k+1)$-th mean curvature of the hypersurface, denoted by H_{k+1}, $A \in \mathbb{R}^{(n+2)\times(n+2)}$ is a constant matrix and $b \in \mathbb{R}^{n+2}$ is a constant vector. For the sake of simplifying the notation and unifying the statements of our main results, let us denote by $\mathbb{M}^{n+1}_c$ either the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ if $c = 1$, or the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ if $c = -1$. In this new situation, the codimension of the manifold M^n in the (pseudo)-Euclidean space $\mathbb{R}^{n+2}_q$ where it is lying is 2, which increases the difficulty of the problem. In the case where A is self-adjoint and $b = 0$ we are able to give the following classification result.

Theorem 1.2. *Let $x : M^n \to \mathbb{M}^{n+1}_c \subset \mathbb{R}^{n+2}_q$ be an orientable hypersurface immersed either into the Euclidean sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ (if $c = 1$) or into the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ (if $c = -1$), and let L_k be the linearized operator of the $(k+1)$-th mean curvature of M, for some fixed $k = 0, \ldots, n-1$. Then the immersion satisfies the condition $L_k x = Ax$ for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ if and only if it is one of the following hypersurfaces:*

(1) *a hypersurface having zero $(k+1)$-th mean curvature and constant k-th mean curvature;*
(2) *an open piece of a standard Riemannian product $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{S}^{n+1}$, $0 < r < 1$, if $c = 1$;*
(3) *an open piece of a standard Riemannian product $\mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{H}^{n+1}$, $r > 0$, if $c = -1$.*

Let us recall that every compact hypersurface immersed into the hyperbolic space $\mathbb{H}^{n+1}$ has an elliptic point, that is, a point where all the principal curvatures are positive (for a proof see, for instance, [5, Lemma 8]). The same happens for every compact hypersurface immersed into an open hemisphere $\mathbb{S}^{n+1}_+$ (see, for instance, [1, Section 3] for a proof in the case $n = 2$, although the proof works also in the general n-dimensional case). In particular, this implies that there exists no compact hypersurface either in $\mathbb{H}^{n+1}$ or in $\mathbb{S}^{n+1}_+$ with vanishing $(k+1)$-th mean curvature, for every $k = 0, \ldots, n-1$. Since the standard Riemannian products $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{S}^{n+1}$ are not contained in an open hemisphere, then we have the following non-existence result as a consequence of our Theorem 1.2.

Corollary 1.3. *There exists no compact orientable hypersurface either in* $\mathbb{H}^{n+1}$ *or in* $\mathbb{S}^{n+1}_+$ *satisfying the condition* $L_k x = Ax$ *for some self-adjoint constant matrix* $A \in \mathbb{R}^{(n+2)\times(n+2)}$*, where* L_k *stands for any of the linearized operators of the higher order mean curvatures.*

When $k = 1$ the operator L_1 is the operator $\square$ introduced by Cheng and Yau in [9] for the study of hypersurfaces with constant scalar curvature. In that case, since the scalar curvature of M is given by $n(n-1)(c + H_2)$ (see equation (2)) we get the following consequence.

Corollary 1.4. *Let* $x : M^n \to \mathbb{M}^{n+1}_c \subset \mathbb{R}^{n+2}_q$ *be an orientable hypersurface immersed either into the Euclidean sphere* $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ *(if* $c = 1$*) or into the hyperbolic space* $\mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ *(if* $c = -1$*), and let* $\square$ *be the Cheng and Yau operator on* M*. Then the immersion satisfies the condition* $\square x = Ax$ *for some self-adjoint constant matrix* $A \in \mathbb{R}^{(n+2)\times(n+2)}$ *if and only if it is one of the following hypersurfaces:*

(1) *a hypersurface having constant scalar curvature* $n(n-1)c$ *and constant mean curvature;*

(2) *an open piece of a standard Riemannian product* $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{S}^{n+1}$*,* $0 < r < 1$*, if* $c = 1$*;*

(3) *an open piece of a standard Riemannian product* $\mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{H}^{n+1}$*,* $r > 0$*, if* $c = -1$*.*

In particular, when $n = 2$, and taking into account that the only surfaces either in $\mathbb{S}^3$ or $\mathbb{H}^3$ having constant mean curvature and constant Gaussian (or scalar) curvature equal to the Gaussian curvature of the ambient space are the totally geodesic ones, we obtain the following result.

Corollary 1.5. *Let* $x : M^2 \to \mathbb{M}^3_c \subset \mathbb{R}^4_q$ *be an orientable surface immersed either into the Euclidean sphere* $\mathbb{S}^3 \subset \mathbb{R}^4$ *(if* $c = 1$*) or into the hyperbolic space* $\mathbb{H}^3 \subset \mathbb{R}^4_1$ *(if* $c = -1$*), and let* $L_1 = \square$ *be the Cheng and Yau operator of* M*. Then the immersion satisfies the condition* $\square x = Ax$ *for some self-adjoint constant matrix* $A \in \mathbb{R}^{(4)\times(4)}$ *if and only if it is one of the following surfaces:*

(1) *an open piece of either a totally geodesic round sphere* $\mathbb{S}^2 \subset \mathbb{S}^3$ *or a standard Riemannian product* $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r) \subset \mathbb{S}^3$, $0 < r < 1$, *if* $c = 1$*;*

(2) *an open piece of either a totally geodesic hyperbolic plane* $\mathbb{H}^2 \subset \mathbb{H}^3$ *or a standard Riemannian product* $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^1(r) \subset \mathbb{H}^3$, $r > 0$, *if* $c = -1$.

Remark 1.6. A different but related result to our Theorem 1.2 has been proved recently by Yang and Liu in [19]. In fact, instead of assuming that A is self-adjoint, they assume that H_k is constant and reach the same classification. Specifically, they use the method of moving frames to derive the basic equations for the hypersurface and then, following the techniques introduced by Alías, Ferrández and Lucas in [3] for the case $k = 0$ and extended by Alías and Gürbüz in [4] for general k, they prove that the hypersurface must be one of the standard examples.

On the other hand, in the case where A is self-adjoint and $b \neq 0$ we are able to prove the following classification result.

Theorem 1.7. *Let* $x : M^n \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ *be an orientable hypersurface immersed either into the Euclidean sphere* $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ *(if* $c = 1$*) or into the hyperbolic space* $\mathbb{H}^{n+1} \subset \mathbb{R}_1^{n+2}$ *(if* $c = -1$*), and let* L_k *be the linearized operator of the* $(k+1)$*-th mean curvature of* M*, for some fixed* $k = 0, \ldots, n-1$*. Assume that* H_k *is constant. Then the immersion satisfies the condition* $L_k x = Ax + b$ *for some self-adjoint constant matrix* $A \in \mathbb{R}^{(n+2)\times(n+2)}$ *and some non-zero constant vector* $b \in \mathbb{R}^{n+2}$ *if and only if:*

(i) $c = 1$ *and it is an open piece of a totally umbilical round sphere* $\mathbb{S}^n(r) \subset \mathbb{S}^{n+1}$, $0 < r < 1$.

(ii) $c = -1$ *and it is one of the following hypersurfaces in* $\mathbb{H}^{n+1}$*:*

- (1) *an open piece of a totally umbilical hyperbolic space* $\mathbb{H}^n(-r)$, $r > 1$,
- (2) *an open piece of a totally umbilical round sphere* $\mathbb{S}^n(r)$, $r > 0$,
- (3) *an open piece of a totally umbilical Euclidean space* $\mathbb{R}^n$.

2. Preliminaries

Throughout this paper we will consider both the case of hypersurfaces immersed into the Euclidean sphere

$$\mathbb{S}^{n+1} = \{x = (x_0, \ldots, x_{n+1}) \in \mathbb{R}^{n+2} : \langle x, x\rangle = 1\},$$

and the case of hypersurfaces immersed into the hyperbolic space $\mathbb{H}^{n+1}$. In this last case, it will be appropriate to use the Minkowski space model of hyperbolic space. Write $\mathbb{R}^{n+2}_1$ for $\mathbb{R}^{n+2}$, with coordinates $(x_0, \ldots, x_{n+1})$, endowed with the Lorentzian metric

$$\langle , \rangle = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2.$$

Then

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2}_1 : \langle x, x\rangle = -1, x_0 > 0\}$$

is a complete spacelike hypersurface in $\mathbb{R}^{n+2}_1$ with constant sectional curvature -1 which provides the Minkowski space model for the hyperbolic space.

In order to simplify our notation, we will denote by $\mathbb{M}^{n+1}_c$ either the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ if $c = 1$, or the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ if $c = -1$. We will also denote by $\langle , \rangle$, without distinction, both the Euclidean metric on $\mathbb{R}^{n+2}$ and the Lorentzian metric on $\mathbb{R}^{n+2}_1$, as well as the corresponding (Riemannian) metrics induced on $\mathbb{M}^{n+1}_c$ and on M. Consider $x : M^n \to \mathbb{M}^{n+1}_c \subset \mathbb{R}^{n+2}_q$ (with $q = 0$ if $c = 1$, and $q = 1$ if $c = -1$) a connected orientable hypersurface immersed into $\mathbb{M}^{n+1}_c$ with Gauss map N. Throughout this paper we will denote by ∇^o, $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\mathbb{R}^{n+2}_q$, $\mathbb{M}^{n+1}_c$ and M, respectively. Then, the basic Gauss and Weingarten formulae of the hypersurface are written as

$$\nabla^o{}_X Y = \overline{\nabla}_X Y - c\langle X, Y\rangle x = \nabla_X Y + \langle SX, Y\rangle N - c\langle X, Y\rangle x$$

and

$$SX = -\overline{\nabla}_X N = -\nabla^o{}_X N,$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$, where $S : \mathcal{X}(M) \to \mathcal{X}(M)$ stands for the shape operator (or Weingarten endomorphism) of M with respect to the chosen orientation N. As is well known, S defines a self-adjoint linear operator on each tangent plane T_pM, and its eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are n algebraic invariants given by

$$s_k(p) = \sigma_k(\kappa_1(p), \ldots, \kappa_n(p)), \quad 1 \leq k \leq n,$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \ldots x_{i_k}.$$

Observe that the characteristic polynomial of S can be writen in terms of the s_k's as

$$Q_S(t) = \det(tI - S) = \sum_{k=0}^{n} (-1)^k s_k t^{n-k}, \tag{1}$$

where $s_0 = 1$ by definition. The k-th mean curvature H_k of the hypersurface is then defined by

$$\binom{n}{k} H_k = s_k, \quad 0 \leq k \leq n.$$

In particular, when $k = 1$ $H_1 = (1/n)\sum_{i=1}^n \kappa_i = (1/n)\mathrm{trace}(S) = H$ is nothing but the mean curvature of M, which is the main extrinsic curvature of the hypersurface. On the other hand, H_2 defines a geometric quantity which is related to the (intrinsic) scalar curvature of M. Indeed, it follows from the Gauss equation of M that its Ricci curvature is given by

$$\mathrm{Ric}(X,Y) = (n-1)c\langle X,Y\rangle + nH\langle SX,Y\rangle - \langle SX,SY\rangle, \quad X,Y \in \mathcal{X}(M),$$

and then the scalar curvature of M is

$$\begin{aligned} \mathrm{tr}(\mathrm{Ric}) &= n(n-1)c + n^2H^2 - \mathrm{tr}(S^2) \\ &= n(n-1)c + \left(\sum_{i=1}^n \kappa_i\right)^2 - \sum_{i=1}^n \kappa_i^2 = n(n-1)(c+H_2). \end{aligned} \tag{2}$$

In general, when k is odd the curvature H_k is extrinsic (and its sign depends on the chosen orientation), while when k is even the curvature H_k is intrinsic and its value does not depend on the chosen orientation.

The classical Newton transformations $P_k : \mathcal{X}(M) \to \mathcal{X}(M)$ are defined inductively from the shape operator S by

$$P_0 = I \quad \text{and} \quad P_k = s_k I - S\circ P_{k-1} = \tbinom{n}{k}H_k I - S\circ P_{k-1},$$

for every $k = 1\ldots,n$, where I denotes the identity in $\mathcal{X}(M)$. Equivalently,

$$P_k = \sum_{j=0}^k (-1)^j s_{k-j}S^j = \sum_{j=0}^k (-1)^j \tbinom{n}{k-j}H_{k-j}S^j.$$

Note that by the Cayley-Hamilton theorem, we have $P_n = 0$ from (1). Observe also that when k is even, the definition of P_k does not depend on the chosen orientation, but when k is odd there is a change of sign in the definition of P_k.

Let us recall that each $P_k(p)$ is also a self-adjoint linear operator on each tangent plane T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_k(p)$ can be simultaneously diagonalized: if $\{e_1,\ldots,e_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\kappa_1(p),\ldots,\kappa_n(p)$, respectively, then they are also the eigenvectors of $P_k(p)$ with corresponding eigenvalues given by

$$\mu_{i,k}(p) = \frac{\partial\sigma_{k+1}}{\partial x_i}(\kappa_1(p),\ldots,\kappa_n(p)) = \sum_{i_1<\cdots<i_k, i_j\neq i} \kappa_{i_1}(p)\cdots\kappa_{i_k}(p), \tag{3}$$

for every $1 \le i \le n$. From here it can be easily seen that

$$\mathrm{trace}(P_k) = (n-k)s_k = c_kH_k, \tag{4}$$

$$\text{trace}(S \circ P_k) = (k+1)s_{k+1=}c_k H_{k+1}, \tag{5}$$

and

$$\text{trace}(S^2 \circ P_k) = (s_1 s_{k+1} - (k+2)s_{k+2}) = \tbinom{n}{k+1}(nH_1 H_{k+1} - (n-k-1)H_{k+2}), \tag{6}$$

where

$$c_k = (n-k)\tbinom{n}{k} = (k+1)\tbinom{n}{k+1}.$$

These properties are all algebraic, and they can be found, for instance, in [15]. There is still another non-algebraic property of P_k that we need, which can be found, for instance, in [14, Lemma A] and [16, Equation (4.4)] (see also [4, page 118]). The property we need is the following equation,

$$\text{tr}(P_k \circ \nabla_X S) = \langle \nabla s_{k+1}, X \rangle = \tbinom{n}{k+1}\langle \nabla H_{k+1}, X \rangle, \quad \text{for} \quad X \in \mathcal{X}(M), \tag{7}$$

where ∇S denotes the covariant differential of S,

$$\nabla S(Y, X) = (\nabla_X S)Y = \nabla_X(SY) - S(\nabla_X Y), \quad X, Y \in \mathcal{X}(M).$$

Associated to each Newton transformation P_k, we consider the second order linear differential operator $L_k : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ given by

$$L_k(f) = \text{trace}(P_k \circ \nabla^2 f).$$

Here $\nabla^2 f : \mathcal{X}(M) \to \mathcal{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f and given by

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathcal{X}(M).$$

Consider $\{E_1, \ldots, E_n\}$ a local orthonormal frame on M and observe that

$$\begin{aligned} \text{div}(P_k(\nabla f)) &= \sum_{i=1}^n \langle (\nabla_{E_i} P_k)(\nabla f, E_i\rangle + \sum_{i=1}^n \langle P_k(\nabla_{E_i} \nabla f), E_i \rangle \\ &= \langle \text{div} P_k, \nabla f \rangle + L_k(f), \end{aligned}$$

where div denotes here the divergence on M and

$$\text{div} P_k := \text{trace}(\nabla P_k) = \sum_{i=1}^n (\nabla_{E_i} P_k)(E_i).$$

Obviously, $\text{div}\ P_0 = \text{div}\ I = 0$. Now Codazzi equation jointly with (7) imply that $\text{div} P_k = 0$ also for every $k \geq 1$ [14, Lemma B]. To see it observe that, from the inductive definition of P_k, we have

$$(\nabla_{E_i} P_k)(E_i) = \tbinom{n}{k}\langle \nabla H_k, E_i \rangle E_i - (\nabla_{E_i} S \circ P_{k-1})E_i - (S \circ \nabla_{E_i} P_{k-1})E_i,$$

so that

$$\mathrm{div}P_k = \binom{n}{k}\nabla H_k - \sum_{i=1}^{n}(\nabla_{E_i}S)(P_{k-1}E_i) - S(\mathrm{div}P_{k-1}).$$

By Codazzi equation we know that ∇S is symmetric, and then for every $X \in \mathcal{X}(M)$

$$\begin{aligned}\sum_{i=1}^{n}\langle(\nabla_{E_i}S)(P_{k-1}E_i), X\rangle &= \sum_{i=1}^{n}\langle P_{k-1}E_i, (\nabla_{E_i}S)X\rangle \sum_{i=1}^{n}\langle P_{k-1}E_i, (\nabla_X S)E_i\rangle \\ &= \mathrm{tr}(P_{k-1}\circ\nabla_X S) = \binom{n}{k}\langle\nabla H_k, X\rangle.\end{aligned}$$

In other words,

$$\sum_{i=1}^{n}(\nabla_{E_i}S)(P_{k-1}E_i) = \binom{n}{k}\nabla H_k,$$

and then

$$\mathrm{div}P_k = -S(\mathrm{div}P_{k-1}).$$

Since $\mathrm{div}P_0 = 0$, this yields $\mathrm{div}P_k = 0$ for every k. As a consequence, $L_k(f) = \mathrm{div}(P_k(\nabla f))$ is a divergence form differential operator on M.

3. Examples

Let $x : M^n \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ be an orientable hypersurface immersed into $\mathbb{M}_c^{n+1}$, with Gauss map N. For a fixed arbitrary vector $a \in \mathbb{R}^{n+2}$, let us consider the coordinate function $\langle a, x\rangle$ on M. From $\nabla^{\mathrm{o}}a = 0$ we see that

$$X(\langle a, x\rangle) = \langle X, a\rangle = \langle X, a^\top\rangle,$$

for every vector field $X \in \mathcal{X}(M)$, where $a^\top \in \mathcal{X}(M)$ denotes the tangential component of a,

$$a = a^\top + \langle a, N\rangle N + c\langle a, x\rangle x. \tag{8}$$

Then the gradient of $\langle a, x\rangle$ on M is given by $\nabla\langle a, x\rangle = a^\top$. By taking covariant derivative in (8) and using the Gauss and Weingarten formulae, we also have from $\nabla^{\mathrm{o}}a = 0$ that

$$\nabla_X\nabla\langle a, x\rangle = \nabla_X a^\top = \langle a, N\rangle SX - c\langle a, x\rangle X, \tag{9}$$

for every tangent vector field $X \in \mathcal{X}(M)$. Therefore, by (5) we find that

$$L_k\langle a, x\rangle = \langle a, N\rangle\mathrm{tr}(S\circ P_k) - c\langle a, x\rangle\mathrm{tr}(P_k) = c_k H_{k+1}\langle a, N\rangle - cc_k H_k\langle a, x\rangle. \tag{10}$$

That is

$$L_k x = c_k H_{k+1}N - cc_k H_k x. \tag{11}$$

Example 3.1. It follows from (11) that every hypersurface with vanishing $(k+1)$-th mean curvature and having constant k-th mean curvature H_k trivially satisfies $L_k x = Ax + b$ with $A = -cc_k H_k I_{n+2} \in \mathbb{R}^{(n+2)\times(n+2)}$ and $b = 0$.

Example 3.2. (Totally umbilical hypersurfaces in $\mathbb{S}^{n+1}$). As is well-known, the totally umbilical hypersurfaces of $\mathbb{S}^{n+1}$ are the n-dimensional round spheres of radius $0 < r \leq 1$ which are obtained by intersecting $\mathbb{S}^{n+1}$ with affine hyperplanes. Specifically, take $a \in \mathbb{R}^{n+2}$ a unit constant vector and, for a given $\tau \in (-1, 1)$, let

$$M_\tau = \{x \in \mathbb{S}^{n+1} : \langle a, x\rangle = \tau\} = \mathbb{S}^n(\sqrt{1-\tau^2}).$$

Then M_τ is a totally umbilical hypersurface in $\mathbb{S}^{n+1}$ with Gauss map $N(x) = (1/\sqrt{1-\tau^2})(a - \tau x)$ and shape operator $S = \tau/\sqrt{1-\tau^2} I$. In particular, its higher order mean curvatures are given by

$$H_k = \frac{\tau^k}{(1-\tau^2)^{k/2}}, \quad k = 0, \ldots, n.$$

Therefore, by equation (11) we see that M_τ satisfies the condition $L_k x = Ax + b$ for every $k = 0, \ldots, n-1$, with

$$A = \frac{-c_k \tau^k}{(1-\tau^2)^{(k+2)/2}} I_{n+2} \quad \text{and} \quad b = \frac{c_k \tau^{k+1}}{(1-\tau^2)^{(k+2)/2}} a.$$

In particular, $b = 0$ only when $\tau = 0$, and then $M_0 = \mathbb{S}^n$ is a totally geodesic round sphere.

Example 3.3. (Totally umbilical hypersurfaces in $\mathbb{H}^{n+1}$). Similarly to the case of the sphere, the totally umbilical hypersurfaces of $\mathbb{H}^{n+1}$ are also obtained by intersecting $\mathbb{H}^{n+1}$ with affine hyperplanes of $\mathbb{R}^{n+2}_1$, but in this case there are three different types of hypersurfaces, depending on the causal character of the hyperplane. To be more precise, take $a \in \mathbb{R}^{n+2}_1$ a non-zero constant vector such that $\langle a, a\rangle \in \{1, 0, -1\}$, and, for a given $\tau \in \mathbb{R}$, let

$$M_\tau = \{x \in \mathbb{H}^{n+1} : \langle a, x\rangle = \tau\}.$$

Then, when $\langle a, a\rangle + \tau^2 > 0$, M_τ is a totally umbilical hypersurface in $\mathbb{H}^{n+1}$. Observe that when $\langle a, a\rangle = 1$ there is no restriction on the value of τ and $M_\tau = \mathbb{H}^n(-\sqrt{1+\tau^2})$ is a hyperbolic n-space of radius $-\sqrt{1+\tau^2}$. On the other hand, if $\langle a, a\rangle = -1$ then $|\tau| > 1$ and $M_\tau = \mathbb{S}^n(\sqrt{\tau^2 - 1})$ is a round n-sphere of radius $\sqrt{\tau^2-1}$. Finally, when $\langle a, a\rangle = 0$ then $\tau \neq 0$ and $M_\tau = \mathbb{R}^n$ is a Euclidean space.

The Gauss map of M_τ is given by $N(x) = (1/\sqrt{\langle a, a\rangle + \tau^2})(a + \tau x)$, its shape operator is $S = -\tau/\sqrt{\langle a, a\rangle + \tau^2} I$, and its higher order mean curvatures are given by

$$H_k = \frac{(-1)^k \tau^k}{(\langle a, a\rangle + \tau^2)^{k/2}}, \quad k = 0, \ldots, n.$$

Therefore, by equation (11) we see that M_τ satisfies the condition $L_k x = Ax + b$ for every $k = 0, \ldots, n-1$, with

$$A = \frac{(-1)^k c_k \langle a, a\rangle \tau^k}{(\langle a, a\rangle + \tau^2)^{(k+2)/2}} I_{n+2} \quad \text{and} \quad b = \frac{(-1)^{k+1} c_k \tau^{k+1}}{(\langle a, a\rangle + \tau^2)^{(k+2)/2}} a.$$

In particular, $b = 0$ only when $\tau = 0$, and then $M_0 = \mathbb{H}^n$ is a totally geodesic hyperbolic space. On the other hand, the totally umbilical Euclidean spaces in $\mathbb{H}^{n+1}$ (corresponding to the case $\langle a, a\rangle = 0$) satisfy the condition $L_k x = Ax + b$ with $A = 0$.

Example 3.4. (Standard Riemannian products in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$). Here we will consider the case where M is a standard Riemannian product; that is, M is either the Riemannian product $\mathbb{S}^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{S}^{n+1}$ with $0 < r < 1$, or the Riemannian product $\mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}^{n-m}(r) \subset \mathbb{H}^{n+1}$ with $r > 0$, for a certain $m = 1, \ldots, n-1$. After a rigid motion of the ambient space, we may consider that M is defined by the equation

$$M = \{x \in \mathbb{M}_c^{n+1} \ : x_{m+1}^2 + \cdots x_{n+1}^2 = r^2\}.$$

In that case, the Gauss map on M is

$$N(x) = \left(\frac{-cr}{\sqrt{1-cr^2}} x_0, \ldots, \frac{-cr}{\sqrt{1-cr^2}} x_m, \frac{\sqrt{1-cr^2}}{r} x_{m+1}, \ldots, \frac{\sqrt{1-cr^2}}{r} x_{n+1} \right).$$

and its the principal curvatures are

$$\kappa_1 = \cdots = \kappa_m = \frac{cr}{\sqrt{1-cr^2}}, \quad \kappa_{m+1} = \cdots = \kappa_n = \frac{-\sqrt{1-cr^2}}{r}.$$

In particular, the higher order mean curvatures are all constant. Therefore, using (11) we get that

$$L_k x = (\lambda x_0, \ldots, \lambda x_m, \mu x_{m+1}, \ldots, \mu x_{n+1})$$

where λ and μ are both constants,

$$\lambda = \frac{-cc_k H_{k+1} r}{\sqrt{1-cr^2}} - cc_k H_k, \quad \mu = \frac{c_k H_{k+1}\sqrt{1-cr^2}}{r} - cc_k H_k.$$

That is, M satisfies the condition $L_k x = Ax + b$ with $b = 0$ and

$$A = \mathrm{diag}[\lambda, \ldots, \lambda, \mu, \ldots, \mu].$$

4. Some Computations and First Auxiliary Results

In Section 3 we have computed the operator L_k acting on the coordinate functions of a hypersurface. On the other hand, consider now the coordinate functions of its Gauss map N, that is, the function $\langle a, N\rangle$ on M, where $a \in \mathbb{R}^{n+2}$ is a fixed arbitrary vector. From $\nabla^o a = 0$ we also see that

$$X(\langle a, N\rangle) = -\langle SX, a\rangle = -\langle X, S(a^\top)\rangle$$

for every vector field $X \in \mathcal{X}(M)$, so that

$$\nabla\langle a, N\rangle = -S(a^\top).$$

Therefore, from (9) we get

$$\begin{aligned} \nabla_X(\nabla\langle a, N\rangle) &= -\nabla_X(Sa^\top) = -\nabla S(a^\top, X) - S(\nabla_X a^\top) \\ &= -(\nabla_X S)a^\top - \langle a, N\rangle S^2 X + c\langle a, x\rangle SX. \end{aligned} \tag{12}$$

By Codazzi equation we know that ∇S is symmetric and then

$$\nabla S(a^\top, X) = \nabla S(X, a^\top) = (\nabla_{a^\top} S)X.$$

Therefore using this in (12), jointly with (6) and (7), we get

$$\begin{aligned} L_k\langle a, N\rangle &= -\mathrm{tr}(P_k \circ \nabla_{a^\top} S) - \langle a, N\rangle \mathrm{tr}(S^2 \circ P_k) + c\langle a, x\rangle \mathrm{tr}(S \circ P_k) \\ &= -\tbinom{n}{k+1}\langle \nabla H_{k+1}, a\rangle \\ &\quad -\tbinom{n}{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2})\langle a, N\rangle \\ &\quad +cc_kH_{k+1}\langle a, x\rangle. \end{aligned} \tag{13}$$

In other words,

$$\begin{aligned} L_k N &= -\tbinom{n}{k+1}\nabla H_{k+1} \\ &\quad -\tbinom{n}{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2})N \\ &\quad +\tbinom{n}{k+1}c(k+1)H_{k+1}x. \end{aligned} \tag{14}$$

Let us assume that, for a fixed $k = 0, \ldots, n-1$, the immersion

$$x : M^n \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$$

satisfies the condition

$$L_k x = Ax + b, \tag{15}$$

for a constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and a constant vector $b \in \mathbb{R}^{n+2}$. From (11) we get that

$$Ax = -b + c_k H_{k+1} N - cc_k H_k x = -b^\top + (c_k H_{k+1} - \langle b, N\rangle)N - c(c_k H_k + \langle b, x\rangle)x, \tag{16}$$

where $b^\top \in \mathcal{X}(M)$ denotes the tangential component of b. Now, if we take covariant derivative in (15) and use the equation (11) as well as Weingarten formula, we obtain

$$AX = -c_k H_{k+1} SX - cc_k H_k X + c_k \langle \nabla H_{k+1}, X\rangle N - cc_k \langle \nabla H_k, X\rangle x \tag{17}$$

for every tangent vector field $X \in \mathcal{X}(M)$. On the other hand, taking into account that

$$L_k(fg) = (L_k f)g + f(L_k g) + 2\langle P_k(\nabla f), \nabla g\rangle, \quad f, g \in \mathcal{C}^\infty(M),$$

we also get from (10) and (13) that

$$\begin{aligned}
&L_k(L_k\langle a, x\rangle)\\
=\ & -c_k\tbinom{n}{k+1} H_{k+1}\langle \nabla H_{k+1}, a\rangle - 2c_k\langle (S\circ P_k)(\nabla H_{k+1}), a\rangle - 2cc_k\langle P_k(\nabla H_k), a\rangle\\
& -c_k\left(\tbinom{n}{k+1} H_{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}) + cc_kH_kH_{k+1} - L_kH_{k+1}\right)\langle a, N\rangle\\
& +c_k\left(cc_kH_{k+1}^2 + c_kH_k^2 - cL_kH_k\right)\langle a, x\rangle.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&L_k(L_k x)\\
=\ & -c_k\tbinom{n}{k+1} H_{k+1}\nabla H_{k+1} - 2c_k(S\circ P_k)(\nabla H_{k+1}) - 2cc_kP_k(\nabla H_k)\\
& -c_k\left(\tbinom{n}{k+1} H_{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}) + cc_kH_kH_{k+1} - L_kH_{k+1}\right)N\\
& +c_k\left(cc_kH_{k+1}^2 + c_kH_k^2 - cL_kH_k\right)x.
\end{aligned}$$

From here, by applying the operator L_k on both sides of (15) and using again (11), we have

$$\begin{aligned}
&H_{k+1}AN\\
=\ & -\tbinom{n}{k+1} H_{k+1}\nabla H_{k+1} - 2(S\circ P_k)(\nabla H_{k+1}) - 2cP_k(\nabla H_k)\\
& -\left(\tbinom{n}{k+1} H_{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}) + cc_kH_kH_{k+1} - L_kH_{k+1}\right)N\\
& +\left(cc_kH_{k+1}^2 + c_kH_k^2 - cL_kH_k\right)x + cH_kAx.
\end{aligned} \tag{18}$$

Using here (16), we get

$$
\begin{aligned}
&H_{k+1}AN \\
&= -\tbinom{n}{k+1}H_{k+1}\nabla H_{k+1} - 2(S\circ P_k)(\nabla H_{k+1}) - 2cP_k(\nabla H_k) - cH_k b^{\top} \\
&\quad - \Big(\tbinom{n}{k+1}H_{k+1}(nH_1H_{k+1} - (n{-}k{-}1)H_{k+2}) + cH_k\langle b,N\rangle - L_kH_{k+1}\Big)N \\
&\quad + \left(cc_kH_{k+1}^2 - cH_k\langle b,x\rangle - cL_kH_k\right)x.
\end{aligned} \tag{19}
$$

4.1. The case where A is self-adjoint

From (17) we have

$$\langle AX, Y\rangle = \langle X, AY\rangle \tag{20}$$

for every tangent vector fields $X, Y \in \mathcal{X}(M)$. In other words, the endomorphism determined by A is always self-adjoint when restricted to the tangent hyperplanes of the hypersurface. Therefore, A is self-adjoint if and only if the three following equalities hold

$$\langle AX, x\rangle = \langle x, AX\rangle \quad \text{for every } X \in \mathcal{X}(M), \tag{21}$$

$$\langle AX, N\rangle = \langle X, AN\rangle \quad \text{for every } X \in \mathcal{X}(M), \tag{22}$$

and

$$\langle AN, x\rangle = \langle N, Ax\rangle. \tag{23}$$

From (16) and (17) it easily follows that (21) is equivalent to

$$\nabla\langle b, x\rangle = b^{\top} = c_k\nabla H_k, \tag{24}$$

that is, $\langle b, x\rangle - c_kH_k$ is constant on M. On the other hand, from (17) and (18), and using also (24), it follows that, at points where $H_{k+1} \neq 0$, (22) is equivalent to

$$
\begin{aligned}
&\frac{2}{H_{k+1}}(S\circ P_k)(\nabla H_{k+1}) + (k+2)\tbinom{n}{k+1}\nabla H_{k+1} = \\
&\quad -\frac{c}{H_{k+1}}\left(2P_k(\nabla H_k) + c_kH_k\nabla H_k\right).
\end{aligned} \tag{25}
$$

Finally, using again (24) we have by (10) that

$$L_kH_k = \frac{1}{c_k}L_k\langle b, x\rangle = H_{k+1}\langle b, N\rangle - cH_k\langle b, x\rangle. \tag{26}$$

Observe also that

$$\langle Ax, x\rangle = -\langle b, x\rangle - cc_k H_k.$$

Therefore, from (18) we get that

$$\begin{aligned} H_{k+1}\langle AN, x\rangle &= c_k H_{k+1}^2 + cc_k H_k^2 - L_k H_k + cH_k\langle Ax, x\rangle \\ &= c_k H_{k+1}^2 - H_{k+1}\langle b, N\rangle \\ &= H_{k+1}\langle N, Ax\rangle. \end{aligned}$$

Thus we have that, at points where $H_{k+1}\neq 0$, the first two equalities (21) and (22) imply the third one (23).

Now we are ready to prove the following auxiliary result.

Lemma 4.1. *Let $x : M^n \to \mathbb{M}_c^{n+1}\subset \mathbb{R}_q^{n+2}$ be an orientable hypersurface satisfying the condition $L_k x = Ax + b$, for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}^{n+2}$. Then H_k is constant if and only if H_{k+1} is constant.*

Proof. Assume that H_k is constant and let us consider the open set

$$\mathcal{U} = \{p \in M : \nabla H_{k+1}^2(p) \neq 0\}.$$

Our objective is to show that $\mathcal{U}$ is empty. Assume that $\mathcal{U}$ is non-empty. From (25) we have that

$$\frac{2}{H_{k+1}}(S\circ P_k)(\nabla H_{k+1}) + (k+2)\binom{n}{k+1}\nabla H_{k+1} = 0 \quad \text{on} \quad \mathcal{U}.$$

Equivalently,

$$(S\circ P_k)(\nabla H_{k+1}) = -\frac{k+2}{2}\binom{n}{k+1}H_{k+1}\nabla H_{k+1} \quad \text{on} \quad \mathcal{U}.$$

Then, reasoning exactly as Alías and Gürbüz in [4, Lemma 5] (starting from equation (23) in [4]) we conclude that H_{k+1} is locally constant on $\mathcal{U}$, which is a contradiction. Actually, the proof in [4] works also here word by word, with the only observation that, since we are assuming that H_k is constant, (17) reduces now to

$$AX = -c_k H_{k+1} SX - cc_k H_k X + c_k\langle \nabla H_{k+1}, X\rangle N.$$

Therefore, instead of having $AE_i = -c_k H_{k+1}\kappa_i E_i$, now we have $AE_i = -c_k(H_{k+1}\kappa_i + cH_k)E_i$ for every $m+1 \leq i \leq n$ (see the last paragraph of the proof of [4, Lemma 5]). But H_k being constant, that makes no difference to the reasoning.

Conversely, assume now that H_{k+1} is constant and let us consider the open set

$$\mathcal{V} = \{p \in M : \nabla H_k^2(p) \neq 0\}.$$

Our objective now is to show that $\mathcal{V}$ is empty. Let us consider first the case where $H_{k+1} = 0$ and assume that $\mathcal{V}$ is non-empty. In this case, by (24) and (26), (19) reduces to

$$-2cP_k(\nabla H_k) - cc_k H_k \nabla H_k - cH_k \langle b, N\rangle N = 0.$$

Thus, $\langle b, N\rangle = 0$ on $\mathcal{V}$. By (16) this gives $\langle AN, x\rangle = \langle N, Ax\rangle = 0$ and, since $\langle AN, X\rangle = \langle N, AX\rangle = 0$ for every $X \in \mathcal{X}(M)$, we obtain that $AN = \langle AN, N\rangle N$; that is, N is an eigenvector of A with corresponding eigenvalue $\lambda = \langle AN, N\rangle$. In particular, λ is locally constant on $\mathcal{V}$. Therefore,

$$\begin{aligned} AX &= -cc_k H_k X - cc_k \langle \nabla H_k, X\rangle x \\ AN &= \lambda N \\ Ax &= -c_k \nabla H_k - c(2c_k H_k + \alpha)x, \end{aligned}$$

where $\alpha = \langle b, x\rangle - c_k H_k$ and λ are both locally constant on $\mathcal{V}$. Then,

$$\mathrm{tr}(A) = -ncc_k H_k + \lambda - c(2c_k H_k + \alpha) = \text{constant},$$

which implies that H_k is locally constant on $\mathcal{V}$, which is a contradiction.

On the other hand, if $H_{k+1} \neq 0$ is constant and we assume that $\mathcal{V}$ is non-empty, then from (25) we have that

$$2P_k(\nabla H_k) + c_k H_k \nabla H_k = 0 \quad \text{on} \quad \mathcal{V}.$$

Equivalently,

$$P_k(\nabla H_k) = -\frac{c_k}{2} H_k \nabla H_k \quad \text{on} \quad \mathcal{V}. \tag{27}$$

Here, we will follow a similar reasoning to that in [4, Lemma 5]. Consider $\{E_1, \ldots, E_n\}$ a local orthonormal frame of principal directions of S such that $SE_i = \kappa_i E_i$ for every $i = 1, \ldots, n$, and then

$$P_k E_i = \mu_{i,k} E_i,$$

with

$$\mu_{i,k} = \sum_{j=0}^{k} (-1)^j \binom{n}{k-j} H_{k-j} \kappa_i^j = \sum_{i_1 < \cdots < i_k, i_j \neq i} \kappa_{i_1} \cdots \kappa_{i_k}. \tag{28}$$

Therefore, writing

$$\nabla H_k = \sum_{i=1}^{n} \langle \nabla H_k, E_i \rangle E_i$$

we see that (27) is equivalent to

$$\langle \nabla H_k, E_i \rangle \left(\mu_{i,k} + \frac{c_k}{2} H_k \right) = 0 \quad \text{on} \quad \mathcal{V}$$

for every $i = 1, \ldots, n$. Thus, for every i such that $\langle \nabla H_k, E_i \rangle \neq 0$ on $\mathcal{V}$ we get

$$\mu_{i,k} = -\frac{c_k}{2} H_k. \tag{29}$$

This implies that $\langle \nabla H_k, E_i \rangle = 0$ necessarily for some i. Otherwise, we would have (29) for every $i = 1, \ldots, n$, which would imply

$$c_k H_k = \operatorname{tr}(P_k) = \sum_{i=1}^{n} \mu_{i,k} = -\frac{n c_k}{2} H_k,$$

and thus $H_k = 0$ on $\mathcal{V}$, which is a contradiction.

Therefore, re-arranging the local orthonormal frame if necessary, we may assume that for some $1 \leq m < n$ we have $\langle \nabla H_k, E_i \rangle \neq 0$ for $i = 1, \ldots, m$, $\langle \nabla H_k, E_i \rangle = 0$ for $i = m+1, \ldots, n$, and $\kappa_1 < \kappa_2 < \cdots < \kappa_m$. The integer m measures the number of linearly independent principal directions of ∇H_k, and ∇H_k is a principal direction of S if and only if $m = 1$. From (29) we know that

$$\mu_{1,k} = \cdots = \mu_{m,k} = -\frac{c_k}{2} H_k \neq 0 \quad \text{on} \quad \mathcal{V}. \tag{30}$$

Thus, by (28) it follows that $\kappa_1 < \kappa_2 < \cdots < \kappa_m$ are m distinct real roots of the following polynomial equation of degree k,

$$Q(t) = \sum_{j=0}^{k} (-1)^j \binom{n}{k-j} H_{k-j} t^j = -\frac{c_k}{2} H_k.$$

In particular $m \leq k$. On the other hand, each κ_i is also a root of the characteristic polynomial of S, which can be written as

$$Q_S(t) = (-1)^k t^{n-k} Q(t) + \sum_{j=k+1}^{n} (-1)^j \binom{n}{j} H_j t^{n-j}.$$

Then, $\kappa_1 < \kappa_2 < \cdots < \kappa_m$ are also m distinct real roots of the following polynomial equation of degree $n-k$,

$$(-1)^{k+1} \frac{c_k}{2} H_k t^{n-k} + \sum_{j=k+1}^{n} (-1)^j \binom{n}{j} H_j t^{n-j} = 0.$$

In particular, $m \leq n-k$, that is, $n-m \geq k$. Now we claim that

$$\mu_{1,k} = \cdots = \mu_{m,k} = \sum_{m<i_1<\cdots<i_k} \kappa_{i_1} \cdots \kappa_{i_k}. \tag{31}$$

The proof of (31) follows exactly as the proof of equation (29) in [4] and we omit it here.

Finally, from equation (17) we have

$$AE_i = -c_k(H_{k+1}\kappa_i + cH_k)E_i$$

for every $m+1 \leq i \leq n$. Therefore, every $-c_k(H_{k+1}\kappa_i + cH_k)$ with $i = m+1, \ldots n$ is a constant eigenvalue α_i of the constant matrix A. Then,

$$\kappa_i = -\frac{\alpha_i + cc_kH_k}{c_kH_{k+1}} \quad \text{for every } i = m+1, \ldots n$$

and from (31) and (30) we get that

$$-\frac{c_k}{2}H_k = \sum_{m<i_1<\cdots<i_k} \kappa_{i_1} \cdots \kappa_{i_k} = \frac{(-1)^k}{c_k^k H_{k+1}^k} \sum_{m<i_1<\cdots<i_k} (\alpha_{i_1} + cc_kH_k) \cdots (\alpha_{i_k} + cc_kH_k)$$

on $\mathcal{V}$. But this means that H_k is locally constant on $\mathcal{V}$, which is a contradiction with the definition of $\mathcal{V}$. This finishes the proof of Lemma 4.1. ■

5. Proof of Theorem 1.2

We have already checked in Section 3 that each one of the hypersurfaces mentioned in Theorem 1.2 does satisfy the condition $L_k x = Ax$ for a self-adjoint constant matrix A. Conversely, let us assume that $x : M^n \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ satisfies the condition $L_k x = Ax$ for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$. Since $b = 0$, from (24) we get that H_k is constant on M. Thus, by Lemma 4.1 we know that H_{k+1} is also constant on M. If $H_{k+1} = 0$ there is nothing to prove. Then, we may assume that H_{k+1} is a non-zero constant and H_k is also constant. Then from (17) and (18) we obtain

$$AX = -c_kH_{k+1}SX - cc_kH_kX \tag{32}$$

for every tangent vector field $X \in \mathcal{X}(M)$, and

$$AN = \alpha N + c_k\left(cH_{k+1} + \frac{H_k^2}{H_{k+1}}\right)x + c\frac{H_k}{H_{k+1}}Ax, \tag{33}$$

with

$$\alpha = -\binom{n}{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}) - cc_kH_k.$$

Taking covariant derivative in (33) and using (32) we have for every $X \in \mathcal{X}(M)$

$$\begin{aligned}\nabla^{o}{}_X(AN) &= \langle\nabla\alpha, X\rangle N - \alpha SX + c_k\left(cH_{k+1} + \frac{H_k^2}{H_{k+1}}\right)X + c\frac{H_k}{H_{k+1}}AX \\ &= \langle\nabla\alpha, X\rangle N + \binom{n}{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}SX + cc_kH_{k+1}X.\end{aligned}$$

On the other hand, from (32) we also find that

$$\nabla^{o}{}_X(AN) = A(\nabla^{o}{}_X N) = -A(SX) = c_kH_{k+1}S^2X + cc_kH_kSX$$

It follows from here that $\langle\nabla\alpha, X\rangle = 0$ for every $X \in \mathcal{X}(M)$, that is, α is constant on M, and also that the shape operator S satisfies the following quadratic equation

$$S^2 + \lambda S - cI = 0,$$

where

$$\lambda = \frac{\alpha}{c_kH_{k+1}} + 2c\frac{H_k}{H_{k+1}} = \text{constant}.$$

As a consequence, either M is totally umbilical in $\mathbb{M}_c^{n+1}$ (but not totally geodesic, because of $H_{k+1} \neq 0$) or M is an isoparametric hypersurface of $\mathbb{M}_c^{n+1}$ with two constant principal curvatures. The former cannot occur, because the only totally umbilical hypersurfaces in $\mathbb{M}_c^{n+1}$ which satisfy $L_kx = Ax$ with $b = 0$ are the totally geodesic ones (see Examples 3.2 and 3.3). In the latter, from well-known results by Lawson [13, Lemma 2] and Ryan [17, Theorem 2.5] we conclude that M is an open piece of a standard Riemannian product.

6. Proof of Theorem 1.7

We have already checked in Section 3 that each one of the hypersurfaces mentioned in Theorem 1.7 does satisfy the condition $L_kx = Ax + b$ for a self-adjoint constant matrix A. Conversely, let us assume that $x : M^n \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ satisfies the condition $L_kx = Ax + b$ for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some non-zero constant vector $b \in \mathbb{R}^{n+2}$. Since H_k is assumed to be constant, by Lemma 4.1 we know that H_k and H_{k+1} are both constant on M. The case $H_{k+1} = 0$ cannot occur, because in that case we have $b = 0$ (Example 3.1). Therefore, we have that H_{k+1} is a non-zero constant and H_k is also constant. Then from (17) and (18) we obtain

$$AX = -c_kH_{k+1}SX - cc_kH_kX \tag{34}$$

for every tangent vector field $X \in \mathcal{X}(M)$, and

$$AN = \alpha N + c_k\left(cH_{k+1} + \frac{H_k^2}{H_{k+1}}\right)x + c\frac{H_k}{H_{k+1}}Ax, \tag{35}$$

with

$$\alpha = -\binom{n}{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}) - cc_kH_k.$$

Taking covariant derivative in (35) and using (34) we have for every $X \in \mathcal{X}(M)$

$$\begin{aligned}\nabla^{\mathrm{o}}{}_X(AN) &= \langle\nabla\alpha, X\rangle N - \alpha SX + c_k\left(cH_{k+1} + \frac{H_k^2}{H_{k+1}}\right)X + c\frac{H_k}{H_{k+1}}AX\\ &= \langle\nabla\alpha, X\rangle N + \binom{n}{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}SX + cc_kH_{k+1}X.\end{aligned}$$

On the other hand, from (34) we also find that

$$\nabla^{\mathrm{o}}{}_X(AN) = A(\nabla^{\mathrm{o}}{}_X N) = -A(SX) = c_kH_{k+1}S^2X + cc_kH_kSX$$

It follows from here that $\langle\nabla\alpha, X\rangle = 0$ for every $X \in \mathcal{X}(M)$, that is, α is constant on M, and also that the shape operator S satisfies the following quadratic equation

$$S^2 + \lambda S - cI = 0,$$

where

$$\lambda = \frac{\alpha}{c_kH_{k+1}} + 2c\frac{H_k}{H_{k+1}} = \text{constant}.$$

As a consequence, either M is totally umbilical in $\mathbb{M}_c^{n+1}$ or M is an isoparametric hypersurface of $\mathbb{M}_c^{n+1}$ with two constant principal curvatures. In the latter, from well-known results by Lawson [13, Lemma 2] and Ryan [17, Theorem 2.5] we would get that M is an open piece of a standard Riemannian product, but this case cannot occur because they satisfy the condition $L_kx = Ax + b$ with $b = 0$ (Example 3.4).

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