TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 5, pp. 1935-1956, October 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON PERIODIC CONTINUED FRACTIONS OVER $\mathbb{F}_q((X^{-1}))$

H. Ben Amar and M. Mkaouar*

Abstract. Let $\mathbb{F}_q$ be a field with q elements of characteristic p and $\mathbb{F}_q((X^{-1}))$ be the field of formal power series over $\mathbb{F}_q$. Let f be a quadratic formal power series of continued fraction expansion $[b_0; b_1, \ldots, b_s, \overline{a_1, \ldots, a_t}]$, we denote by $t = \mathrm{Per}\,(f)$ the period length of the partial quotients of f. The aim of this paper is to study the continued fraction expansion of Af where A is a polynomial $\in \mathbb{F}_q[X]$. In particular we study the asymptotic behavior of the functions

$$S(N, n) = \sup_{\deg A = N} \sup_{f \in \Lambda_n} \mathrm{Per}\,(Af) \text{ and } R(N) = \sup_{n \geq 1} \frac{S(N, n)}{n},$$

where Λ_n is the set of quadratic formal power series of period n in $\mathbb{F}_q((X^{-1}))$.

1. Introduction

In 1974, Cohen [1] studied the function $S(N, n) = \sup_{\mathrm{Per}\,(x)=n} \mathrm{Per}\,(Nx)$ where N is a positive integer, x is a quadratic irrational and $\mathrm{Per}\,(Nx)$ is the length of the period of the continued fraction expansion of Nx. He made use of an algorithm for computing the continued fraction expansion of Nx and defined a projective space which permits to evaluate $S(N, n)$ and to study the function $R(N) = \sup_{n \geq 1} \dfrac{S(N, n)}{n}$. Later, Cusick [2] studied the length of the period of the product of a positive integer with a quadratic irrational by using Raney's algorithm (see [5]). The aim of this paper is to give a similar result to the one of Cohen in the case of formal power series over a finite fields $\mathbb{F}_q$ by using Cohen's [1] and Mendès France's [3, 4] methods.

Received July 3, 2008, accepted February 21, 2009.
Communicated by Tzu-Yueh Wang.
2000 *Mathematics Subject Classification*: 11A55.
Key words and phrases: Continued fraction.
The authors were supported in part by the Tunisian Project CMCU 05S1505.
*Corresponding author.

Let p be a prime, and let $\mathbb{F}_q$ be a field with q elements of characteristic p. Moreover, let $\mathbb{F}_q[X]$ be the ring of polynomials over $\mathbb{F}_q$ and $\mathbb{F}_q(X)$ its field of fractions. The field $\mathbb{F}_q((X^{-1}))$ of formal power series over $\mathbb{F}_q$ is defined by

$$\mathbb{F}_q((X^{-1})) = \{ \sum_{n \geq n_0} f_n X^{-n} : \quad f_n \in \mathbb{F}_q, \ n_0 \in \mathbb{Z}\}.$$

Let $f = \sum_{n \geq n_0} f_n X^{-n}$ where $n_0 \in \mathbb{Z}$. We denote by $[f]$ the polynomial part of f and $\{f\}$ its fractional part. We define a non archimedean absolute value on $\mathbb{F}_q((X^{-1}))$ by $\mid f \mid = e^{-n_0}$, for any $f \in \mathbb{F}_q((X^{-1}))$. It is clear that, for any $P \in \mathbb{F}_q[X]$, $\mid P \mid = e^{\deg P}$ and, for any $Q \in \mathbb{F}_q[X]$, such that $Q \neq 0$, $\mid \dfrac{P}{Q} \mid = e^{\deg P - \deg Q}$.

We can write the continued fraction expansion of an irrational $f \in \mathbb{F}_q((X^{-1}))$ in the form

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0; a_1, a_2, \ldots],$$

where a_i is a polynomial of degree ≥ 1 for each $i \geq 1$ and $a_0 \in \mathbb{F}_q[X]$. The sequence $(a_i)_{i \geq 0}$ is called the sequence of partial quotients of f.

We say that the formal power series f has a t-periodic continued fraction expansion or the continued fraction expansion of f is ultimately periodic of period t if the sequence $(a_i)_{i \geq 0}$ is ultimately periodic of period t. We denote by $\mathrm{Per}\,(f) = t$ and write $f = [a_0; a_1, \ldots, a_s, \overline{a_{s+1}, \ldots, a_{s+t}}]$ for the continued fraction expansion of f. We say that the formal power series f has a pure periodic continued fraction expansion of period t if the sequence $(a_i)_{i \geq 0}$ is purely periodic of period t and write $f = [\overline{a_1; \ldots, a_t}]$. Let $f \in \mathbb{F}_q((X^{-1}))$, then f is quadratic if and only if the continued fraction expansion of f is periodic. We define the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ and $(Q_n)_{n \in \mathbb{N}}$ by :

$$P_{-1} = 1, \ P_0 = a_0 \text{ and } P_{n+1} = a_{n+1} P_n + P_{n-1},$$

and

$$Q_{-1} = 0, \ Q_0 = 1 \text{ and } Q_{n+1} = a_{n+1} Q_n + Q_{n-1}.$$

The fraction $\dfrac{P_n}{Q_n}$ is called n-th convergent of f. It is clear that $\dfrac{P_n}{Q_n} = [a_0; a_1, \ldots, a_n]$.

Let $(P'_n)_{n \in \mathbb{N}}$ and $(Q'_n)_{n \in \mathbb{N}}$ be the sequences associated to the periodic part of f, i.e. $\dfrac{P'_n}{Q'_n} = [\overline{a_{s+1}, \ldots, a_{s+t}}]$, we call $M = \begin{pmatrix} P'_t & P'_{t-1} \\ Q'_t & Q'_{t-1} \end{pmatrix}$ the matrix associated to the quadratic formal power series f.

Let J, H be two polynomials in $\mathbb{F}_q[X]$ and $[b_0; b_1, \ldots, b_s]$ the continued fraction expansion of $\frac{J}{H}$, we denote by $\left[\frac{J}{H}\right] = b_0, b_1, \ldots, b_s$ and $\psi\left(\frac{J}{H}\right) = s$ the length of continued fraction expansion of $\frac{J}{H}$ and

$$\left[c_0; \ldots, c_i, \left[\frac{J}{H}\right], c_{i+1}, \ldots\right] = [c_0; \ldots, c_i, b_0, b_1, \ldots, b_s, c_{i+1}, \ldots],$$

and $\left[\frac{J}{0}\right]$ the empty word. Note that $\psi\left(\overline{\left[\frac{J}{H}\right]}\right) = \psi\left(\left[\frac{J}{H}\right]\right) + 1$.

The paper is organized as follows. In section 2, we give the continued fraction expansion of Af where $f \in \mathbb{F}_q((X^{-1}))$ and A is a nonconstant polynomial in $\mathbb{F}_q[X]$. Section 3 is devoted to the study of the length of the period of the continued fraction expansion of Af given in section 2 where f is quadratic. In section 4, we will construct a new space noted P_A in order to study the functions $S(N, n)$ and $R(N)$ in section 5.

2. Continued Fraction of the Product of a Polynomial with a Formal Power Series

We describe an algorithm which gives the continued fraction expansion of Af where $f \in \mathbb{F}_q((X^{-1}))$ and $A \in \mathbb{F}_q[X]\backslash\mathbb{F}_q$ in the following theorem.

Theorem 2.1. *Let $f \in \mathbb{F}_q((X^{-1}))$ and $A \in \mathbb{F}_q[X]\backslash\mathbb{F}_q$, we write the continued fraction expansion of f in the following form*

$$f = [Ab'_0 + h_0; Ab'_1 + h_1, \ldots, Ab'_n + h_n, \ldots] \tag{2.1}$$

with b'_i, $h_i \in \mathbb{F}_q[X]$ and $\deg(h_i) < \deg(A)$ for each $i \geq 0$. Define the sequences $(H_i)_{i\geq -1}$, $(b''_i)_{i\geq 0}$, $(j_i)_{i\geq 0}$, $(Q^{(i)})_{i\geq -1}$, $(t_i)_{i\geq -1}$, $(u_i)_{i\geq -1}$ and $(\delta_i)_{i\geq -1}$ by : $Q^{(-1)} = Q^{(0)} = 0$, $t_{-1} = u_{-1} = 0$, $\delta_{-1} = 1$, $\delta_0 = A$ and for each $i \geq 0$

- $H_i = \frac{A}{\delta_i}$,
- $b''_i + \frac{j_i}{H_i} = \frac{(-1)^{u_{i-1}}\delta_i h_i - \delta_{i-1}Q^{(i-1)}}{H_i}$, *where b''_i, $j_i \in \mathbb{F}_q[X]$, $\deg(j_i) < \deg(H_i)$, for all $i \geq 1$ and $j_0 = 0$,*
- $t_i + 1 = \psi\left(\frac{j_i}{H_i}\right)$ *and $u_i = u_{i-1} + t_i$,*
- *$Q^{(i)}$ is the denominator of the last but one convergent of the continued fraction expansion of $\frac{j_i}{H_i}$, for each $i \geq 1$,*

- $\delta_{i+1} = \gcd(j_i, H_i)$.

Then,

$$Af = \left[(-1)^{u-1}\delta_0^2 b_0' + b_0''\ ,\left[\frac{H_0}{j_0}\right], \ldots, (-1)^{u_i-1}\delta_i^2 b_i' + b_i''\ ,\left[\frac{H_i}{j_i}\right], \ldots\right].$$

Remark 2.2. We consider the expansion provided by Theorem 2.1 as a generalized continued fraction expansion of Af. However, we note that this algorithm does not give the usual continued fraction expansion of Af. In fact, the term $\lambda_i = (-1)^{u_i-1}\delta_i^2 b_i' + b_i''$ may be in $\mathbb{F}_q$ for some index $i \geq 1$. However, the usual continued fraction expansion of Af can be deduced from λ_i. We deduce the usual continued fraction expansion of Af as follows:

If $\lambda_i = 0$ then $[c_0, \ldots, c_{i-1}\ , 0\ , c_{i+1}, \ldots] = [c_0, \ldots, c_{i-1} + c_{i+1}, \ldots]$.

If $\lambda_i \in \mathbb{F}_q^*$ then

$$[c_0, \ldots, c_{i-1}\ , \lambda_i\ , c_{i+1}, \ldots] = [c_0, \ldots, c_{i-1} + \frac{1}{\lambda_i}\ , -{\lambda_i}^2 c_{i+1} - \lambda_i\ , -\frac{c_{i+2}}{{\lambda_i}^2}, \ldots]$$

because

$$c_{i-1} + \cfrac{1}{\lambda_i + \cfrac{1}{\beta_{i+1}}} = c_{i-1} + \frac{1}{\lambda_i} - \frac{1}{{\lambda_i}^2\beta_{i+1} + \lambda_i} \text{ where } \beta_{i+1} = [c_{i+1}, \ldots].$$

We notice that the length of usual continued fraction expansion of Af is less or equal to the length of the generalized continued fraction expansion given by Theorem 2.1.

We need the following lemma in order to prove Theorem 2.1.

Lemma 2.3. *Let J and H be two polynomials in $\mathbb{F}_q[X]$ such that* $\deg(J) < \deg(H)$, $\delta = \gcd(J, H)$ *and* $\frac{J}{H} = [0; c_1, \ldots, c_s]$, *then for all $Z \in \mathbb{F}_q((X^{-1}))$ we have*

$$\begin{aligned} \frac{J}{H} + \frac{1}{H^2 Z} &= \left[0; c_1, \ldots, c_s, (-1)^s\delta^2 Z - \frac{\delta Q_{s-1}}{H}\right] \\ &= \left[\left[\frac{J}{H}\right], (-1)^s\delta^2 Z - \frac{\delta Q_{s-1}}{H}\right], \end{aligned}$$

where Q_{s-1} is the denominator of the last but one convergent of the continued fraction expansion of $\frac{J}{H}$.

Proof. Let P_s be the numerator of the last convergent of the continued fraction expansion of $\frac{J}{H}$, then

$P_s = \dfrac{J}{\delta}$ and $Q_s = \dfrac{H}{\delta}$. Let $\gamma = (-1)^s \delta^2 Z - \dfrac{\delta Q_{s-1}}{H}$, then

$$\begin{aligned}[0, c_1, \ldots, c_s, \gamma] &= \frac{P_s \gamma + P_{s-1}}{Q_s \gamma + Q_{s-1}} \\ &= \frac{P_s}{Q_s} + \frac{(-1)^s}{Q_s(Q_s\gamma + Q_{s-1})} \\ &= \frac{J}{H} + \frac{1}{H^2 Z}.\end{aligned}$$

Proof of Theorem 2.1. We first prove by induction that $H_i \in \mathbb{F}_q[X]$. We have $H_0 = 1$, then the claim is true for $i = 0$. Suppose that $H_i \in \mathbb{F}_q[X]$, then $H_i \mid A$. Since $\delta_{i+1} \mid H_i$, thus $\delta_{i+1} \mid A$, which implies $H_{i+1} \in \mathbb{F}_q[X]$. Secondly, we prove by induction that for $i \geq 2$

$$Af = \left[(-1)^{u_{-1}} \delta_0^2 b_0' + b_0'' \,, \begin{bmatrix} \dfrac{H_0}{j_0} \end{bmatrix}, \ldots, \begin{bmatrix} \dfrac{H_{i-1}}{j_{i-1}} \end{bmatrix} \,, (-1)^{u_{i-1}} \delta_i^2 x_i - \delta_i \frac{Q^{(i-1)}}{H_{i-1}} \right].$$

Let x_i be the continued fraction defined for all $i \geq 0$ by

$$Ax_i = [Ab_i' + h_i, \ldots, Ab_s' + h_s, \ldots]. \tag{2.2}$$

The first step of the algorithm is to combine the equations (2.1) and (2.2) for $i = 1$. We obtain

$$Af = A\left(Ab_0' + h_0 + \frac{1}{Ax_1} \right) = A^2 b_0' + Ah_0 + \frac{1}{x_1}.$$

Using the induction formula, for b_i'' given in the statement of Theorem 2.1, we obtain $b_0'' = Ah_0$. This implies that

$$Af = (-1)^{u_{-1}} \delta_0^2 b_0' + b_0'' + \frac{1}{x_1} = \left[(-1)^{u_{-1}} \delta_0^2 b_0' + b_0'',\ \begin{bmatrix} \dfrac{H_0}{j_0} \end{bmatrix},\ x_1 \right],$$

recall that $\begin{bmatrix} \dfrac{H_0}{j_0} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{0} \end{bmatrix}$ is the empty set. Moreover,

$$b_1'' + \frac{j_1}{H_1} = \frac{(-1)^{u_0} \delta_1 h_1 - \delta_0 Q^{(0)}}{H_1} = \frac{h_1}{A},$$

because $Q^{(0)} = 0$, $H_1 = A$, and $t_0 = u_0 = 0$, which yields

$$x_1 = (-1)^{u_0} \delta_1^2 b_1' + b_1'' + \frac{j_1}{H_1} + \frac{1}{H_1^2 x_2}.$$

Applying Lemma 2.3 to the polynomials j_1 and H_1, we obtain

$$\frac{j_1}{H_1}+\frac{1}{H_1^2x_2}=\left[\left[\frac{j_1}{H_1}\right]\ ,(-1)^{u_1}\delta_2^2x_2-\delta_2\frac{Q^{(1)}}{H_1}\right].$$

Since $\dfrac{H_1}{j_1}=[b_1;\ldots,b_{t_1}]$, we have

$$Af=\left[(-1)^{u-1}\delta_0^2b'_0+b''_0\ ,\left[\frac{H_0}{j_0}\right]\ ,(-1)^{u_0}\delta_1^2b'_1+b''_1,\left[\frac{H_1}{j_1}\right],(-1)^{u_1}\delta_2^2x_2-\delta_2\frac{Q^{(1)}}{H_1}\right],$$

then the claim is true for $i=2$. Suppose that

$$Af=\left[(-1)^{u-1}\delta_0^2b'_0+b''_0\ ,\left[\frac{H_0}{j_0}\right],\ldots,\left[\frac{H_{i-1}}{j_{i-1}}\right]\ ,(-1)^{u_{i-1}}\delta_i^2x_i-\delta_i\frac{Q^{(i-1)}}{H_{i-1}}\right],$$

for $i>2$. Since $x_i=b'_i+\dfrac{h_i}{A}+\dfrac{1}{A^2x_{i+1}}$ and $b''_i+\dfrac{j_i}{H_i}=(-1)^{u_{i-1}}\dfrac{\delta_i^2h_i}{A}-\dfrac{\delta_{i-1}Q^{(i-1)}}{H_i}$, it is easy to verify that

$$(-1)^{u_{i-1}}\delta_i^2x_i-\delta_i\frac{Q^{(i-1)}}{H_{i-1}}=(-1)^{u_{i-1}}\delta_i^2b'_i+b''_i+\frac{j_i}{H_i}+\frac{(-1)^{u_{i-1}}}{H_i^2x_{i+1}}.$$

Now, applying Lemma 2.3, we obtain

$$\frac{j_i}{H_i}+\frac{(-1)^{u_{i-1}}}{H_i^2x_{i+1}}=\left[\left[\frac{j_i}{H_i}\right],(-1)^{u_i}\delta_{i+1}^2x_{i+1}-\delta_{i+1}\frac{Q^{(i)}}{H_i}\right].$$

Finally, we state that

$$Af=\left[(-1)^{u-1}\delta_0^2b'_0+b''_0,\left[\frac{H_0}{j_0}\right],\ldots,\left[\frac{H_i}{j_i}\right],(-1)^{u_i}\delta_{i+1}^2x_{i+1}-\delta_{i+1}\frac{Q^{(i)}}{H_i}\right].$$

This process has to be stopped in the case where f is rational, in other words the algorithm stops in the step s if $f=[Ab'_0+h_0\ ,Ab'_1+h_1\ ,\ldots,Ab'_s+h_s]$. Consequently,

$$Af=\left[(-1)^{u-1}\delta_0^2b'_0+b''_0,\left[\frac{H_0}{j_0}\right],\ldots,\left[\frac{H_{s-1}}{j_{s-1}}\right],(-1)^{u_{s-1}}\delta_s^2x_s-\delta_s\frac{Q^{(s-1)}}{H_{s-1}}\right].$$

As $x_s=b'_s+\dfrac{h_s}{A}$, so

$$\begin{aligned}(-1)^{u_{s-1}}\delta_s^2x_s-\delta_s\frac{Q^{(s-1)}}{H_{s-1}}&=(-1)^{u_{s-1}}\delta_s^2b'_s+\frac{(-1)^{u_{s-1}}\delta_sh_s-\delta_{s-1}Q^{(s-1)}}{H_s}\\&=(-1)^{u_{s-1}}\delta_s^2b'_s+b''_s+\frac{j_s}{H_s}\ .\end{aligned}$$

Finally,

$$Af = \left[(-1)^{u_{-1}}\delta_0^2 b'_0 + b''_0, \left[\frac{H_0}{j_0}\right], \ldots, \left[\frac{H_{s-1}}{j_{s-1}}\right], (-1)^{u_{s-1}}\delta_s^2 b'_s + b''_s, \left[\frac{H_s}{j_s}\right]\right].$$

In the case where the continued fraction expansion of f is infinite i.e. $f \notin \mathbb{F}_q(X)$, the algorithm never stops and

$$Af = \left[(-1)^{u_{-1}}\delta_0^2 b'_0 + b''_0 , \left[\frac{H_0}{j_0}\right], \ldots, \left[\frac{H_{i-1}}{j_{i-1}}\right] , (-1)^{u_{i-1}}\delta_i^2 b'_i + b''_i , \left[\frac{H_i}{j_i}\right], \ldots\right].$$

3. Length of the Period of the Continued Fraction of Af

Let f be a quadratic formal power series. We prove that the continued fraction expansion of Af given by Theorem 2.1 is periodic and we study the properties of the period of the continued fraction expansion of Af.

Notation. Let f be a quadratic formal power series and A be a polynomial in $\mathbb{F}_q[X]$. We denote by $P'(Af)$ the period of the continued fraction expansion of Af given by Theorem 2.1.

Proposition 3.1. *Let f be a quadratic formal power series, then the series Af is periodic and the continued fraction expansion of Af given by Theorem 2.1 is also periodic. We have*

$$P'(Af) \geq Per(Af).$$

Proof. Throughout the proof, we will use the notations of Theorem 2.1. Since f is quadratic, it follows that the continued fraction expansion of f is periodic. Let $[a_0; a_1, \ldots, a_m, \overline{a_{m+1}, \ldots, a_{m+n}}]$ be the continued fraction expansion of f. Let k be an integer greater than m and $d = \sup_{1\leq i\leq n} \deg a_{m+i}$. We have $\deg H_k = \deg A - \deg \delta_k \leq \deg A = N$, $\deg j_k < \deg H_k \leq N$ and $\deg\left((-1)^{u_{k-1}}\delta_k^2 b'_k\right) = 2\deg \delta_k + \deg b'_k \leq 2N + d$. Moreover, as $\deg\left(\frac{\delta_{k-1}Q^{(k-1)}}{H_k}\right) < 0$,

$$\deg\left(b''_k\right) \leq \deg\left(\frac{\delta_k h_k}{H_k}\right) = \deg\left(\frac{\delta_k^2 h_k}{A}\right) < \deg\left(\delta_k^2\right) \leq 2N.$$

Since there are only n different values of $k \pmod n$, we conclude that the number of possible values of $\Delta_k = \left((-1)^{u_{k-1}}\delta_k^2 b'_k,\ b''_k,\ j_k,\ H_k,\ k(\mathrm{mod}\ n)\right)$ is finite. Thus there exist two integers l and s such that $\Delta_l = \Delta_s$. Using the induction given in Theorem 2.1, we obtain that

$$\Delta_{l+i} = \Delta_{s+i},\ \forall i \geq 1.$$

Therefore the continued fraction expansion of Af given by Theorem 2.1 is periodic. We can write Af in the form :

$$Af = [a'_0, \ldots, a'_{m'}, \overline{a'_{m'+1}, \ldots, a'_{m'+n'}}],$$

where $\deg a'_i \geq 0$ for all $i \in \mathbb{N}$ and $n' = P'(Af)$. According to Remark 2.2, $[a'_0, \ldots, a'_{m'}, \overline{a'_{m'+1}, \ldots, a'_{m'+n'}}]$ can be transformed to $[a''_0; \ldots, a''_{m''}, \overline{a''_{m''+1}, \ldots, a''_{m''+n''}}]$ where $n'' \leq n'$, $m'' \leq m'$ and $\deg(a''_i) > 0$ for all $i \in \mathbb{N}^*$. We notice that this last continued fraction expansion is the usual one and $n'' = Per(Af)$. Consequently

$$P'(Af) \geq Per(Af).$$

Applied with some conditions on the partial quotients of f, the algorithm given by Theorem 2.1 provides the usual continued fraction expansion of Af.

Proposition 3.2. *Let $f \in \mathbb{F}_q((X^{-1}))$ and let $[a_0; a_1, \ldots, a_s, \ldots]$ be its continued fraction expansion. If $a_0 \neq 0$ and $\deg(a_i) > \deg(A)$, for all $i \geq 1$, then the continued fraction expansion of Af given by Theorem 2.1 is the usual one.*

Proof. It is sufficient to prove that $\deg((-1)^{u_{i-1}}\delta_i^2 b'_i + b''_i) > 0$, for all $i \geq 0$ in order to show that the continued fraction expansion of Af given by Theorem 2.1 is usual. For $i = 0$, we have $(-1)^{u_{-1}}\delta_0^2 b'_0 + b''_0 = A^2 b'_0 + b''_0$. It is clear that if $\deg(a_0) = 0$, then $\deg((-1)^{u_{-1}}\delta_0^2 b'_0 + b''_0) > 0$. Otherwise, $\deg(a_0 A) > 0$ and the result follows by distinguishing two cases:

Case 1. $\deg(a_0) < \deg(A)$. It follows that $h_0 = a_0$ and $b'_0 = 0$. Since $b''_0 + \dfrac{j_0}{H_0} = Ah_0$, we have $\deg(b''_0) = \deg(Aa_0) > 0$ and thus

$$\deg((-1)^{u_{-1}}\delta_0^2 b'_0 + b''_0) = \deg(b''_0) > 0.$$

Case 2. $\deg(a_0) \geq \deg(A)$. We observe that $\deg(b'_0) > 0$ which yields

$$\deg(b''_0) = \deg(Ah_0) < \deg(A^2 b'_0) \text{ and } \deg((-1)^{u_{-1}}\delta_0^2 b'_0 + b''_0) = \deg(A^2 b'_0) > 0.$$

We remark that in all cases $\deg((-1)^{u_{-1}}\delta_0^2 b'_0 + b''_0) > 0$.

Let $i \geq 1$. It is clear that $\deg(b'_i) > 0$. According to Proposition 3.1, we have

$$\deg(b''_i) \leq \deg(\frac{\delta_i h_i}{H_i}) < \deg(\delta_i^2) < \deg(\delta_i^2 b'_i).$$

Consequently,

$$\deg((-1)^{u_{i-1}}\delta_i^2 b'_i + b''_i) = \deg(\delta_i^2 b'_i) > 0.$$

Next we prove the following result.

Proposition 3.3. *Let N, $n \in \mathbb{N}$, if $S(N,n)$ exists then*

$$S(N,n) = \sup_{\deg A = N} \sup_{f \in \Lambda_n} \mathrm{Per}\,(Af) = \sup_{\deg A = N} \sup_{f \in \Lambda_n} P'(Af).$$

Proof. Let A be a polynomial of degree N, $f \in \Lambda_n$ and $[a_0(f); a_1(f), \ldots, a_s(f), \overline{a_{s+1}(f), \ldots, a_{s+n}(f)}]$ its continued fraction expansion. Proposition 3.1 implies that Af is periodic and $\mathrm{Per}\,(Af) \leq P'(Af)$, therefore

$$\sup_{f \in \Lambda_n} \mathrm{Per}\,(Af) \leq \sup_{f \in \Lambda_n} P'(Af).$$

On the other hand, suppose that $\deg\,(a_k(f)) > N$ for all $k \geq 1$ and $a_0(f) \neq 0$ then Proposition 3.2 shows that for all $k \in \mathbb{N}$

$$\deg\,((-1)^{u_{k-1}} \delta_i^2 b'_k + b''_k) > 0,$$

and we state that the continued fraction expansion of Af given by Theorem 2.1 is usual, therefore

$$\mathrm{Per}\,(Af) = P'(Af).$$

Hence,

$$\begin{aligned} \sup_{f \in \Lambda_n} \mathrm{Per}\,(Af) &\geq \sup_{\substack{f \in \Lambda_n \\ (\forall k)(\deg a_k > N)}} \mathrm{Per}\,(Af) \\ &= \sup_{\substack{f \in \Lambda_n \\ (\forall k)(\deg a_k > N)}} P'(Af) \\ &= \sup_{f \in \Lambda_n} P'(Af), \end{aligned}$$

because the length of the period of the continued fraction expansion of Af given by Theorem 2.1 depends only on the sequence (a_k) having degree $\leq N$. Summarizing up, we get

$$S(N,n) = \sup_{\deg A = N} \sup_{f \in \Lambda_n} P'(Af).$$

Remark 3.4. For $N, n \in \mathbb{N}$, the calculation of $S(N,n)$ is finite because the period of the continued fraction expansion given by the Theorem 2.1 depends only on the coefficients taken $(\mathrm{mod}\ A)$. But, for great values of N and n, the calculation becomes difficult.

Example 3.5. *In $\mathbb{F}_2$, for every value of $S(N,n)$, we denote by (A,f) pairs such that $S(N,n) = \mathrm{Per}\,(Af)$, $A \in \mathbb{F}_2[X]$ with $\deg A = N$ and $f \in \Lambda_n$ given by its continued fraction expansion.*

Table 1.

N	n	$S\ (N,n)$	(A,f)
1	1	2	$(X+1,[\overline{X}])$
1	2	4	$(X,[\overline{X,X+1}])$
1	3	6	$(X+1,[\overline{X,X,X+1}])$ $(X,[\overline{X,X+1,X}])$
2	1	4	$(X^2+X+1,[\overline{X^2}])$ $(X^2+X,[\overline{X^2}])$
2	2	4	$(X^2+X+1,[\overline{X^2,X^2+1}])$ $(X^2,[\overline{X^2,X+1}])$ $(X^2+X+1,[\overline{X^2+1,X}])$ $(X^2,[\overline{X,X+1}])$
2	3	6	$(X^3,[\overline{X^3,X^3+X^2+1}])$ $(X^3+1,[\overline{X^3,X^3+X^2+1}])$ $(X^3+X^2,[\overline{X^3,X^3+X^2+1}])$ $(X^3+X^2+X,[\overline{X^3,X^3+X^2+1}])$
3	1	6	$(X^3+X^2,[\overline{X}])$ $(X^3+1,[\overline{X}])$ $(X^3,[\overline{X+1}])$

4. Properties of an Equivalence Relation

Now, we use the Cohen [1] method's, we construct a new space noted P_A. We prove some properties of the space P_A which permits to evaluate $S(N,n)$ in the next section.

Let $E=\{(a,b)\in\mathbb{F}_q[X]\times\mathbb{F}_q[X]$ such that $(a,b)=1\}$ and R_A be an equivalence relation over $E\times E$ defined for all $(a,b)\in E$ and $(a',b')\in E$ by

$$(a,b)R_A(a',b')\Longleftrightarrow ab'\equiv a'b(\text{mod } A),$$

and

$$P_A=E/R_A.$$

We give some notations and properties concerning the space P_A.

- We note by $\overline{(a,b)}$ the value of (a,b) modulo R_A.
- Let $GL(2,\mathbb{F}_q[X])=\left\{M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} : \alpha,\beta,\gamma,\delta\in\mathbb{F}_q[X] \text{ and } \alpha\delta-\beta\gamma=\pm 1\right\}$.

The elements of $GL(2,\mathbb{F}_q[X])$ operate on P_A by quotient to R_A as follows : for all $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL(2,\mathbb{F}_q[X])$ and $\overline{(a,b)}\in P_A$ we have

$$M\overline{(a,b)}=\overline{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}}=\overline{(\alpha a+\beta b,\gamma a+\delta b)}.$$

- Let $a\in\mathbb{F}_q[X]$ and $u\in P_A$, we note by $a+u^{-1}$ or $a+\dfrac{1}{u}$ the result of the action of the matrix $\begin{pmatrix}a & 1\\ 1 & 0\end{pmatrix}$ to u in P_A.
- Let $\Gamma(A)=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL(2,\mathbb{F}_q[X])\ \ \beta\equiv\gamma\equiv\alpha-\delta\equiv 0 \pmod{A}\right\}$.

The following result gives the elementary properties of P_A.

Proposition 4.1. *An element $\overline{(a,c)}$ of P_A verifies the following conditions :*

- $(a,c)\in\mathbb{F}_q[X]\times\mathbb{F}_q[X]$ *such that* $\gcd(a,c)=1$*,*
- $c\mid A$*,*
- a *take all values modulo* $\dfrac{A}{c}$ *such that* $\gcd(a,c)=1$.

We will need the following lemma to prove the above proposition.

Lemma 4.2. *Let a, b and $c\in\mathbb{F}_q[X]$ such that $(a,b)=1$, then there is a $\lambda\in\mathbb{F}_q[X]$ such that*

$$\gcd(a+\lambda b,c)=1.$$

Proof. We will treat two cases.

Case 1. All irreducible factors of c are factors of a.
It is sufficient to take $\lambda=1$. In fact, if $d=\gcd(a+b,c)$ and p is an irreducible factor of d, then $p\mid a+b$ and $p\mid c$. However $p\mid c$ implies $p\mid a$, hence, $p\mid\gcd(a,b)$.

Case 2. There exists an irreducible factor of c which is not a factor of a.
Let

$$\lambda = \prod_{\substack{p \text{ irreducible} \\ p \mid c \text{ and } p \nmid a}} p,$$

$d = \gcd(a + \lambda b, c)$ and p is an irreducible factor of d then $p \mid a + \lambda b$ and $p \mid c$. If $p \mid a$, it gives $p \mid \lambda b$ and as $p \nmid \lambda$, so $p \mid b$ and we are done. If $p \nmid a$, from $p \mid c$, we deduce $p \mid \lambda$ which yields $p \mid a$, a contradiction.

Proof of Proposition 4.1. Let $\overline{(a', b')}$ be an element of P_A and $d = \gcd(b', A)$, then there exist two polynomials P and Q such that $P\dfrac{A}{d} + Q\dfrac{b'}{d} = 1$. As $\gcd\left(Q, \dfrac{A}{d}\right)$ $= 1$, Lemma 4.2 implies that there exists a polynomial λ such that $\gcd\left(Q + \lambda\dfrac{A}{d}, A\right)$ $= 1$. We can choose Q so that $\gcd(Q, A) = 1$ and $Q\dfrac{b'}{d} \equiv 1(\bmod\ \dfrac{A}{d})$. Since it is easy to verify that $\gcd(d'Q, d) = 1$, we get $\overline{(a', b')} = \overline{(a'Q, d)}$ and $d \mid A$.

For the second part of the proposition, let $\overline{(a, c)}$ and $\overline{(a', c')}$ be two elements of P_A verifying

$$\overline{(a, c)} = \overline{(a', c')},\ c \mid A \text{ and } c' \mid A.$$

We see that

$$\begin{aligned} \overline{(a, c)} = \overline{(a', c')} &\iff (a, c)R_A(a', c') \\ &\iff ac' \equiv a'c \ (\bmod\ A) \\ &\iff ac' = a'c + \alpha A, \end{aligned}$$

where α is a not zero polynomial. As $c' \mid A$ and $\gcd(a', c') = 1$, we get $c' \mid c$ and in the same way we see that $c \mid c'$. Hence, $c = c'$ and $a \equiv a' \left(\bmod\ \dfrac{A}{c}\right)$.

Next we prove that the space P_A is finite.

Theorem 4.3. *Let $J = \{P \in \mathbb{F}_q[X];\ P$ is monic and irreducible$\}$. Then*

$$\text{card } P_A \leq (q-1) \mid A \mid \prod_{\substack{P \mid A \\ P \in J}} \left(1 + \frac{1}{\mid P \mid}\right).$$

Proof. Let $f : \mathbb{F}_q[X] \to \mathbb{N}$ be the map defined for all $A \in \mathbb{F}_q[X]$ by

$$f(A) = \sum_{c \mid A} \frac{\mid A/c \mid}{\mid \gcd(c, A/c) \mid} \varphi(\gcd(c, A/c))$$

where φ is the Euler's function defined for all $\alpha \in \mathbb{F}_q[X]$ by

$$\varphi(\alpha) = \text{card } \{r \in \mathbb{F}_q[X],\ \text{monic } :\ \deg r < \deg \alpha \text{ and } \gcd(r, \alpha) = 1\}.$$

It is sufficient to prove that $\text{card } P_A \leq (q-1)f(A)$, f is multiplicative and $f(P^l) = \mid P \mid^l + \mid P \mid^{l-1}$, for all $P \in J$ and l an integer, in order to prove this theorem.

We will prove that $\text{card } P_A \leq (q-1)f(A)$. Let $\alpha = \gcd\left(c, \frac{A}{c}\right)$ and

$$F_c = \left\{ a \in \mathbb{F}_q[X], \text{ monic } : \deg a < \deg \frac{A}{c} \text{ and } \gcd(a,c) = 1 \right\}.$$

We can write a in the form $r + K\alpha$ where $\deg r < \deg \alpha$. It is clear that $a \equiv r \pmod{\alpha}$, which implies that $\gcd(a, \alpha) = \gcd(r, \alpha) = 1$. Therefore

$$F_c = \bigcup_{\substack{\deg K \leq \deg (\frac{A}{\alpha c}) \\ K \text{ monic}}} \{K\alpha + r, \text{ monic } : \deg r < \deg \alpha \text{ and } \gcd(K\alpha + r, c) = 1\}.$$

Let K be a monic polynomial satisfying $\deg K \leq \deg \left(\frac{A}{\alpha c}\right)$ and

$$\begin{aligned}
G_K &= \{K\alpha + r \ : \ \deg r < \deg \alpha \text{ and } \gcd(K\alpha + r, c) = 1\}. \\
\text{card } (G_K) &= \text{card } \{\mathrm{K}\alpha + \mathrm{r} \ : \ \deg \mathrm{r} < \deg \alpha \text{ and } \gcd(\mathrm{K}\alpha + \mathrm{r}, \alpha) = 1\} \\
&\leq \text{card } \{\mathrm{r} \in \mathbb{F}_\mathrm{q}[\mathrm{X}] \ : \ \deg \mathrm{r} < \deg \alpha \text{ and } \gcd(\mathrm{r}, \alpha) = 1\} \\
&\leq (q-1)\varphi(\alpha) \\
&\leq (q-1) \mid \alpha \mid \prod_{P \mid \alpha} \left(1 - \frac{1}{\mid P \mid}\right).
\end{aligned}$$

As the sets G_K form a partition of F_c, we have

$$\text{card } F_c = \mid \frac{A}{\alpha c} \mid \text{card } G_K \leq (q-1) \mid \frac{A}{\alpha c} \mid \varphi(\alpha)$$

and

$$\begin{aligned}
\text{card } P_A &= \sum_{c \mid A} \text{card } F_c \\
&\leq (q-1) \sum_{c \mid A} \frac{\mid A/c \mid}{\mid \gcd(c, A/c) \mid} \varphi(\gcd(c, A/c)) \\
&\leq (q-1) f(A).
\end{aligned}$$

We will now prove that f is a multiplicative map by using that

$$\gcd\left(c_1 c_2, \frac{M}{c_1}\frac{N}{c_2}\right) = \gcd\left(c_1, \frac{M}{c_1}\right) \gcd\left(c_2, \frac{N}{c_2}\right)$$

$$\text{and } \varphi\left(\gcd\left(c_1c_2, \frac{M}{c_1}\frac{N}{c_2}\right)\right) = \varphi\left(\gcd\left(c_1, \frac{M}{c_1}\right)\right)\varphi\left(\gcd\left(c_2, \frac{N}{c_2}\right)\right).$$

It is obvious that $\beta \in \mathbb{F}_q^*$ and $A \in \mathbb{F}_q[X]$ implies $f(\beta A) = f(A)$.

Let $P \in J$ and l be an integer. Then

$$\begin{aligned} f(P^l) &= \sum_{d \mid P^l} \frac{\mid P^l/d \mid}{\mid \gcd(d, P^l/d) \mid}\varphi(\gcd(d, P^l/d)) \\ &= \sum_{j=0}^{l} \frac{\mid P^{l-j} \mid}{\mid \gcd(P^j, P^{l-j}) \mid}\varphi(\gcd(P^j, P^{l-j})) \\ &= 1 + \sum_{j=1}^{l-1} \frac{\mid P^{l-j} \mid}{\mid P^{\min(j,l-j)} \mid}\varphi(P^{\min(j,l-j)}) + \mid P \mid^l \\ &= 1 + \mid P \mid^l + \sum_{j=1}^{l-1}(\mid P \mid^{l-j} - \mid P \mid^{l-j-1}) \\ &= \mid P \mid^l + \mid P \mid^{l-1}. \end{aligned}$$

Finally, if we write A in the form $\lambda P_1^{\alpha_1} \dots P_s^{\alpha_s}$ where $P_i \in J$ for all $i = 1, \dots, s$ and $\lambda \in \mathbb{F}_q$. We obtain

$$\begin{aligned} f(A) &= f(\prod_{i=1}^{s} P_i^{\alpha_i}) \\ &= \prod_{i=1}^{s} f(P_i^{\alpha_i}) \\ &= \prod_{i=1}^{s} (\mid P_i \mid^{\alpha_i} + \mid P_i \mid^{\alpha_i - 1}) \\ &= \mid A \mid \prod_{i=1}^{s}\left(1 + \frac{1}{\mid P_i \mid}\right) \\ &= \mid A \mid \prod_{\substack{P \mid A \\ P \in J}} \left(1 + \frac{1}{\mid P \mid}\right). \end{aligned}$$

5. On the Functions $S(N, n)$ and $R(N)$

We give an upper bound of $\operatorname{Per}(Af)$, which leads it to prove the existence of the functions $S(N, n)$ and $R(N)$.

Definition 5.1. Let $\alpha,\ \beta \in \mathbb{F}_q((X^{-1}))$, we write $\alpha \equiv \beta \pmod{\mathbb{F}_q[X]}$ if $\alpha - \beta \in \mathbb{F}_q[X]$. Note that the relation $\equiv$ is an equivalence relation.

Proposition 5.2. *Consider the mapping*

$$\begin{aligned} \phi_A : \quad P_A &\longrightarrow \mathbb{F}_q(X) \setminus \mathbb{F}_q[X] \\ \overline{(a,c)} &\longmapsto \phi_A(\overline{(a,c)}) \equiv \frac{ac}{A} (\text{mod } \mathbb{F}_q[X]), \end{aligned}$$

this mapping is well-defined and

$$\text{Im}\phi_A = \left\{ \frac{a}{A} \ (\text{mod } \mathbb{F}_q[X]) \ : \ \deg a < \deg A \right\}.$$

Proof. Let $\overline{(a,c)}$ and $\overline{(a',c')}$ be two elements of P_A such that $\overline{(a,c)} = \overline{(a',c')}$, then the proof of Proposition 4.1 implies that $c = c'$ and $a \equiv a' \pmod{A/c}$, it yields that $\frac{ac}{A} \equiv \frac{a'c}{A} (\text{mod } \mathbb{F}_q[X])$ and the map ϕ_A is well-defined. We write

$$\begin{aligned} \text{Im}\phi_A &= \{\phi_A(\overline{(a,c)}) : \ \overline{(a,c)} \in P_A\} \\ &= \Bigg\{ \frac{ac}{A} \ (\text{mod } \mathbb{F}_q[X]) : a \text{ takes all the values} \\ &\qquad \left(\text{mod}\, \frac{A}{c}\right) \text{ such that } \gcd(a,c) = 1 \Bigg\}. \end{aligned}$$

It is clear that $\left\{ \frac{a}{A} \ (\text{mod } \mathbb{F}_q[X]) \ : \ \deg a < \deg A \right\} \subset \text{Im}\phi_A$.

Conversely, let $\phi_A(\overline{(a,c)}) \equiv \frac{ac}{A} \ (\text{mod } \mathbb{F}_q[X])$ be an element of $\text{Im}\phi_A$. Then $\deg a < \deg \left(\frac{A}{c}\right)$ and $\deg (ac) < \deg A$. Now, if we take $a' = ac$, we obtain

$$\phi_A(\overline{(a,c)}) \equiv \frac{a'}{A} \ (\text{mod } \mathbb{F}_q[X]) \text{ and } \deg a' < \deg A.$$

Finally, we conclude that

$$\text{Im}\phi_A = \left\{ \frac{a}{A} \ (\text{mod } \mathbb{F}_q[X]) \ : \ \deg a < \deg A \right\}.$$

Theorem 5.3. *We use the same notations as in Theorem 2.1. Let $(v_k)_{k\in\mathbb{N}}$ be a sequence of elements of P_A defined by*

$$v_0 = \overline{(1,A)} \text{ and } v_{k+1} = a_{k+1} + v_k^{-1}. \tag{5.1}$$

Then for all $k \geq 0$ we have

$$v_k = \overline{\left(e_k, \frac{A}{H_k}\right)}, \text{ where } \gcd\left(e_k, \frac{A}{H_k}\right) = 1, \ e_k \equiv (-1)^{u_{k-1}} J_k (\text{mod } H_k)$$

and

$$\phi_A(v_k) \equiv (-1)^{u_{k-1}} \frac{j_k}{H_k} (\text{mod } \mathbb{F}_q[X]).$$

Remark 5.4. *Using the inductions (5.1), we can write v_k for all $k \geq 0$ in the following form*

$$v_k = a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1 (\mathrm{mod}\ A)}}},$$

in fact, we know that

$$v_k = a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1 + \cfrac{1}{v_0}}}},$$

and $a_1 + \dfrac{1}{v_0} = \overline{(a_1 + A, 1)} \equiv \overline{(a_1, 1)} \pmod{A}$ *noted by* $a_1 (\mathrm{mod}\ A)$.

Proof of Theorem 5.3. We prove this theorem by induction. We have $H_0 = 1$ and $j_0 = 0$. If we take $e_0 = 1$, we obtain

$$v_0 = \overline{(1, A)} = \overline{\left(e_0, \frac{A}{H_0}\right)}$$

and the claim holds for $k = 0$. Let k be a positive integer and suppose that $v_k = \overline{\left(e_k, \dfrac{A}{H_k}\right)}$, $\gcd\left(e_k, \dfrac{A}{H_k}\right) = 1$ and $e_k \equiv (-1)^{u_{k-1}} J_k \pmod{H_k}$. This implies that

$$\begin{aligned} v_{k+1} &= a_{k+1} + v_k^{-1} \\ &= \overline{\begin{pmatrix} a_{k+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_k \\ A/H_k \end{pmatrix}} \\ &= \overline{\left(a_{k+1} e_k + \frac{A}{H_k}, e_k\right)}. \end{aligned}$$

Moreover, we notice that

$$\gcd(e_k, A) = \gcd\left(e_k, \frac{A}{H_k}\right) \gcd(e_k, H_k) = \gcd(e_k, H_k) = \delta_{k+1}.$$

Thus there exists a polynomial P' such that $P'\left(\dfrac{e_k}{\delta_{k+1}}\right) \equiv 1 \left(\mathrm{mod}\ \dfrac{A}{\delta_{k+1}}\right)$, which gives $\gcd\left(P', \dfrac{A}{\delta_{k+1}}\right) = 1$. Applying Lemma 4.2, we see that there exists a polynomial λ such that $\gcd\left(P' + \lambda\frac{A}{\delta_{k+1}}, A\right) = 1$. Therefore there exists a polynomial P such that

$$\gcd(A,P) = 1 \text{ and } P\left(\frac{e_k}{\delta_{k+1}}\right) \equiv 1 \left(\bmod\ \frac{A}{\delta_{k+1}}\right).$$

We remark that $Pe_k \equiv \delta_{k+1} (\bmod\ A)$, which shows that

$$v_{k+1} = \overline{\left(P\left(a_{k+1}e_k + \frac{A}{H_k}\right), \delta_{k+1}\right)} \text{ and } (-1)^{u_{k-1}}P\left(\frac{j_k}{\delta_{k+1}}\right) \equiv 1 \left(\bmod\ \frac{H_k}{\delta_{k+1}}\right).$$

On the other hand,

$$P_{t_k}^{(k)}Q^{(k)} - Q_{t_k}^{(k)}P^{(k)} = (-1)^{t_k - 1},$$

where $P_{t_k}^{(k)} = \frac{j_k}{\delta_{k+1}}$ and $Q_{t_k}^{(k)} = \frac{H_k}{\delta_{k+1}}$, then $(-1)^{t_k-1}Q^{(k)}\left(\frac{j_k}{\delta_{k+1}}\right) \equiv 1 \left(\bmod \frac{H_k}{\delta_{k+1}}\right)$. Consequently

$$P \equiv (-1)^{u_k - 1}Q^{(k)} (\bmod\ \frac{H_k}{\delta_{k+1}}).$$

Thus there exists $T \in \mathbb{F}_q[X]$ such that $P = (-1)^{u_k-1}Q^{(k)} + T\frac{H_k}{\delta_{k+1}}$. Now,

$$\begin{aligned} P\left(a_{k+1}e_k + \frac{A}{H_k}\right) &\equiv h_{k+1}\delta_{k+1} + \left((-1)^{u_k-1}Q^{(k)} + T\frac{H_k}{\delta_{k+1}}\right)\frac{A}{H_k} \ (\bmod\ A) \\ &\equiv h_{k+1}\delta_{k+1} - (-1)^{u_k}Q^{(k)}\delta_k + T\frac{A}{\delta_{k+1}} \ (\bmod\ A) \\ &\equiv h_{k+1}\delta_{k+1} - (-1)^{u_k}Q^{(k)}\delta_k \ \left(\bmod\ \frac{A}{\delta_{k+1}}\right) \\ &= (-1)^{u_k}j_{k+1} \\ &\equiv e_{k+1} \ (\bmod\ \frac{A}{\delta_{k+1}}) \end{aligned}$$

and

$$v_{k+1} = \overline{(e_{k+1}, \delta_{k+1})} = \overline{\left(e_{k+1}, \frac{A}{H_{k+1}}\right)}.$$

Next we prove the upper bound of $\mathrm{Per}\,(Af)$.

Theorem 5.5. *Consider the quadratic formal power series* $f = [a_0; a_1, \ldots, a_m, \overline{a_{m+1}, \ldots, a_{m+n}}]$ *and let* M *be the matrix associated to* f. *Let* $(v_k)_{k\in\mathbb{N}}$ *be the sequence of elements of* P_A *associated to* f *as in Theorem 5.3* $(w_k)_{k\in\mathbb{N}}$ *be the* *sequence of elements of* P_A *defined by* $w_k = v_{m+k}$ *for all* $k \geq 0$ *and* $\lambda_0(M) = \lambda_0(A, M)$ *the least positive integer satisfying* $M^{\lambda_0(A,M)} \in \Gamma(A)$. *Then*

$$P'(Af) \mid \sum_{1 \leq k \leq n\lambda_0(A,M)} (\psi(\Phi_A(w_k)) + 1),$$

particularly,

$$\mathrm{Per}\,(Af) \leq \sum_{1 \leq k \leq n\lambda_0(A,M)} (\psi(\Phi_A(w_k)) + 1).$$

Proof. We first prove the existence of $\lambda_0(M)$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the matrix associated to f and $\overline{M} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$ where $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{d}$ are the values of a, b, c and d taken $(\mathrm{mod}\ A)$ respectively, $\overline{M}$ is an element of the finite group $GL(2, \mathbb{F}_q[X]/_{<A>})$, therefore the subgroup $< \overline{M} >$ generated by $\overline{M}$ is finite. Thus there exists an integer λ such that $\overline{M}^{\lambda} = \overline{Id}$ and $M^{\lambda} \in \Gamma(A)$. We conclude that there exists an integer $\lambda_0(M)$ which is the least integer verifying $M^{\lambda_0(M)} \in \Gamma(A)$.

Remark 5.6. *Refereing to the later result, we remark that que*

$$\lambda_0(A, M) \leq\mid GL(2, \mathbb{F}_q[X]/_{<A>}) \mid\leq (\deg A)^4.$$

We will need the following lemma in order to complete the proof of this theorem.

Lemma 5.7. *The following assertions are equivalent.*

(i) k_0 *is the least positive integer satisfying* $w_{k_0} = w_0$ *and* $n \mid k_0$.

(ii) *for all* $p \geq 0$, $H_{k_0+m+p} = H_{m+p}$ *and* $e_{k_0+m+p} \equiv e_{m+p} \pmod{H_{m+p}}$.

(iii) *The sequence* $(w_k)_{k\in\mathbb{N}}$ *is purely periodic with period* k_0.

Proof. (i) $\Longrightarrow$ (ii) A simple induction on p gives the result.

(ii)$\Longrightarrow$ (iii) Let $p \geq 0$ and $\alpha > 0$, it is clear that $w_{p+\alpha k_0} = w_0$ and $(w_k)_{k\in\mathbb{N}}$ is periodic of period k_0.

(iii)$\Longrightarrow$ (i) Suppose that $(w_k)_{k\in\mathbb{N}}$ is periodic with period k_0. Thus $w_{k_0} = w_0$ and it remains to verify that $n \mid k_0$. If $w_k = w_{k'}$ and $k \equiv k' \pmod{n}$ then $w_{|k-k'|} = w_0$. As $\mid k - k' \mid\equiv 0 (\mathrm{mod}\ n)$, let k_0 be the least integer satisfying $k_0 \equiv 0 (\mathrm{mod}\ n)$ and as $w_{k_0} = w_0$, it follows that $n \mid k_0$. ∎

Now, we will prove that there exists an integer k_0 which is the least integer satisfying $w_{k_0} = w_0$ and $n \mid k_0$. Let k be a given integer and

$$\Gamma = \{w_{k'} \in P_A \ : \ k \equiv k' (\mathrm{mod}\ n)\} \subset P_A.$$

As Γ is finite, there exist two integers l and h such that $w_l = w_h$ and $l \equiv h (\bmod\ n)$, it implies that $w_{|l-h|} = w_0$ and $\mid l-h \mid \equiv 0 (\bmod\ n)$. Consequently, there exists an integer t such that $w_t = w_0$ and $n \mid t$. Let k_0 be the least positive integer satisfying $w_{k_0} = w_0$ and $n \mid k_0$. By using Lemma 5.7, we conclude that k_0 is a period of the sequences $(w_k)_{k\in\mathbb{N}}$ and $(H_{m+k})_{k\in\mathbb{N}}$ such that $n \mid k_0$. Therefore

$$P'(Af) \mid k_0. \tag{5.2}$$

Moreover, we can verify that $w_{\lambda_0(M)n} = (t_M)^{\lambda_0(M)}(w_0)$. In fact,

$$\begin{aligned} w_n &= a_n + \frac{1}{w_{n-1}} \\ &= \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} w_{n-1} \\ &= \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} w_0 \\ &= (a_n)\dots(a_1)w_0, \end{aligned}$$

where (a_i) is the matrix $\begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$ for all $i \in \{1,\dots,n\}$. Since $t_M = (a_n)\dots(a_1)$ and the sequence $(a_n)_{n\in\mathbb{N}}$ is periodic of period n, we conclude that

$$w_{\lambda_0(M)n} = (a_n)\dots(a_1)\dots(a_n)\dots(a_1)w_0 = (t_M)^{\lambda_0(M)}(w_0).$$

As $\lambda_0(M)$ is the least integer satisfying $M^{\lambda_0(M)} \in \Gamma(A)$, $(t_M)^{\lambda_0(M)} \in \Gamma(A)$, implies that $w_{\lambda_0(M)n} = w_0$ and $k_0 \mid \lambda_0(M)n$. Finally, by using (5.2), we conclude that $P'(Af) \mid \lambda_0(M)n$ and we can write Af in the form

$$Af = \Bigg[(-1)^{u-1}\delta_0^2 b'_0 + b''_0, \left[\frac{H_0}{j_0}\right], \dots, \overline{(-1)^{u_m}\delta_{m+1}^2 b'_{m+1} + b''_{m+1}\ , \left[\frac{H_{m+1}}{j_{m+1}}\right],}$$

$$\overline{\dots, (-1)^{u_{m+n\lambda_0(M)-1}}\delta^2_{m+n\lambda_0(M)} b'_{m+n\lambda_0(M)} + b''_{m+n\lambda_0(M)}\ , \left[\frac{H_{m+n\lambda_0(M)}}{j_{m+n\lambda_0(M)}}\right]}\Bigg].$$

From the above, it is clear that for $1 \le i \le n\lambda_0(M)$ the number of terms in $(-1)^{u_{m+i-1}}\delta_{m+i}^2 b'_{m+i} + b''_{m+i}\ , \left[\frac{H_{m+i}}{j_{m+i}}\right]$ is equal to

$$2 + \psi\Big(\frac{H_{m+i}}{j_{m+i}}\Big) = 1 + \psi\Big(\frac{j_{m+i}}{H_{m+i}}\Big) = \psi\Big(\overline{\left[\frac{j_{m+i}}{H_{m+i}}\right]}\Big).$$

We conclude that

$$P'(Af) \mid \sum_{1 \le k \le n\lambda_0(M)} \psi\Big(\Big[\overline{\frac{j_{m+k}}{H_{m+k}}}\Big] \Big).$$

Finally, as $\Phi_A(w_k) \equiv (-1)^{u_{m+k-1}} \dfrac{j_{m+k}}{H_{m+k}} \pmod{\mathbb{F}_q[X]}$, for all $k = 1, \ldots, n\lambda_0(M)$, we arrive at

$$P'(Af) \mid \sum_{1 \le k \le n\lambda_0(M)} \psi([\overline{\Phi_A(w_k)}]).$$

Corollary 5.8.

$$S(N, n) = \sup_{M \in \mathcal{M}} \sup_{\deg(A)=N} \sum_{1 \le k \le n\lambda_0(A,M)} \psi([\overline{\Phi_A(w_k)}]),$$

where $\mathcal{M}$ is the family of the matrices associated to the formal power series belonging to Λ_n.

Proof. Let A be a polynomial of degree N and $f \in \Lambda_n$, then by applying Theorem 5.5, we obtain

$$\mathrm{Per}\,(Af) \le \sum_{1 \le k \le n\lambda_0(A,M)} \psi([\overline{\Phi_A(w_k)}]).$$

Therefore,

$$\sup_{f \in \Lambda_n} \mathrm{Per}\,(Af) \le \sup_{M \in \mathcal{M}} \sum_{1 \le k \le n\lambda_0(A,M)} \psi([\overline{\Phi_A(w_k)}]),$$

hence

$$S(N, n) \le \sup_{M \in \mathcal{M}} \sup_{\deg(A)=N} \sum_{1 \le k \le n\lambda_0(A,M)} \psi([\overline{\Phi_A(w_k)}]).$$

Remark 5.6 implies that if $M \in \mathcal{M}$, then $\sup\limits_{\deg(A)=N} \sum\limits_{1 \le k \le n\lambda_0(A,M)} \psi([\overline{\Phi_A(w_k)}])$ is an integer less than $\sup\limits_{\deg(A)=N} \sum\limits_{1 \le k \le nN^4} \psi([\overline{\Phi_A(w_k)}])$, which gives that the set $\left\{ \sup\limits_{\deg(A)=N} \sum\limits_{1 \le k \le n\lambda_0(A,M)} \psi([\overline{\Phi_A(w_k)}]),\ M \in \mathcal{M} \right\}$ is a bounded subset of $\mathbb{N}$. Thus there exists $M_0 \in \mathcal{M}$ and $A \in \mathbb{F}_q[X]$ of degree N such that

$$\sup_{M \in \mathcal{M}} \sup_{\deg(A)=N} \sum_{1 \le k \le n\lambda_0(A,M)} \psi([\overline{\Phi_A(w_k)}]) = \sum_{1 \le k \le n\lambda_0(A,M_0)} \psi([\overline{\Phi_A(w_k)}]),$$

where M_0 is the matrix associated to the quadratic and periodic formal power series f_0 of period n with continued fraction expansion $[a_0; \ldots, a_m, \overline{a_{m+1}, \ldots, a_{m+n}}]$. Let $f_{0,A}$ be the formal power series defined by

$$f_{0,A} = [a_0; a_1(A+1), \ldots, a_m(A+1), \overline{a_{m+1}(A+1), \ldots, a_{m+n}(A+1)}]$$

and let $M_{0,A}$ be the matrix associated to $f_{0,A}$. As $\deg\ (a_i(A+1)) > \deg\ A$ for all $i \geq 1$, Proposition 3.2 implies that the continued fraction expansion of $Af_{0,A}$ given by Theorem 2.1 is the usual one. Therefore $\mathrm{Per}\,(Af_{0,A}) = P'(Af_{0,A})$. On the other hand, it is easy to see that $\overline{M}_{0,A} = \overline{M}_0$. In fact,

$$M_0^{\lambda_0(A,M_0)} \in \Gamma(A) \Longleftrightarrow \overline{M}_0^{\lambda_0(A,M_0)} = aI \Longleftrightarrow \overline{M}_{0,A}^{\lambda_0(A,M_0)} = aI$$

and thus $\lambda_0(A, M_0) = \lambda_0(A, M_{0,A})$. Hence

$$\begin{aligned} \sum_{1 \leq k \leq n\lambda_0(A,M_0)} \psi([\overline{\Phi_A(w_k)}]) &= \sum_{1 \leq k \leq n\lambda_0(A,M_{0,A})} \psi([\overline{\Phi_A(w_k)}]) \\ &= \mathrm{Per}\,(Af_{0,A}) \\ &\leq S(N,n). \end{aligned}$$

Theorem 5.9. *let* $\Omega(N) = \sup\limits_{\deg\,(A)=N} \sum\limits_{u \in P_A} \psi([\overline{\Phi_A(u)}])$. *The functions* $S(N,n)$ *and* $R(N)$ *exist and satisfy*

$$S(N,n) \leq n\Omega(N),\ R(N) \leq \Omega(N).$$

Proof. Corollary 5.8 implies that

$$\begin{aligned} S(N,n) &= \sup_{M \in \mathcal{M}} \sup_{\deg\,(A)=N} \sum_{1 \leq k \leq n\lambda_0(A,M)} \psi([\overline{\Phi_A(w_k)}]) \\ &= \sup_{M \in \mathcal{M}} \sup_{\deg\,(A)=N} \sum_{1 \leq k \leq n} \sum_{0 \leq \lambda < \lambda_0(A,M)} \psi([\overline{\Phi_A(w_{\lambda n+k})}]). \end{aligned}$$

If $w_k = w_{k'}$ and $k \equiv k' \pmod{n}$, then $w_{|k-k'|} = w_0$, which yields that for a given integer k, the $w_{\lambda n+k}$ are a different elements of P_A for all $0 \leq \lambda < \lambda_0(N, M)$. Therefore

$$\begin{aligned} S(N,n) &\leq \sup_{M \in \mathcal{M}} \sup_{\deg\,(A)=N} \sum_{1 \leq k \leq n} \sum_{u \in P_A} \psi([\overline{\Phi_A(u)}]) \\ &\leq n\Omega(N). \end{aligned}$$

Finally,

$$R(N) \leq \Omega(N).$$

Example 5.10. Let now $q = 2$, we give some values of $\Omega(N)$.

Table 2.

N	n	$S(N,n)$	$\Omega(N)$
1	1	2	2
1	2	4	2
1	3	6	2
2	1	4	10
2	2	4	10
2	3	6	10
3	1	6	29

We conclude from Table 2 and Theorem 5.9 that $R(1) = 2$.

Acknowledgments

The authors thank the referee for many detailed comments and suggestions.

References

1. H. Cohen, Multiplication par un entier d'une fraction continue periodique, *Acta Arith.*, **23** (1974), 129-148.
2. T. W. Cusick, Integer multiples of periodic continued fractions, *Pacific Math.*, **78** (1978), 47-60.
3. M. Mendès France, Sur les fractions continues limitées, *Acta. Arith*, **23** (1973), 207-215.
4. M. Mendès France, *The depth of a rational number*, Colloquia Mathematica Societatis Janos Bolyai. 13 Topics IN Number Theory, Debrecen (UNGARY), (1974), pp. 183-194.
5. G. N. Raney, On continued fractions and finite automata, *Math. Ann.*, **206** (1973), 265-283.

H. Ben Amar and M. Mkaouar
Faculté des Sciences de Sfax,
BP 802, Sfax 3018,
Tunisia
E-mail: Hela.BenAmar@issatgb.rnu.tn
mohamed.mkaouar@fss.rnu.tn