TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 5, pp. 1903-1924, October 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

DECOMPOSITION OF CENTRO-AFFINE COVARIANTS OF POLYNOMIAL DIFFERENTIAL SYSTEMS

Dahira Dali and Sui Sun Cheng

Abstract. Starting from a minimal system of the ideal of centro-affine covariants of a polynomial differential system, we develop an algorithmic method to reduce the polynomial decomposition of a given centro-affine covariant of this system to a linear decomposition by constructing a matrix whose size depends on the type of the given covariant. This method avoids the Aronhold symbolic calculation and offers new means to calculate syzygies and can be used to describe the algebra of the centro-affine covariants. We also give many examples in the case where the system is a planar polynomial quadratic differential system.

1. Motivation

Among the many tools that are used in the studies of equations, the theory of algebraic invariants was among the important ones. This theory was developed by many authors and in 1897, Hilbert [7] even gave an introductory course in the University of Göttingen. Almost 100 years later in 1982, Sibirskii [11] wrote a book on the theory of invariants for differential equations. In particular, he explained how invariants are important in the classification of differential systems. He also gave necessary and sufficient conditions for the existence of centers, as well as many other qualitative and *geometric properties* of systems of differential equations with quadratic nonlinearities. These conditions are formulated in the form of certain polynomials (of degree one, two and three) from the elements of a minimal polynomial basis of the 'centro-affine' invariants. Thus the theory of invariant is proven useful in the qualitative studies of polynomial differential systems. It also allows us to characterize geometric properties of these systems by invariant conditions with

Received October 2, 2008, accepted February 17, 2009.
Communicated by Jen-Chih Yao.
2000 *Mathematics Subject Classification*: 34C14, 15A72.
Key words and phrases: Polynomial differential systems, Linear group, Invariant, Covariant, Ideal, Algebra, Syzygies, Generating family.

the help of algebraic or semi-algebraic relations depending on the coefficients of the differential systems.

In the case where the algebra of invariants is of finite type, the Aronhold symbolism calculation [11] based on the computation of determinants and the Gröbner basis approach (see e.g. [3]) based on the test of membership in an ideal give us methods to describe invariant conditions in terms of polynomial combinations. Such methods, however, are not easy. Indeed, even for planar quadratic differential systems, the invariants are polynomials of 12 indeterminates and minimal generators are of degrees $1, ...,$ or 7.

In this work, starting from a minimal system of generators (or simply from a system of generators) of the ideal of algebraic invariants of a polynomial differential system we will develop an algorithmic method to reduce the polynomial decomposition of invariants to a linear one. We will apply this method to determine syzygies between these invariants. We will also give many examples in the case of a planar quadratic polynomial system. Our choice is motived by the existing knowledge of these systems which are objects of numerous scientific investigations including those by Poincare [10] and Liapunov [8].

2. Preliminaries

Using Einstein's notation (see e.g. [11]), the complete planar polynomial quadratic differential system of finite dimension n and of degree at most k with coefficients in a field $\mathbb{k}$ of characteristic zero can be written as

(1)
$$\begin{aligned}\frac{dx^j}{dt} &= a^j + a^j_{\alpha_1}x^{\alpha_1} + a^j_{\alpha_1\alpha_2}x^{\alpha_1}x^{\alpha_2}\\ &\quad + a^j_{\alpha_1\cdots\alpha_r}x^{\alpha_1}\cdots x^{\alpha_r},\ j,\alpha_1,\alpha_r \in \{1,...,n\}, 1 \le r \le k,\end{aligned}$$

where for $j = 1, ..., n$ and for $2 \le r \le k$, the tensor $a^j_{\alpha_1\cdots\alpha_r}$ (1 time contravariant and r times covariant) is symmetric with respect to the lower subscripts.

Let S be the set of all coefficients on the right hand side and $x = (x^1, ..., x^n)$ be the vector of the unknown variables of (1). Let G be a group of linear transformations on $\mathbb{k}^n$. A polynomial function $C : S \times \mathbb{k}^n \to \mathbb{k}$ is a covariant with respect to the group G, or G-covariant if there exists a character λ of the group G, such that

$$\forall q \in G,\ \forall a \in S,\ C(\rho(q)a, qx) = \lambda(q)C(a, x),$$

where ρ is a representation of the considered group on S. If $\lambda \equiv 1$, the covariant is said to be absolute, otherwise it is said to be relative. In the case of the linear group $GL(n)$, $\lambda(q) = \det(q)^{-\varkappa}$, where $\varkappa$ is an integer ([5] [11]), $\varkappa$ is called the weight of the covariant $C(a, x)$. If the polynomial $C(a, x)$ is independent of x, then it is said to be a G-invariant. Recall that the set of G-covariants of (1) is an algebra over the field $\mathbb{k}$.

A G-covariant $C(a,x)$ is said to be *reducible* if it can be expressed as a polynomial function of G-covariants of (the same or) lower degree. If $C(a,x)$ is *reducible*, we simply write $C(a,x) \equiv 0$ (modulo G-covariants of lower degree). A finite family B of G-covariants of (1) is called a system of generators if any G-covariant of (1) can be expressed as a sum of products of constants and elements in B. A finite family B of G-covariants of the system (1) is a system of generators of the G-covariants of the system if every G-covariant of (1) is *reducible* to zero modulo B. A system B of generators is said to be minimal if none of them is generated by the others.

Let $K = \{K_\lambda |\, \lambda \in \Lambda\}$ be the set of all G-covariants of (1). Let $Q(K)$ be a polynomial in K. If the relation $Q(K) = 0$ is an identity with respect to the variables from a and x, but not an identity with respect to the elements from K, then the relation $Q(K) = 0$ is called a *syzygy relation* for the elements in K, and $Q(K)$ the corresponding *syzygy* for (1). A *syzygy relation* for the elements in a subset of K is similarly defined. A finite family $\mathcal{S}$ of *syzygies relation* for the elements in K is *generating* if the set of its corresponding *syzygies* is a system of generators of all *syzygies* for (1) and is *free* if this set is minimal. It is a *basis* if it is free and generating.

Let G be the linear group $GL(n)$. The action of the group $GL(n)$ on $\mathbb{k}^n$: $(q,x) \to qx$, induces a representation $\rho : GL(n) \to GL(S)$ defined by

$$\begin{aligned}
\rho(q)(a^j) &= \sum_{i=1}^{n} q_i^j a^i, \\
\rho(q)(a^j_{\alpha_1}) &= \sum_{i=1}^{n} \sum_{j_1=1}^{n} q_i^j p^{j_1}_{\alpha_1} a^i_{j_1}, \\
\rho(q)(a^j_{\alpha_1\alpha_2}) &= \sum_{i=1}^{n} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} q_i^j p^{j_1}_{\alpha_1} p^{j_2}_{\alpha_2} a^i_{j_1 j_2}, \\
&\ \vdots \\
\rho(q)(a^j_{\alpha_1\cdots\alpha_r}) &= \sum_{i=1}^{n} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_r=1}^{n} q_i^j p^{j_1}_{\alpha_1} \cdots p^{j_r}_{\alpha_r} a^i_{j_1\cdots j_r},
\end{aligned}$$

where $j, \alpha_1, ..., \alpha_r \in \{1, ..., n\}$, $r = 1, ..., k$, and q is a matrix of $GL(n)$ and p its inverse.

The $GL(n)$-covariants of (1) are called centro-affine covariants. If a centro-affine covariant does not depend on x, then it is called a centro-affine invariant.

For examples (see [11]), for the planar polynomial quadratic differential system

$$\frac{dx^j}{dt} = a^j + a^j_{\alpha_1}x^{\alpha_1} + a^j_{\alpha_1\alpha_2}x^{\alpha_1}x^{\alpha_2},\ j, \alpha_1, \alpha_2 \in \{1, 2\} \tag{2}$$

(that is, system (1) where $\Bbbk = \mathbf{R}$ and $n = 2$), $I_1 = tr(a^i_j)_{i,j=1,2} = a^1_1 + a^2_2$ (or $I_1 = a^\alpha_\alpha$), $\det(a^i_j) = a^1_1a^2_2 - a^1_2a^2_1$ (or $\det(a^j_i) = \frac{1}{2}a^p_r a^q_s \varepsilon_{pq}\varepsilon^{rs}$ where $\varepsilon_{pq} = q - p$ and $\varepsilon^{rs} = s - r$), and $K_1 = (a^1_{11} + a^2_{12})x^1 + (a^1_{12} + a^2_{22})x^2$ (or $K_1 = a^\alpha_{\alpha\beta}x^\alpha$) and $K_{21} = a^1x^2 - a^2x^1$(or $K_{21} = a^px^q\varepsilon_{pq}$) are centro-affine covariants of (2). Other invariants (see e.g. [11, pp. 143-144]) include $I_1, ..., I_{36}$:

$$
\begin{array}{ll}
I_1 = a^\alpha_\alpha & I_{10} = a^\alpha_p a^\beta_\delta a^\gamma_\mu a^\delta_{\alpha q}\varepsilon^{pq} \\
I_2 = a^\alpha_\beta a^\beta_\alpha & I_{11} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\gamma} a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs} \\
I_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma}\varepsilon^{pq} & I_{12} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\delta} a^\mu_{\gamma\mu}\varepsilon^{pq}\varepsilon^{rs} \\
I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma}\varepsilon^{pq} & I_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs} \\
I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta}\varepsilon^{pq} & I_{14} = a^\alpha_p a^\beta_r a^\gamma_{\alpha q} a^\delta_{\beta s} a^\mu_{\gamma\delta} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs} \\
I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta}\varepsilon^{pq} & I_{15} = a^\alpha_{pr} a^\beta_{qk} a^\gamma_{\alpha s} a^\delta_{\delta l} a^\mu_{\beta\gamma} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl} \\
I_7 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\beta s} a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs} & I_{16} = a^\alpha_p a^\beta_r a^\gamma_\delta a^\delta_{\alpha q} a^\mu_{\beta s} a^\nu_{\gamma\tau} a^\tau_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs} \\
I_8 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\delta s} a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs} & I_{17} = a^\alpha_{\alpha\beta} a^\beta \\
I_9 = a^\alpha_{pr} a^\beta_{\beta q} a^\gamma_{\gamma s} a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs} & I_{18} = a^p_\alpha a^\alpha a^q \varepsilon_{pq} \\
I_{19} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma & I_{28} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} a^\delta a^\mu \\
I_{20} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma & I_{29} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma_{\delta\mu} a^\delta a^\mu \\
I_{21} = a^p_{\alpha\beta} a^\alpha a^\beta a^q \varepsilon_{pq} & I_{30} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\delta} a^\delta_{\gamma\mu} a^\mu \varepsilon^{pq} \\
I_{22} = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} a^\gamma a^\delta & I_{31} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\mu} a^\delta_{\gamma\delta} a^\mu \varepsilon^{pq} \\
I_{23} = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} a^\gamma a^\delta & I_{32} = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\mu} a^\delta_{\gamma\delta} a^\mu \varepsilon^{pq} \\
I_{24} = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta} a^\delta & I_{33} = a^\alpha_{\beta\nu} a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} a^\delta a^\mu a^\nu \\
I_{25} = a^\alpha_{\alpha p} a^\beta_{\gamma q} a^\gamma_{\beta\delta} a^\delta \varepsilon^{pq} & I_{34} = a^\alpha_{\mu p} a^\beta_{\alpha q} a^\gamma_{\beta\nu} a^\delta_{\gamma\delta} a^\mu a^\nu \varepsilon^{pq} \\
I_{26} = a^\alpha_{\alpha p} a^\beta_{\gamma q} a^\gamma_{\beta\delta} a^\delta \varepsilon^{pq} & I_{35} = a^\alpha_p a^\beta_\nu a^\gamma_{\alpha q} a^\delta_{\beta\mu} a^\mu_{\gamma\delta} a^\nu \varepsilon^{pq} \\
I_{27} = a^p_\alpha a^\alpha_{\beta\gamma} a^\beta a^\gamma a^q \varepsilon_{pq} & I_{36} = a^\alpha_{pr} a^\beta_{\nu q} a^\gamma_{\alpha s} a^\delta_{\beta\gamma} a^\mu_{\delta\mu} a^\nu \varepsilon^{pq}\varepsilon^{rs}
\end{array}
$$

and covariants (see e.g. [11, 13, 2]) include $K_1, ..., K_{33}$:

$$
\begin{array}{lll}
K_1 = a^\alpha_{\alpha\beta} x^\beta & K_{12} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} x^\delta x^\mu & K_{23} = a^p a^q_{\alpha\beta} x^\alpha x^\beta \varepsilon_{pq} \\
K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq} & K_{13} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma_{\delta\mu} x^\delta x^\mu & K_{24} = a^p a^q_\alpha a^\alpha_{\beta\gamma} x^\beta x^\gamma \varepsilon_{pq} \\
K_3 = a^\alpha_\beta a^\beta_{\alpha\gamma} x^\gamma & K_{14} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\delta} a^\delta_{\gamma\mu} x^\mu \varepsilon^{pq} & K_{25} = a^\alpha a^\beta a^p_{\alpha\beta} x^q \varepsilon_{pq} \\
K_4 = a^\alpha_\gamma a^\beta_{\alpha\beta} x^\gamma & K_{15} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\mu} a^\delta_{\gamma\delta} x^\mu \varepsilon^{pq} & K_{26} = a^\alpha a^\beta_{\alpha\delta} a^\gamma_{\beta\gamma} x^\delta
\end{array}
$$

$K_5 = a^p_{\alpha\beta} a^\alpha x^\beta x^q \varepsilon_{pq}$ $\quad K_{16} = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\mu} a^\delta_{\gamma\delta} x^\mu \varepsilon^{pq}$ $\quad K_{27} = a^\alpha a^\beta_{\alpha\gamma} a^\gamma_{\beta\delta} x^\delta$

$K_6 = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} x^\gamma x^\delta$ $\quad K_{17} = a^\alpha_{\beta\nu} a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} x^\delta x^\mu x^\nu$ $\quad K_{28} = a^\alpha a^\beta a^p_\gamma a^\gamma_{\alpha\beta} x^q \varepsilon_{pq}$

$K_7 = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} x^\gamma x^\delta$ $\quad K_{18} = a^\alpha_{\mu p} a^\beta_{\alpha q} a^\gamma_{\beta\nu} a^\delta_{\gamma\delta} x^\mu x^\nu \varepsilon^{pq}$ $\quad K_{29} = a^\alpha a^\beta_\delta a^\gamma_{\alpha\mu} a^\delta_{\beta\delta} x^\mu$

$K_8 = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta} x^\delta$ $\quad K_{19} = a^\alpha_p a^\beta_\nu a^\gamma_{\alpha q} a^\delta_{\beta\mu} a^\mu_{\gamma\delta} x^\nu \varepsilon^{pq}$ $\quad K_{30} = a^\alpha a^\beta_\gamma a^\gamma_{\alpha\mu} a^\delta_{\beta\delta} x^\mu$

$K_9 = a^\alpha_{\alpha p} a^\beta_{\gamma q} a^\gamma_{\beta\delta} x^\delta \varepsilon^{pq}$ $\quad K_{20} = a^\alpha_{pr} a^\beta_{\nu q} a^\gamma_{\alpha s} a^\delta_{\beta\gamma} a^\mu_{\delta\mu} x^\nu \varepsilon^{pq} \varepsilon^{rs}$ $\quad K_{31} = a^\alpha a^\beta_{\alpha\gamma} a^\gamma_{\beta\delta} a^\delta_{\mu\nu} x^\mu x^\nu$

$K_{10} = a^\alpha_{\alpha p} a^\beta_{\delta q} a^\gamma_{\beta\gamma} x^\delta \varepsilon^{pq}$ $\quad K_{21} = a^p x^q \varepsilon_{pq}$ $\quad K_{32} = a^\alpha a^\beta a^\gamma_{\alpha\beta} a^\delta_{\mu\nu} a^\mu_{\gamma\delta} x^\nu$

$K_{11} = a^p_\alpha a^\alpha_{\beta\gamma} a^\beta x^\gamma x^q \varepsilon_{pq}$ $\quad K_{22} = a^p_\alpha a^\alpha x^q \varepsilon_{pq}$ $\quad K_{33} = a^\alpha a^\beta_{p\alpha} a^\gamma_{q\beta} a^\delta_{\gamma\nu} a^\mu_{\delta\mu} x^\nu \varepsilon^{pq}$

where $\varepsilon^{pq} = \varepsilon_{pq} = q - p$.

Two basic facts about centro-affine covariants of (2) are known.

Theorem 1. ([6]). *Any system of generators of centro-affine covariants of* (*1*) *is made up of polynomial expressions of the coefficients of these systems and the vector* x *obtained from the tensorial operations of alternation or total contraction.*

For examples, $I_1 = a^\alpha_\alpha$ and $K_1 = a^\alpha_{\alpha\beta} x^\alpha$ are obtained from total contraction, $\det(a^j_i) = \frac{1}{2} a^p_r a^q_s \varepsilon_{pq} \varepsilon^{rs}$ and $K_{21} = a^p x^q \varepsilon_{pq}$ are obtained from alternation, and $I_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma} \varepsilon^{pq}$ and $K_{23} = a^p a^q_{\alpha\beta} x^\alpha x^\beta \varepsilon_{pq}$ are obtained from alternation and total contraction.

Theorem 2. ([11, 13, 2]). *The family* $B = \{I_1, \ldots, I_{36}, K_1, \ldots, K_{33}\}$ *form a minimal system of generators of the ideal of centro-affine invariants and covariants of (2), and* $E = \{I_1, \ldots, I_{36}\}$ *form a minimal system of generators of the ideal of centro-affine invariants of planar polynomial quadratic differential systems (2).*

3. Algebra of Centro-affine Covariants

An important question is how invariant conditions of differential systems (1) can be expressed in convenient manners. We have already mentioned the Aronhold symbolism method [11] and the method of Gröbner basis. Here we describe an alternate method.

Let $B = \{C_1, \ldots, C_s\}$ be a minimal system of generators of the ideal of centro-affine covariants (or centro-affine invariants) of (1). Since each element in the family B is a homogeneous polynomial in a and x, in view of Theorem 1, each centro-affine covariant C (respectively centro-affine invariant) of (1) is of the form

$$C = \sum_r c_r p_r(C_1, ..., C_s),$$

where each c_r is a scalar in the field $\mathbb{k}$ and $p_r(C_1, ..., C_s) = C_1^{\lambda_1} \cdots C_s^{\lambda_s}$ with exponents $\lambda_1, ..., \lambda_s \in \mathbf{N}$, where $\mathbf{N}$ is the set of nonnegative integers.

This motivates the following definitions. A centro-affine covariant of (1) is said to be of type (or of the multi-degree) $(d_0, d_1, d_2, ..., d_r, \delta)$ if it is homogeneous of degree d_0 in relation to a^j, of degree d_1 in relation to a^j_α, of degree d_2 in relation to $a^j_{\alpha_1\alpha_2}$, ..., of degree d_r in relation to $a^j_{\alpha_1\cdots\alpha_r}$ and of degree δ in relation to the contravariant vector x. The integer δ is called the order of the centro-affine covariant. An invariant is a centro-affine covariant of order $\delta = 0$, hence it is conveniently said to be of type $(d_0, d_1, d_2, ..., d_r)$.

For examples, $K_1 = a^\alpha_{\alpha\beta}x^\alpha$ is of type $(0, 0, 1, 1)$, $I_1 = a^\alpha_\alpha$ is of type $(0, 1, 0)$, $I_2 = a^\alpha_\beta a^\beta_\alpha$ is of type $(0, 2, 0)$, $I_1^2 = (a^\alpha_\alpha)^2$ is of type $(0, 2, 0)$, $I_{17} = a^\alpha a^\alpha_{\alpha\beta}$ is of type $(1, 0, 1)$, $I_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma}\varepsilon^{pq}$ is of type $(0, 1, 2)$, $I_2 I_3 = (a^\alpha_\beta a^\beta_\alpha)(a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma}\varepsilon^{pq})$ is of type $(0, 3, 2)$ and $K_{21} = a^p x^q \varepsilon_{pq}$ is of type $(1, 0, 0, 1)$. For later uses, let's record here the respective types $T_1, T_2, ..., T_{36}$ of centro-affine invariants $I_1, ..., I_{36}$ of E:

$$T_1 = (0, 1, 0), T_2 = (0, 2, 0), T_3 = T_4 = T_5 = (0, 1, 2), T_6 = (0, 2, 2), T_7 = T_8 = T_9 = (0, 0, 4),$$

$$T_{10} = (0, 3, 2), T_{11} = T_{12} = T_{13} = (0, 1, 4), T_{14} = (0, 2, 4), T_{15} = (0, 0, 6), T_{16} = (0, 3, 4),$$

$$T_{17} = (1, 0, 1), T_{18} = (2, 1, 0), T_{19} = T_{20} = (1, 1, 1), T_{21} = (3, 0, 1), T_{22} = T_{23} = (2, 0, 2),$$

$$T_{24} = (1, 2, 1), T_{25} = T_{26} = (1, 0, 3), T_{27} = (3, 1, 1), T_{28} = T_{29} = (2, 1, 2),$$

$$T_{30} = T_{31} = T_{32} = (1, 1, 3), T_{33} = (3, 0, 3), T_{34} = (2, 0, 4), T_{35} = (1, 2, 3), T_{36} = (1, 0, 5).$$

We will use $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$ ($\mathcal{A}_{(d_0,d_1,d_2,...,d_r)}$) to denote the set of centro-affine covariants (respectively invariants) of type $(d_0, d_1, d_2, ..., d_r, \delta)$ (respectively $(d_0, d_1, d_2, ..., d_r)$).

The differential system (1) can be identified as the direct sum of tensorial subspaces

$$\mathcal{T}^1_0 \oplus \mathcal{T}^1_1 \oplus \mathcal{T}^1_2 \oplus \cdots \oplus \mathcal{T}^1_r,\ 1 \le r \le k,$$

where for $r = 1, ..., k$, $\mathcal{T}^1_r$ denotes the space of tensors 1 time contravariant and r times covariants. $\mathcal{T}^1_r$ corresponds to the homogenous part of degree r of the polynomials of the right hand side of system (1)). If $\mathcal{A}$ denotes the $\mathbb{k}$-algebra of centro-affine covariants (respectively invariants) of these systems, $\mathcal{A}$ is a direct sum of the vectorial subspaces $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$ (respectively $\mathcal{A}_{(d_0,d_1,d_2,...,d_r)}$), where $d_0, d_1, d_2, ..., d_r, \delta \in \mathbf{N}$.

Note that the algebra $\mathcal{A}$ of the covariants is graded, that is,

$$\mathcal{A} = \bigoplus\nolimits_{d_0,d_1,d_2,...,d_r,\delta\in\mathbf{N}} \mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)},$$

and that the centro-affine invariants can be considered as particular centro-affine covariants (by letting $\delta = 0$). We will therefore limit our study on the centro-affine covariants.

Each centro-affine covariant C can be written as

$$C = \sum_{d_0,d_1,d_2,\dots,d_r,\delta\in\mathbf{N}} c_{(d_0,d_1,d_2,\dots,d_r,\delta)}C_{(d_0,d_1,d_2,\dots,d_r,\delta)}$$

where $c_{(d_0,d_1,d_2,\dots,d_r,\delta)} \in \mathbb{k}$, and $C_{(d_0,d_1,d_2,\dots,d_r,\delta)} \in \mathcal{A}_{(d_0,d_1,d_2,\dots,d_r,\delta)}$ can be written as

$$C_{(d_0,d_1,d_2,\dots,d_r,\delta)} = \sum_{\lambda_1,\lambda_2,\dots,\lambda_r\in\mathbf{N}} c_{\lambda_1,\lambda_2,\dots,\lambda_r} C_1^{\lambda_1}\cdots C_s^{\lambda_S}.$$

where $c_{\lambda_1,\lambda_2,\dots,\lambda_r}\in\mathbb{k}$ and $\lambda_1, \lambda_2, \dots, \lambda_r\in\mathbf{N}$, $C_1^{\lambda_1}\cdots C_s^{\lambda_S}$ is of type $(d_0, d_1, d_2, \dots, d_r, \delta)$ which can be written as $(d_0, d_1, d_2, \dots, d_r, \delta) = \lambda_1 TC_1 + \lambda_2 TC_2 + \cdots + \lambda_s TC_s$ since $C_1, C_2, \dots, C_s$ are homogenous polynomials of multidegree $TC_1, TC_2, \dots, TC_s$ respectively. It is easy to see that the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,\dots,d_r,\delta)}$ is *generated* by centro-affine covariants of the form: $C_1^{\lambda_1}\cdots C_s^{\lambda_S}$ where $\lambda_1, \lambda_2, \dots, \lambda_r$ are nonnegative integers such that $(d_0, d_1, d_2, \dots, d_r, \delta) = \lambda_1 TC_1 + \lambda_2 TC_2 + \cdots + \lambda_s TC_s$.

Now, let us consider the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,\dots,d_r,\delta)}$. A covariant $C_{(d_0,d_1,d_2,\dots,d_r,\delta)}$ for (1) of type $(d_0, d_1, d_2, \dots, d_r, \delta)$ can be written as a finite sum $C_{(d_0,d_1,d_2,\dots,d_r,\delta)} = \sum c_{\lambda_1\cdots\lambda_s} C_1^{\lambda_1}\cdots C_s^{\lambda_S}$, where $\lambda_1, \dots, \lambda_s$ are nonnegative integers such the homogenous centro-affine covariants $C_1^{\lambda_1}\cdots C_s^{\lambda_S}$ of (1) are of type $(d_0, d_1, d_2, \dots, d_r, \delta)$. For the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2)}$, we have $C_{(d_0,d_1,d_2)} = \sum c_{\lambda_1\cdots\lambda_{36}} I_1^{\lambda_1} \cdots I_{36}^{\lambda_{36}}$, where $\lambda_1, \dots, \lambda_{36}$ are nonnegative integers such the homogenous centro-affine invariants $I_1^{\lambda_1}\cdots I_{36}^{\lambda_{36}}$ of (2) are of type (d_0, d_1, d_2). Let's determine $\lambda_1, \dots, \lambda_{36} \in N$ such that $I_1^{\lambda_1}\cdots I_{36}^{\lambda_{36}}$ is of type (d_0, d_1, d_2). Note that $I_1 = a^\alpha_\alpha$ is of type $T_1 = (0, 1, 0)$ since it is of degree $d_0 = 0$ in relation to a^j, of degree $d_1 = 1$ in relation to a^j_α and of degree $d_2 = 0$ in relation to $a^j_{\alpha\beta}$. Thus $I_1^{\lambda_1} = (a^\alpha_\alpha)^{\lambda_1}$ is homogenous of degree $d_0 = 0$ in relation to a^j, of degree $d_1 = \lambda_1$ in relation to a^j_α, and of degree $d_2 = 0$ in relation to $a^j_{\alpha\beta}$. In other words, it is of type $(0, \lambda_1, 0) = \lambda_1(0, 1, 0) = \lambda_1 T_1$. The same reasoning leads us to: $I_2^{\lambda_2}$ is of type $(0, 2\lambda_2, 0) = \lambda_2(0, 2, 0) = \lambda_2 T_2, \dots$, and $I_{36}^{\lambda_{36}}$ is of type $(\lambda_{36}, 0, 5\lambda_{36}) = \lambda_{36}(1, 0, 5) = \lambda_{36} T_{36}$.

Consider $I_1^{\lambda_1}\cdots I_{36}^{\lambda_{36}}$. It has degree d_0 in relation to a^j and is the sum of degrees of $I_1^{\lambda_1}, \dots, I_{36}^{\lambda_{36}}$ in relation to a^j; and has degree d_1 in relation to a^j_α which is the sum of degrees of $I_1^{\lambda_1}, \dots, I_{36}^{\lambda_{36}}$ in relation to a^j_α. It has degree d_2 in relation to $a^j_{\alpha\beta}$ and is the sum of degrees of $I_1^{\lambda_1}, \dots, I_{36}^{\lambda_{36}}$ in relation to $a^j_{\alpha\beta}$. Thus (d_0, d_1, d_2) is the sum of types $(0, \lambda_1, 0)$, ..., $(\lambda_{36}, 0, 5\lambda_{36})$:

$$\begin{aligned}(d_0, d_1, d_2) &= (0, \lambda_1, 0) + \cdots + (\lambda_{36}, 0, 5\lambda_{36}) = \lambda_1(0, 1, 0) + \cdots + \lambda_{36}(1, 0, 5)\\ &= \lambda_1 T_1 + \cdots + \lambda_{36} T_{36}.\end{aligned}$$

For example, let us determine the type of $I_1^3 I_8 I_{17}^5$. I_1^3 is of type $3(0,1,0) = (0,3,0)$ (since I_1 is of type $(1,0,1)$), I_8 is of type $(0,0,4)$ and I_{17}^5 is of type $5(1,0,1) = (5,0,5)$ (since I_{17} is of type $(1,0,1)$). Hence $I_1^3 I_8 I_{17}^5$ is of type $(0,3,0) + (0,0,4) + (5,0,5) = (5,3,9) = 3T_1 + T_8 + 3T_{17}$, $I_5^3 I_{11}^2$ is of type $3T_5 + 2T_{11}$, and $I_4^3 I_{12}^7 I_{25}^2 I_{31}$ is of type $3T_4 + 7T_{12} + 2T_{25} + T_{31}, ...,$ etc.

Let $\mathcal{F}_{(d_0,d_1,d_2,...,d_r,\delta)} = \{f_1, \ldots, f_\tau\}$ be a finite generating family of the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$ where $(d_0, d_1, d_2, ..., d_r, \delta) = \lambda_1 TC_1 + \lambda_2 TC_2 + \cdots + \lambda_s TC_s$.

It is easy to see that the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$ is *generated* by centro-affine covariants of the form:

$$f = C_1^{\lambda_1} \cdots C_s^{\lambda_s}, \ \lambda_1, ..., \lambda_s \in \mathbf{N},$$

where

(3) $$(d_0, d_1, d_2, ..., d_r, \delta) = \lambda_1 TC_1 + \lambda_2 TC_2 + \cdots + \lambda_s TC_s.$$

To determine the type of $C_1^{\lambda_1} \cdots C_s^{\lambda_s}$, we need to determine $\lambda_1, ..., \lambda_s \in \mathbf{N}$ such that

$$(d_0, d_1, d_2, ..., d_r, \delta) = \lambda_1 TC_1 + \lambda_2 TC_2 + \cdots + \lambda_s TC_s.$$

For example, since $(0,2,0) = 2T_1 = T_2$, we see that there are two *generators* I_1^2 and I_2 for $\mathcal{A}_{(0,2,0)}$. As another example, $\mathcal{F}_{(1,1,1)} = \{I_{20}, I_{19}, I_1 I_{17}\}$ since $(1,1,1) = T_{20} = T_{19} = T_1 + T_{17}, ...,$ and $\mathcal{F}_{(1,1,3)} = \big\{I_{32}, I_{31}, I_{30}, I_5 I_{17}, I_4 I_{17}, I_3 I_{17}, I_1 I_{26}, I_1 I_{25}\big\}$ since

$$(1,1,3) = T_{32} = T_{31} = T_{30} = T_5 + T_{17} = T_4 + T_{17} = T_3 + T_{17} = T_1 + T_{26} = T_1 + T_{25}.$$

Let $T_1, ..., T_{36}$ be the types of $I_1, ..., I_{36}$ computed in the previous section. Given $T = (d_0, d_1, d_2)$, we search $\lambda_1, ..., \lambda_{36}$ in $\mathbf{N}$ such that $(d_0, d_1, d_2) = \lambda_1 T_1 + \cdots + \lambda_{36} T_{36}$. We first remark that for the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2)}$, if $(d_0, d_1, d_2) = \lambda_1 T_1 + \cdots + \lambda_{36} T_{36}$, then, $0 \le \lambda_1 \le d_1$ since $T_1 = (0,1,0)$; $0 \le \lambda_2 \le [d_1/2]$ since $T_2 = (0,2,0)$; ...; $0 \le \lambda_{28} \le [\min(d_0/2, d_1, d_2/2)]$ since $T_{28} = (2,1,2)$; ...; $0 \le \lambda_{36} \le [\min(d_0, d_2/5)]$ since $T_{36} = (1,0,5)$.

Thus if we let

$$t_i[j+1] = \begin{cases} \frac{d_j}{TC_i[j+1]} & \text{if } TC_i[j+1] \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, ..., s$ and $j = 0, 1, 2$; and let

$$\alpha_i = \left[\min_{k=1,2,3} (t_i[k] \neq 0)\right], \ i = 1, ..., s,$$

where $[x]$ denotes the greatest integer part of x, then seeking $\lambda_1, ..., \lambda_s$ in $\mathbf{N}$ such that $(d_0, d_1, d_2, ..., d_r, \delta) = \lambda_1 TC_1 + \lambda_2 TC_2 + \cdots + \lambda_s TC_s$ is the same as searching $\lambda_1, ..., \lambda_s$ in $\mathbf{N}$ such that

$$0 \le \lambda_1 \le \alpha_1, ..., 0 \le \lambda_s \le \alpha_s \text{ and } (d_0, d_1, d_2, ..., d_r, \delta) = \lambda_1 TC_1 + \lambda_2 TC_2 + \cdots + \lambda_s TC_s.$$

For $I_1, ..., I_{36}$, the integers $\alpha_1, ..., \alpha_{36}$ are calculated as follows:

$$\begin{aligned}
T_1 &= (0,1,0) \to t_1 = \left(0, \frac{d_1}{T_1[2]}, 0\right) = \left(0, \frac{d_1}{1}, 0\right) = (0, d_1, 0),\\
\alpha_1 &= [\min_{k=1,2,3}(t_1[k] \neq 0)] = [\min(d_1)] = d_1;\\
T_2 &= (0,2,0) \to t_2 = \left(0, \frac{d_1}{T_2[2]}, 0\right) = \left(0, \frac{d_1}{2}, 0\right),\\
\alpha_2 &= [\min_{k=1,2,3}(t_2[k] \neq 0)] = \left[\min\left(\frac{d_1}{2}\right)\right] = \left[\frac{d_1}{2}\right];\\
&\vdots\\
T_{28} &= (2,1,2) \to t_{28} = \left(\frac{d_0}{T_{28}[1]}, \frac{d_1}{T_{28}[2]}, \frac{d_2}{T_{28}[3]}\right) = \left(\frac{d_0}{2}, \frac{d_1}{1}, \frac{d_2}{2}\right) = \left(\frac{d_0}{2}, d_1, \frac{d_2}{2}\right),\\
\alpha_{28} &= [\min_{k=1,2,3}(t_{28}[k] \neq 0)] = \left[\min\left(\frac{d_0}{2}, \frac{d_1}{1}, \frac{d_2}{2}\right)\right] = \left[\min\left(\frac{d_0}{2}, d_1, \frac{d_2}{2}\right)\right];\\
T_{36} &= (1,0,5) \to t_{36} = \left(\frac{d_0}{T_{36}[1]}, 0, \frac{d_2}{T_{36}[3]}\right) = \left(\frac{d_0}{1}, 0, \frac{d_2}{5}\right) = \left(d_0, 0, \frac{d_2}{5}\right),\\
\alpha_{36} &= [\min_{k=1,2,3}(t_{36}[k] \neq 0)] = \left[\min\left(\frac{d_0}{1}, \frac{d_2}{5}\right)\right] = \left[\min\left(d_0, \frac{d_2}{5}\right)\right].
\end{aligned}$$

Let's consider an example where $(d_0, d_1, d_2) = (2, 3, 4)$. By the above calculations, we see that

$$\begin{aligned}
0 &\le \lambda_1 \le \alpha_1 = [\min_{k=1,2,3}(t_1[k] \neq 0)] = \left[\min\left(\frac{3}{1}\right)\right] = 3,\\
0 &\le \lambda_2 \le \alpha_2 = [\min_{k=1,2,3}(t_2[k] \neq 0)] = \left[\min\left(\frac{3}{2}\right)\right] = 1,\\
&\quad \cdots\\
0 &\le \lambda_{28} \le \alpha_{28} = [\min_{k=1,2,3}(t_{28}[k] \neq 0)] = [\min\left(\frac{2}{2}, 3, \frac{4}{2}\right)] = 1,\\
&\quad \cdots\\
0 &\le \lambda_{36} \le \alpha_{36} = [\min_{k=1,2,3}(t_{36}[k] \neq 0)] = \left[\min\left(\frac{2}{1}, \frac{4}{5}\right)\right] = 0.
\end{aligned}$$

Given a type $(d_0, d_1, d_2, ..., d_r, \delta)$, we may now compute the finite *generating family*

$$\mathcal{F}_{(d_0,d_1,d_2,...,d_r,\delta)} = \{f_1, ..., f_s\}$$

of the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$.

Algorithm 1. Compute $\mathcal{F}_{(d_0,d_1,d_2,...,d_r,\delta)} = \{f_1, ..., f_s\}$, the *generating family* of the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$).

(1) Enter $TC_1, ..., TC_s$ defined before. Enter a given type $TC = (d_0, d_1, d_2, ..., d_r, \delta)$ and an index $l = 0$.

(2) For $i = 1, ..., s$ and $j = 0, ..., r$,

$$t_i[j+1] = \begin{cases} \frac{d_j}{TC_i[j+1]} & \text{if } TC_i[j+1] \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(3) For $i = 1, ..., s$,

$$\alpha_i = \left[\min_{k=1,r+1} (t_i[k] \neq 0) \right].$$

(4) While $\lambda_1 \leq \alpha_1, ..., \lambda_s \leq \alpha_s$, if $(d_0, d_1, d_2, ..., d_r, \delta) = \lambda_1 TC_1 + \cdots + \lambda_s TC_s$ then $l = l + 1$ and $f_{l+1} = C_1^{\lambda_1} \cdots C_s^{\lambda_s}$.

(5) Return $\mathcal{F}_{(d_0,d_1,d_2,...,d_r,\delta)} = \{f_1, ..., f_s\}$.

For the differential system (2), the elements $f_1, ..., f_s$ are the products of homogenous polynomials of 12 indeterminates! For this reason, in the next section, we will develope an alternate algorithmic method to express the centro-affine covariants of (1) which avoids polynomial products and polynomial sums.

4. Development of Centro-affine Covariants

In view of Theorem 1 a covariant C of $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$ is a tensor

$$(\mathcal{T}_0^1)^{\otimes d_0} \otimes (\mathcal{T}_1^1)^{\otimes d_1} \otimes \cdots \otimes (\mathcal{T}_s^1)^{\otimes d_r} \otimes \Bbbk^{\otimes \delta},\ 1 \leq r \leq k,$$

obtained from alternation or total contraction. This motivates the following definitions.

Let $t_1^0, ..., t_{j_0}^0$ be the j_0 coefficients of the tensor a^j (1 time contravariant and 0 time covariant), and for $l = 1, ..., r$ where $1 \leq r \leq k$, $t_1^l, ..., t_{j_l}^l$ the j_l coefficients of the tensor $a^j_{\alpha_1 \cdots \alpha_l}$, $0 \leq l \leq r \leq k$ (1 time contravariant and l time covariants).

For $p_0 = (p_1^0, ..., p_{j_0}^0) \in \mathbf{N}^{j_0}$, we use $(a^j)^{p_0}$ to denote the product $(t_1^0)^{p_1^0} \cdots (t_{j_0}^0)^{p_{j_0}^0}$, for $p_1 = (p_1^1, ..., p_{j_1}^1) \in \mathbf{N}^{j_1}$ we use $(a^j_{\alpha_1})^{p_1}$ to denote the product $(t_1^1)^{p_1^0} \cdots (t_{j_1}^1)^{p_{n_{j_1}}^1}$, for $p_2 = (p_1^2, ..., p_{j_2}^2) \in \mathbf{N}^{j_2}$, we use $(a^j_{\alpha_1\alpha_2})^{p_2}$ to denote the product $(t_1^1)^{p_1^0} \cdots (t_{j_2}^1)^{p_{j_2}^1}$, etc.

For $p_r = (p_1^r, ..., p_{j_r}^r) \in \mathbf{N}^{j_r}$, where $1 \le r \le k$, we use $(a^j_{\alpha_1\cdots\alpha_r})^{p_r}$ to denote the product $(t_1^r)^{p_1^r} \cdots (t_{j_r}^r)^{p_{j_r}^1}$, and for $\alpha = (\delta_1, ..., \delta_n) \in \mathbf{N}^n$, we use $(x)^\alpha$ to denote the product $(x^1)^{\delta_1} \cdots (x^n)^{\delta_n}$.

A monomial associated with (1) is a finite product of the form

$$(a^j)^{p_0}(a^j_{\alpha_1})^{p_1}(a^j_{\alpha_1\alpha_2})^{p_2}\cdots(a^j_{\alpha_1\cdots\alpha_r})^{p_r}(x)^\alpha,\ 1 \le r \le k.$$

In general, a monomial is not a centro-affine covariant of (1). If it is a monomial of a centro-affine covariant of type $(d_0, d_1, d_2, ..., d_r, \delta)$, where $(p_1^0, ..., p_{j_0}^0)$, $(p_1^2, ..., p_{j_2}^2)$, $(p_1^1, ..., p_{j_1}^1)$, ... , $(p_1^r, ..., p_{j_r}^r)$, $1 \le r \le k$, and $(\delta_1, ..., \delta_n)$ are respectively the partitions[1] of the nonnegative integers $d_0, d_1, d_2, ..., d_r$ and δ, and j_r is the number of coefficient of the tensor $a^j_{\alpha_1\cdots\alpha_r}$ where $1 \le r \le k$, the monomial

$$(a^j)^{d_0}(a^j_{\alpha_1})^{d_1}(a^j_{\alpha_1\alpha_2})^{d_2}\cdots(a^j_{\alpha_1\cdots\alpha_r})^{d_r}(x)^\delta, 1 \le r \le k,$$

will be called a monomial of type $(d_0, d_1, d_2, ..., d_r, \delta)$. For example, among the monomials

$$a^{i_1}_{j_1}a^{i_2}_{j_2} : (a_1^1)^2, a_1^1a_2^1, a_1^1a_2^2, (a_2^1)^2, a_2^1a_1^2, a_2^1a_2^2, (a_1^2)^2, a_1^2a_2^2, (a_2^2)^2,$$

monomials of type $(0, 2, 0)$ are: $(a_1^1)^2, a_1^1a_2^2, a_2^1a_1^2$ and $(a_2^2)^2$.

Monomials are cumbersome to write. To simplify matters, let us first order the tensorial coefficients $a^j, a^j_{\alpha_1}, a^j_{\alpha_1\alpha_2}, ..., a^j_{\alpha_1\cdots\alpha_r}$ where $1 \le r \le k$, $j, \alpha_1, ..., \alpha_r \in \{1, ..., n\}$ of (1) and the components $x^1, ..., x^n$ of the contravariant vector x in the following manner: $a^j \prec a^\ell_{i_1\ldots i_s} \prec x^i$ for all $i, j, \ell \in \{1, ..., n\}$; $a^j \prec a^\ell$ if $j < \ell$ for all $i, j, \ell \in \{1, ..., n\}$; and $a^j_{j_1\ldots j_{s_1}} \prec a^\ell_{\ell_1\ldots\ell_{s_2}}$ if $s_1 < s_2$ or ($s_1 = s_2$ and the first non null component of the vector $(j, j_1, \ldots, j_{s_1}) - (\ell, \ell_1, \ldots, \ell_{s_2}) = (j - \ell, j_1 - \ell_1, \ldots, j_{s_1} - \ell_{s_2})$ is negative).

For the planar quadratic differential system (2), one has

$$a^1 \prec a^2 \prec a_1^1 \prec a_2^1 \prec a_1^2 \prec a_2^2 \prec a_{11}^1 \prec a_{12}^1 \prec a_{22}^1 \prec a_{11}^2 \prec a_{12}^2 \prec a_{22}^2 \prec x^1 \prec x^2. \tag{4}$$

The set of all monomials will be denoted by $\mathcal{M}$, while the set of all monomials of type $(d_0, d_1, d_2, ..., d_r, \delta)$ will be denoted by $\mathcal{M}_{(d_0,d_1,d_2,...,d_r,\delta)}$. If we define

$$\begin{aligned} &(a^j)^{p_0}(a^j_{\alpha_1})^{p_1}(a^j_{\alpha_1\alpha_2})^{p_2}\cdots(a^j_{\alpha_1\cdots\alpha_r})^{p_r}(x)^\delta \\ &\times(a^j)^{q_0}(a^j_{\alpha_1})^{q_1}(a^j_{\alpha_1\alpha_2})^{q_2}\cdots(a^j_{\alpha_1\cdots\alpha_r})^{q_r}(x)^\mu \\ = &(a^j)^{p_0+q_0}(a^j_{\alpha_1})^{p_1+q_1}(a^j_{\alpha_1\alpha_2})^{p_2+q_2}\cdots(a^j_{\alpha_1\cdots\alpha_r})^{p_r+q_r}(x)^{\delta+\mu} \end{aligned}$$

[1] $(\alpha_1, ..., \alpha_m)$ is a partition of the nonnegative integer β if $\alpha_1, ..., \alpha_m$ are nonnegative integers such that $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_m$.

for $i = 1, ..., r$, and $p_i, q_i \in \mathbf{N}^{j_r}$ $(1 \leq r \leq k)$, then $\mathcal{M}$ is a monoid with the identity 1.

Since the number of partitions of the nonnegative integers $d_0, d_1, d_2, ..., d_r$, $(1 \leq r \leq k)$, and δ are finite, $\mathcal{M}_{(d_0,d_1,d_2,...,d_r,\delta)}$ is a finite set and hence can be written as $\{m_1, ..., m_{n_0}\}$, where n_0 is the number of elements of $\mathcal{M}_{(d_0,d_1,d_2,...,d_r,\delta)}$. For example,

$$\mathcal{M}_{(0,2,0)} = \left\{(a_1^1)^2, a_1^1 a_2^2, a_2^1 a_1^2, (a_2^2)^2\right\} \text{ and } n_0 = 4.$$

Recall that a monomial order for a monoid is a binary relation that is (i) total, (ii) compatible with the product, and (iii) well ordered (so that any nonempty subset of the monoid has a smallest element). By treating the tensorial coefficients as 'alphabets', the total ordering defined by (4) can be extended to a total lexicographic ordering for the set $\mathcal{M}$ in the usual manner (see e.g. [12, pp. 373-375]):

$$\begin{aligned} & (a^j)^{p_0}(a^j_{\alpha_1})^{p_1}(a^j_{\alpha_1\alpha_2})^{p_2}\cdots(a^j_{\alpha_1\cdots\alpha_r})^{p_r}(x)^\alpha \\ & \prec (a^j)^{q_0}(a^j_{\alpha_1})^{q_1}(a^j_{\alpha_1\alpha_2})^{q_2}\cdots(a^j_{\alpha_1\cdots\alpha_r})^{q_r}(x)^\mu \\ \Leftrightarrow \quad & \text{the first nonzero component of the vector} \\ & (p_0 - q_0, p_1 - q_1, p_2 - q_2, ..., p_r - q_r, \alpha - \mu) \text{ is positive.} \end{aligned}$$

This ordering is a monomial order, since it suffices and is easy to check that if m_1 and m_2 are two monomials such that $m_1 \prec m_2$ then for any monomial m, one has $mm_1 \prec mm_2$.

Let us consider some examples of centro-affine invariants of planar quadratic differential system (2): $I_1 = a^\alpha_\alpha$ is a sum of a^1_1 and a^2_2. Since $a^1_1 \prec a^2_2$, we may write I_1 as a sum of terms ordered in an increasing manner:

$$I_1 = a^1_1 + a^2_2.$$

We may do the same for $I_2 = a^\alpha_\beta a^\beta_\alpha$, $I_{17} = a^\alpha a^\alpha_{\alpha\beta}$ and $K_1 = a^\alpha_{\alpha\beta} x^\alpha$:

$$\begin{aligned} I_2 &= (a^1_1)^2 + 2a^1_2 a^2_1 + (a^2_2)^2, \\ I_{17} &= a^1 a^1_{11} + a^1 a^2_{11} + a^2 a^1_{12} + a^2 a^2_{22}, \\ K_1 &= a^1_{11} x^1 + a^2_{11} x^1 + a^1_{12} x^2 + a^2_{22} x^2. \end{aligned}$$

In general, let $\prec$ be a monomial order for the monoid $\mathcal{M}$, the lexicographic order for example. Then any centro-affine covariant C can be written as

$$C = \alpha_1 m_1 + \cdots + \alpha_{n_0} m_{n_0}$$

where $m_1 \prec m_2 \prec \cdots \prec m_{n_0}$. Such a sum is called a sum arranged in increasing order.

Theorem 3. *A centro-affine covariant* C *of* $\mathcal{A}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$ $(1 \leq r \leq k)$ *can be represented by a unique vector* v *of* $\mathbf{R}^{n_0}$ *where* n_0 *is the number of elements of* $\mathcal{M}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$.

The proof is based on the fact that a covariant C of type $(d_0, d_1, d_2, ..., d_r, \delta)$, $(1 \leq r \leq k)$ can be written as a sum arranged in increasing order as $C = \alpha_1 m_1 + \cdots + \alpha_{n_0} m_{n_0}$, where n_0 is the number of elements of $\mathcal{M}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$. The choice $v = (\alpha_1, \ldots, \alpha_{n_0})$ implies that v is unique because $m_1 \prec \cdots \prec m_{n_0}$.

We will say that v is the vector representing the covariant $C_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$. For example, I_1^2 and $I_2 \in \mathcal{M}_{(0,2,0)}$ and $\mathcal{M}_{(0,2,0)} = \left\{(a_1^1)^2, a_1^1 a_2^2, a_2^1 a_1^2, (a_2^2)^2\right\}$, and since

$$I_1^2 = (a_\alpha^\alpha)^2 = (a_1^1)^2 + 2a_1^1 a_2^2 + (a_2^2)^2,$$
$$I_2 = a_\beta^\alpha a_\alpha^\beta = (a_1^1)^2 + 2a_2^1 a_1^2 + (a_2^2)^2,$$

we see that I_1^2 is represented by $(1, 2, 0, 1)$ and I_2 by $(1, 0, 2, 1)$.

A covariant of $\mathcal{A}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$ is a tensor of rank $N = d_0 + 2d_1 + 3d_2 + d_0 + \cdots + (r-1)d_r + \delta$. We represent each monomial m in $\mathcal{M}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$ by its corresponding list of its N indices taking into account contractions and alternations. The idea is to construct all lists that correspond to our monomials without permutations (in computation, permutations and polynomial operations are not interesting because of complexity). It is known that the number of permutations of N numbers taken from $\{1, 2, ..., n\}$ is $n^N = n^{d_0+2d_1+3d_2+d_0+\cdots+(r-1)d_r+\delta}$. The lists of indices of monomials in $\mathcal{M}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$ of N numbers taken from $\{1, 2, ..., n\}$ can be regarded as rows of a matrix denoted by M.

We illustrate our ideas by means of examples. Let us first compute M for the vectorial subspace $\mathcal{A}_{(0,2,0)}$ that is generated by $\mathcal{F}_{(0,2,0)} = \left\{I_1^2, I_2\right\}$.

We have to determine the associated vectors v_1 and v_2 of I_1^2 and I_2 respectively. First, we determine the set $\mathcal{M}_{(0,2,0)}$ of the monomials of invariants of type $(0, 2, 0)$ (but avoiding polynomial product I_1^2 and polynomial sum): $I_1^2 = (a_\alpha^\alpha)^2 = a_\alpha^\alpha a_\beta^\beta$ and $I_2 = a_\beta^\alpha a_\alpha^\beta$ since $\alpha, \beta = 1, 2$ and since a monomial of type $(0, 2, 0)$ may be $(a_1^1)^2 = a_1^1 a_1^1$, $a_1^1 a_2^2$, $a_2^1 a_1^2$, or $(a_2^2)^2 = a_2^2 a_2^2$. Then each monomial can be identified as a member among a list of permutations of four numbers taken from $\{1, 2\}$. For example, $(a_1^1)^2 = a_1^1 a_1^1$ can be identified as 1111 and $a_2^1 a_1^2$ as 1221.

Let us represent each monomial m in $\mathcal{M}_{(d_0,d_1,d_2)}$ by its corresponding list of its indices. The idea is to construct all lists that correspond to our monomials without permutations. Since the number of indices is $N = d_0 + 2d_1 + 3d_2$, the number of all possibilities is $2^N = 2^{d_0+2d_1+3d_2}$.

The lists of permutations of N numbers taken from $\{1, 2\}$ can be regarded as rows of the matrix

$$A = A[i,j] = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 2 \\ 1 & 1 & 1 & \cdots & 1 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 2 & 2 \\ 1 & 1 & 1 & \cdots & 2 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 2 & 1 & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

If we identify the numbers 1 and 2 as the binary digits 0 and 1 respectively, then we can see that the N-word $111\cdots111$ can be identified as the binary N-word $000\cdots000$, the N-word $111\cdots112$ as the binary N-word $000\cdots001$, etc. Since the set of all binary N-words can be generated by starting with the N-word $000\cdots000$, and then adding the binary N-word $000\cdots001$ successively to it, we see that the matrix A can be mechanically generated easily.

For I_1^2 and I_2, $N = 4$ and $2^N = 2^4 = 16$. Thus the rows of A are

$$1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222,$$
$$2111, 2112, 2121, 2122, 2211, 2212, 2221, 2222.$$

Since $I_1^2 = a_\alpha^\alpha a_\beta^\beta$ can be identified as $\alpha\alpha\beta\beta$ and $I_2 = a_\beta^\alpha a_\alpha^\beta$ can be written as $\alpha\beta\beta\alpha$, we take only lists that satisfy $A[i,1] = A[i,2]$ and $A[i,3] = A[i,4]$, or, $A[i,1] = A[i,4]$ and $A[i,2] = A[i,3]$. This gives us, after simplification, a modified A made up of

$$l_1 = 1111,\ l_2 = 1122,\ l_3 = 1221,\ l_4 = 2112 = l_3,\ l_5 = 2211 = l_2,\ l_6 = 2222.$$

Now we have all monomials of type $(0,2,0)$, although they are not ordered. After ordering the monomials of type $(0,2,0)$ obtained above, we have

$$m_1 = l_1 = 1111,\ m_2 = l_2 = 1122,\ m_3 = l_3 = 1221,\ m_4 = l_6 = 2222,$$

i.e., we have $\mathcal{M}_{(0,2,0)}$ and $n_0 = 4$. We may now construct the matrix M with columns v_i. Take I_1^2 for example, we compute its associated vector v_1.

Enter $v_1 = [0,0,0,0]$ and A.

For $\alpha = 1, 2$ do

for $\beta = 1, 2$ do

for $i = 1, 4$ do if $[\alpha, \alpha, \beta, \beta] = [A[i,1], A[i,2], A[i,3], A[i,4]]$

then $v_1[i+1] := v_1[i] + 1$

return v_1.

By means of the above algorithm, we may see that the vectors associated with I_1^2 and I_2 are respectively

$$v_1 = (1, 2, 0, 1),$$

and

$$v_2 = (1, 0, 2, 1),$$

which can also be seen from $I_1^2 = m_1 + 2m_2 + m_4$ and $I_2 = m_1 + 2m_3 + m_4$. In this example, the tensorial alternation is not involved. When alternation is also involved, we may construct the matrix A in the same manner without taking into account $\varepsilon_{pq} \cdots \varepsilon_{rs}$ ($\varepsilon^{pq} \cdots, \varepsilon^{rs}$) but we delete all lists corresponding to $p = q$ or $r = s$ since $\varepsilon_{pp} = 0$ (respectively $\varepsilon^{rr} = 0$), and taking into account that $\varepsilon_{pq} \cdots \varepsilon_{rs} = 1$ or -1 (respectively $\varepsilon^{pq} \cdots \varepsilon^{rs} = 1$ or -1) when we compute the matrix M. For example, the vectorial subspace $\mathcal{A}_{(0,1,2)}$ is *generated* by $\mathcal{F}_{(0,1,2)} = \{I_3, I_4, I_5\}$. Let us show how we may determine the vector w_1 associated with $I_3 = a_p^\alpha a_{\alpha q}^\beta a_{\beta\gamma}^\gamma \varepsilon^{pq}$. Here $N = d_0 + 2d_1 + 3d_2 = 2 + 6 = 8$. Construct A in the same manner for $\mathcal{F}_{(0,2,0)}$ without taking into account ε^{pq} but delete all lists corresponding to $p = q$ since $\varepsilon^{11} = \varepsilon^{22} = 0$.

We construct A in the same manner as that in the case of $\mathcal{F}_{(0,2,0)}$. A is the matrix of the lists of permutations of $N = d_0 + 2d_1 + 3d_2 = 2 + 6 = 8$ numbers taken from $\{1, 2\}$. We delete all lists which do not correspond to monomials of $\mathcal{M}_{(0,1,2)}$ of elements of $\mathcal{F}_{(0,1,2)} = \{I_3, I_4, I_5\}$ and delete all lists that correspond to $p = q$.

As an example, note that $I_3 = a_p^\alpha a_{\alpha q}^\beta a_{\beta\gamma}^\gamma \varepsilon^{pq}$ can be written as $\alpha p \beta \alpha q \gamma \beta \gamma$ (we do not take ε^{pq} in the list), lists of the form $\alpha 1 \beta \alpha 1 \gamma \beta \gamma$ are deleted from A because $p = q = 1$ the same thing for lists of the form $\alpha 2 \beta \alpha 2 \gamma \beta \gamma$, but the coefficient of monomials $a_1^\alpha a_{\alpha 2}^\beta a_{\beta\gamma}^\gamma$ (or $a_2^\alpha a_{\alpha 1}^\beta a_{\beta\gamma}^\gamma$) that correspond to the lists of the form $\alpha 1 \beta \alpha 2 \gamma \beta \gamma$ (or $\alpha 2 \beta \alpha 1 \gamma \beta \gamma$) are 1 (respectively -1).

We determine the vectors w_1, w_2 and w_3 in a manner similar to the case of I_1^2 and I_2 but we take into account that coefficients are 1 if $p = 1$ and -1 if $p = 2$.

Enter $w_1 = [0, 0, 0, 0]$ and A.
For $\alpha = 1, 2$ do
for $\beta = 1, 2$ do
or $\gamma = 1, 2$ do
for $p = 1, 2$ do
or $q = 1, 2$ do
for $i = 1, 8$ do if $[\alpha, p, \beta, \alpha, q, \gamma, \beta, \gamma] = [A[i, 1], A[i, 2], A[i, 3], A[i, 4], A[i, 5], A[i, 6], A[i, 7], A[i, 8]]$
then
if $p = 1$ then $(w_1[i + 1] := w_1[i] + 1)$ else $(w_1[i + 1] := w_1[i] - 1)$
return w_1.

Let us consider the vectorial subspace $\mathcal{A}(0,2,0)$. It's generated by centro-affine invariants of the form $I_1^{\lambda_1}\cdots I_{36}^{\lambda_{36}}$ where $\lambda_1,\ldots,\lambda_{36}\in\mathbf{N}$ satisfying (3) for $(d_0,d_1,d_2)=(0,2,0)$ which is possible for $\lambda_1=2$ and $\lambda_2=0$, or, $\lambda_1=0$ and $\lambda_2=1$. $\mathcal{A}(0,2,0)$ is then generated by I_1^2 and I_2 whose associated vectors $v_1=(1,2,0,1)$ and $v_2=(1,0,2,1)$ are linearly independent.

Let's summarize step by step how a given contravariant for $\mathcal{A}(0,2,0)$ can be developed

(1) We search all invariants $I_1^{\lambda_1}\cdots I_{36}^{\lambda_{36}}$, where $\lambda_1,\ldots,\lambda_{36}\in\mathbf{N}$, of type $(0,2,0)$ satisfying (3) for $(d_0,d_1,d_2)=(0,2,0)$. We get I_1^2 and I_2.

(2) We determine $\mathcal{M}_{(0,2,0)}$ the set of monomials of centro-affine invariants of type (d_0,d_1,d_2) while developing monomials of the form $a_\alpha^\alpha a_\beta^\beta$ and $a_\beta^\alpha a_\alpha^\beta$. We get $\mathcal{M}_{(0,2,0)}=\left\{(a_1^1)^2, a_1^1a_2^2, a_2^1a_1^2, (a_2^2)^2\right\}$ and $n_0=4$.

(3) Order $\mathcal{M}_{(0,2,0)}$ by a monomial order. We get $\mathcal{M}_{(0,2,0)}=\{m_1,m_2,m_3,m_4\}$.

(4) Decompose I_1^2 and I_2 in $\mathcal{M}_{(0,2,0)}=\{m_1,m_2,m_3,m_4\}$. We get $I_1^2=m_1+2m_2+m_4$ and $I_2=m_1+2m_3+m_4$.

(5) Calculate v_1 and v_2 the vectors associated respectively to I_1^2 and I_2. We get $v_1,v_2\in\mathbf{R}^{n_0}=\mathbf{R}^4$ such that $v_1=$ coefficients of I_1^2 and $v_2=$ coefficients of I_2. We get $v_1=(1,2,0,1)$ and $v_2=(1,0,2,1)$.

(6) Return M.

Now, let us give an algorithm to develop a given centro-affine contravariant of type $(d_0,d_1,d_2,...,d_r,\delta)$ of the differential system (1).

Algorithm 2. Enter $(d_0,d_1,d_2,...,d_r,\delta)$ and $TC_1,\ldots,TC_s$.

Step 1. Compute the finite *generating family* $\mathcal{F}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}=\{f_1,f_2,\ldots,f_s\}$ of elements of the form $C_1^{\lambda_1}\cdots C_s^{\lambda_s}$, where $\lambda_1,...,\lambda_s\in\mathbf{N}$, such that $(d_0,d_1,d_2,...,d_r,\delta)=\lambda_1TC_1+\cdots+\lambda_sTC_s$.

Step 2. Determine $\mathcal{M}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$ the set of monomials of $f_1,f_2,\ldots,f_s$, of type $(d_0,d_1,d_2,...,d_r,\delta)$ (they are of the form $(a^j)^{p_0}(a^j_{\alpha_1})^{p_1}(a^j_{\alpha_1\alpha_2})^{p_2}\cdots(a^j_{\alpha_1\cdots\alpha_r})^{p_r}(x)^\alpha$) and its size n_0.

Step 3. Order $\mathcal{M}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$ by monomial order.

Step 4. For $i=1,...,s$, decompose f_i in $\mathcal{M}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}=\{m_1,...,m_{n_0}\}$, $f_i=\beta_1^im_1+\cdots+\beta_{n_0}^im_{n_0}$.

Step 5. For $i = 1, ..., s$, calculate v_i in $\mathbf{R}^{n_0}$, v_i = coefficients f_i in $\mathcal{M}_{(d_0,d_1,d_2,...,d_r,\delta)}$, $v_i = (\beta_1^i, ..., \beta_{n_0}^i)$.

Step 6. Construct the matrix M formed by the vectors v_i, $i = 1, ..., s$, that is, $M = (v_1, ..., v_s)$.

Step 7. Return M.

We will say that M obtained from Algorithm 2 is the matrix associated with the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$.

Now we are able to develop a given centro-affine covariant C of type $(d_0, d_1, d_2, ..., d_r, \delta)$ while constructing its associated vector v and therefore it suffices to solve in $\mathbf{R}^s$ the equation $M\lambda = v$ after reducing M with the help of an appropriate software in linear algebra. Let us return to the vectorial subspace $\mathcal{A}_{(0,2,0)}$ for illustration. It is generated by I_1^2 and I_2, and its associated matrix formed from its corresponding vectors $v_1 = (1, 2, 0, 1)$ and $v_2 = (1, 0, 2, 1)$ is of rank 2. Thus I_1^2 and I_2 form an algebraic basis for the vectorial subspace $\mathcal{A}_{(0,2,0)}$. Recall that $\det(a_j^i)_{j=1,2}$ is a centro-affine invariant. Using Aronhold symbolic calculation we find $\det(a_j^i)_{j=1,2} = \frac{1}{2}(I_1^2 - I_2)$. On the other hand, we can apply Algorithm 2 to compute its associated vector v ($\det(a_j^i)_{j=1,2} = a_1^1 a_2^2 - a_2^1 a_1^2, v = (0, 1, -1, 0)$) and to decompose it in v_1 and v_2 to obtain $v = \frac{1}{2}(v_1 - v_2)$. That is, $\det(a_j^i)_{j=1,2} = \frac{1}{2}(I_1^2 - I_2)$. This method does not need the Aronhold symbolic calculation and can be used to describe the algebra of centro-affine covariants of (1) and to decompose any given invariant of these systems.

We can construct centro-affine covariants of these differential systems by using the fundamental theorem of Gurevich and apply the Algorithm 2 to determine the matrix associated with each corresponding vectorial subspace and to reduce this matrix or to determine its *syzygies* for obtaining its algebraic basis.

5. Syzygies

In this section we will apply our algorithms to compute *syzygy relations* between centro-affine covariants of polynomial differential systems of the form (1).

For the subset $\{I_1, ..., I_{16}, K_1, ..., K_{20}\}$, a minimal system of generators for the ideal of syzygies relating its elements is known (see e.g., [11, Theorem 17.1], [4]). For instance, one such generator is $I_9K_5 - K_6K_{10} + K_1^2K_9$, and the corresponding syzygy relation is

$$I_9K_5 - K_6K_{10} + K_1^2K_9 = 0.$$

The question then arises as to how any *syzygy* between elements of the expanded set $B = \{I_1, ..., I_{36}, K_1, ..., K_{33}\}$ can be determined. In the following, starting from the basis $E = \{I_1, ..., I_{36}\}$ of the centro-affine invariants of system (1), we will develop an algorithmic method that permits us to calculate a *syzygy* as a linear combination of centro-affine covariants.

A *syzygy* S for K (or for (1)) is said to be homogeneous of type $(d_0, d_1, d_2, ..., d_r, \delta)$ if it is a linear combination of homogeneous centro-affine covariants in the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,...,d_r\delta)}$.

For example: $S = I_9^2 I_8 - 2I_{15}^2 - I_7^3 + I_7^2 I_9 - I_7 I_9^2$ is a homogenous *syzygy* of type $(0, 0, 12)$ since it is a linear combination of homogenous invariants $I_9^2 I_8, I_{15}^2, I_7^3, I_7^2 I_9$ and $I_7 I_9^2$, all of type $(0, 0, 12)$.

Lemma 1. *If a syzygy S for (1) can be written as a finite real linear combination of homogenous invariants $S_1, ..., S_{s_0}$ for (1) such that their types are mutually different, then $S_1, ..., S_{s_0}$ are necessarily syzygies.*

Proof. Suppose $S = \lambda_1 S_1 + \cdots + \lambda_s S_{s_0}$ such that their types are mutually different. Since S is syzygy, we have $0 = \lambda_1 S_1 + \cdots + \lambda_s S_{s_0}$. If there exist $i_0 \in \{1, 2, ..., s\}$ such that $S_{i_0} \neq 0$, then $\lambda_{i_0} S_{i_0} = -\lambda_{j_1} S_{j_1} - \cdots - \lambda_{j_r} S_{j_r}$ which is impossible.

This lemma implies that each syzygy can be decomposed in the direct sum

$$\bigoplus{}_{d_0,d_1,d_2,...,d_r,\delta\in\mathbf{N}} \mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)},$$

and it is sufficient to determine *syzygy relation* of a given type.

A homogenous *syzygy* S for (2) can be written as a real linear combination of homogenous *syzygies* S_i of type $(d_0^i, d_1^i, d_2^i, \delta^i)$ which is a real linear combination of homogenous centro-affine invariants of the form $I_1^{\lambda_1} \cdots I_{36}^{\lambda_{36}}$ of the same type $(d_0^i, d_1^i, d_2^i, \delta^i)$.

Let us consider the vectorial subspace $\mathcal{A}(d_0, d_1, d_2)$. A *syzygy* S for (2) of type (d_0, d_1, d_2) can be written as a finite sum $S_i = \sum_{j=1}^s c_j f_j$, where $f_1, ..., f_s \in \mathcal{F}(d_0, d_1, d_2)$.

Lemma 2. *If $h_1, h_2, \ldots, h_{s_0} \in \mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$ and $v_1, v_2, \ldots, v_{s_0}$ their respective associated vectors then there exists a vanishing linear combination of the vectors $v_1, ..., v_{s_0} \in \mathbf{R}^{n_0}$ such that $\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{s_0} v_{s_0} = 0$, if, and only if, $S = \beta_1 h_1 + \beta_2 h_2 + \ldots + \beta_{s_0} h_{s_0}$ is a syzygy of type $(d_0, d_1, d_2, ..., d_r, \delta)$.*

Proof. Let $h_i = \alpha_1^i m_1 + \ldots + \alpha_{n_0}^i m_{n_0}$, $v_i = (\alpha_1^i, \ldots, \alpha_{n_0}^i)$ where $\alpha_1^i, ..., \alpha_{n_0}^i \in \mathbf{R}$, $i = 1, ..., s_0$. Then

$$\begin{aligned} \sum_{i=1}^{s_0} \beta_i h_i &= \sum_{i=1}^{s_0} \beta_i (\alpha_1^i m_1 + \ldots + \alpha_{n_0}^i m_{n_0}) \\ &= \left(\sum_{i=1}^{s_0} \beta_i \alpha_1^i \right) m_1 + \ldots + \left(\sum_{i=1}^{s_0} \beta_i \alpha_{n_0}^i \right) m_{n_0} = 0 \end{aligned}$$

if, and only if,

$$\sum_{i=1}^{s_0} \beta_i \alpha_1^i = \cdots = \sum_{i=1}^{s_0} \beta_i \alpha_{n_0}^i = 0$$

if, and only if

$$\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{s_0} v_{s_0} = 0.$$

Let *S* be a *syzygy* for (1) of type $(d_0, d_1, d_2, ..., d_r, \delta)$. If M is the matrix associated with the corresponding vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_s)$ in $\Bbbk^s$ such $M\beta = 0$, then $\beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_s f_s = 0$ is a corresponding *relation syzygy.*

If we reduce the matrix M, the equation $M\beta = 0$ gives as a basis of *syzygies* of the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2,...,d_r,\delta)}$. Therefore, this algorithm can be used to determine *syzygies* relating elements of any subset $\Omega = \{g_1, ..., g_m\}$ of centro-affine covariants of (1). We write Ω as $\Omega = \cup\Omega_{T_i}$ of subsets of elements of Ω that has the same type T_i. Since a *syzygy* can be written as a linear combination of homogenous *syzygies,* it suffices to apply our algorithm for each subset Ω_{T_i}.

An application of *relation syzygies* is to deduce an algebraic basis from a given *generating family* of invariants. Let us give some examples.

Let $\mathcal{S}_{(d_0,d_1,d_2)}$ be the algebraic basis of *syzygies* of the vectorial subspace $\mathcal{A}_{(d_0,d_1,d_2)}$. The *generating family* $\mathcal{F}_{(0,2,0)} = \{I_1^2, I_2\}$ is an algebraic basis for $\mathcal{A}_{(0,2,0)}$. By the methods described in the previous section, we may compute the generating family $\mathcal{F}_{(1,2,5)}$:

$$\begin{aligned}\mathcal{F}_{(1,2,5)} = \{ & I_1I_{13}I_{17}, I_1I_{12}I_{17}, I_1I_{11}I_{17}, I_1I_9I_{20}, I_1I_9I_{19}, I_1I_8I_{20}, I_1I_8I_{19}, \\ & I_1I_7I_{20}, I_1I_7I_{19}, I_1I_5I_{26}, I_1I_5I_{25}, I_1I_4I_{26}, I_1I_4I_{25}, I_1I_3I_{26}, \\ & I_1I_3I_{25}, I_1^2I_{36}, I_1^2I_9I_{17}, I_1^2I_8I_{17}, I_1^2I_7I_{17}, I_2I_{36}, I_2I_9I_{17}, I_2I_8I_{17}, I_2I_7I_{17}\},\end{aligned}$$

and the algebraic basis $\mathcal{S}_{(1,2,5)}$:

$$\begin{aligned}\mathcal{S}_{(1,2,5)} = \{ & I_1I_9I_{19} - I_1I_{13}I_{17} + I_1I_{12}I_{17} - I_1I_3I_{26} - I_1I_9I_{20}; \\ & 2I_1I_9I_{19} + I_1^2I_7I_{17} - 2I_1I_{13}I_{17} + 2I_1I_4I_{26} - I_1^2I_9I_{17}\}\end{aligned}$$

$\mathcal{A}_{(1,2,5)}$ is generated by $\mathcal{F}_{(1,2,5)}$ but its elements $I_1I_9I_{19}$, $I_1I_{13}I_{17}$, $I_1I_{12}I_{17}$, $I_1I_3I_{26}$, $I_1I_9I_{20}$ are linearly dependent because $I_1I_9I_{19} - I_1I_{13}I_{17} + I_1I_{12}I_{17} - I_1I_3I_{26} - I_1I_9I_{20} = 0$. Then one of them can be expressed by the others, for example, $I_1I_9I_{20} = I_1I_9I_{19} - I_1I_{13}I_{17} + I_1I_{12}I_{17} - I_1I_3I_{26}$, so we delete it from the *generating family* $\mathcal{F}_{(1,2,5)}$. The same holds for $I_1I_9I_{19}$, $I_1^2I_7I_{17}$, $2I_1I_{13}I_{17}$, $2I_1I_4I_{26}$, $I_1^2I_9I_{17}$ but we must choose a different one from the element we deleted before. For example, if we choose to delete $I_1I_9I_{19}$ (or $I_1I_{13}I_{17}$) using one *relation syzygy,* we must delete another element using necessarily a second *relation syzygy* and this element must be different from the first element deleted.

By means of our methods described above, a list of some other *generating families* can be given:

$\mathcal{F}_{(0,1,0)}: I_1$
$\mathcal{F}_{(0,1,2)}: I_5, I_4, I_3$
$\mathcal{F}_{(0,2,0)}: I_2, I_1^2$
$\mathcal{F}_{(0,2,2)}: I_6, I_1I_5, I_1I_4, I_1I_3$
$\mathcal{F}_{(0,3,2)}: I_{10}, I_2I_5, I_2I_4, I_2I_3, I_1I_6, I_1^2I_5, I_1^2I_4, I_1^2I_3$
$\mathcal{F}_{(1,0,1)}: I_{17}$
$\mathcal{F}_{(1,0,3)}: I_{26}, I_{25}$
$\mathcal{F}_{(1,1,1)}: I_{20}, I_{19}, I_1I_{17}$
$\mathcal{F}_{(0,3,0)}: I_1I_2, I_1^3$
$\mathcal{F}_{(1,1,3)}: I_{32}, I_{31}, I_{30}, I_5I_{17}, I_4I_{17}, I_3I_{17}, I_1I_{26}, I_1I_{25}$
$\mathcal{F}_{(1,2,1)}: I_{24}, I_2I_{17}, I_1I_{20}, I_1I_{19}, I_1^2I_{17}$
$\mathcal{F}_{(1,2,3)}: I_{35}, I_6I_{17}, I_5I_{20}, I_5I_{19}, I_4I_{20}, I_4I_{19}, I_3I_{20}, I_3I_{19}, I_2I_{26}, I_2I_{25},$
$I_1I_{32}, I_1I_{31}, I_1I_{30}, I_1I_5I_{17}, I_1I_4I_{17}, I_1I_3I_{17}, I_1^2I_{26}, I_1^2I_{25}$
$\mathcal{F}_{(1,3,1)}: I_2I_{20}, I_2I_{19}, I_1I_{24}, I_1I_2I_{17}, I_1^2I_{20}, I_1^2I_{19}, I_1^3I_{17}$
$\mathcal{F}_{(1,3,3)}: I_{10}I_{17}, I_6I_{20}, I_6I_{19}, I_5I_{24}, I_4I_{24}, I_3I_{24}, I_2I_{32}, I_2I_{31}, I_2I_{30}, I_2I_5I_{17}, I_2I_4I_{17}$
$I_2I_3I_{17}, I_1I_{35}, I_1I_6I_{17}, I_1I_5I_{20}, I_1I_5I_{19}, I_1I_4I_{20}, I_1I_4I_{19}, I_1I_3I_{20}, I_1I_3I_{19},$
$I_1I_2I_{26}, I_1I_2I_{25}, I_1^2I_{32}, I_1^2I_{31}, I_1^2I_{30}, I_1^2I_5I_{17}, I_1^2I_4I_{17}, I_1^2I_3I_{17}, I_1^3I_{26}, I_1^3I_{25}$

$\mathcal{F}_{(2,0,2)}: I_{23}, I_{22}, I_{17}^2$
$\mathcal{F}_{(2,1,0)}: I_{18}$
$\mathcal{F}_{(2,3,0)}: I_2I_{18}, I_1^2I_{18}$
$\mathcal{F}_{(2,1,2)}: I_{29}, I_{28}, I_{17}I_{20}, I_{17}I_{19}, I_1I_{23}, I_1I_{22}, I_1I_{17}^2$
$\mathcal{F}_{(2,2,0)}: I_1I_{18}$
$\mathcal{F}_{(2,2,2)}: I_{20}^2, I_{19}I_{20}, I_{19}^2, I_{17}I_{24}, I_5I_{18}, I_4I_{18}, I_3I_{18}, I_2I_{23}, I_2I_{22},$
$I_2I_{17}^2, I_1I_{29}, I_1I_{28}, I_1I_{17}I_{20}, I_1I_{17}I_{19}, I_1^2I_{23}, I_1^2I_{22}, I_1^2I_{17}^2$
$\mathcal{F}_{(2,3,2)}: I_{20}I_{24}, I_{19}I_{24}, I_6I_{18}, I_2I_{29}, I_2I_{28}, I_2I_{17}I_{20}, I_2I_{17}I_{19}, I_1I_{20}^2, I_1I_{19}I_{20},$
$I_1I_{19}^2, I_1I_{17}I_{24}, I_1I_5I_{18}, I_1I_4, I_{18}, I_1I_3I_{18}, I_1I_2I_{23}, I_1I_2I_{22}, I_1I_2I_{17}^2,$
$I_1^2I_{29}, I_1^2I_{28}, I_1^2I_{17}I_{20}, I_1^2I_{17}I_{19}, I_1^3I_{23}, I_1^3I_{22}, I_1^3I_{17}^2$

$\mathcal{F}_{(3,1,0)}: I_{21}$
$\mathcal{F}_{(3,0,3)}: I_{33}, I_{17}I_{23}, I_{17}I_{22}, I_{17}^3$
$\mathcal{F}_{(3,1,1)}: I_{27}, I_{17}I_{18}, I_1I_{21}$
$\mathcal{F}_{(3,1,3)}: I_{20}I_{23}, I_{20}I_{22}, I_{19}I_{23}, I_{19}I_{22}, I_{18}I_{26}, I_{18}I_{25}, I_{17}I_{29}, I_{17}I_{28}, I_{17}^2I_{20},$
$I_{17}^2I_{19}, I_5I_{21}, I_4I_{21}, I_3I_{21}, I_1I_{33}, I_1I_{17}I_{23}, I_1I_{17}I_{22}, I_1I_{17}^3$
$\mathcal{F}_{(3,2,1)}: I_{18}I_{20}, I_{18}I_{19}, I_2I_{21}, I_1I_{27}, I_1I_{17}I_{18}, I_1^2I_{21}$
$\mathcal{F}_{(3,2,3)}: I_{23}I_{24}, I_{22}I_{24}, I_{20}I_{29}, I_{20}I_{28}, I_{19}I_{29}, I_{19}I_{28}, I_{18}I_{32}, I_{18}I_{31}, I_{18}I_{30}, I_{17}I_{20}^2,$
$I_{17}I_{19}I_{20}, I_{17}I_{19}^2, I_{17}^2I_{24}, I_6I_{21}, I_5I_{27}, I_5I_{17}I_{18}, I_4I_{24}, I_4I_{17}I_{18}, I_3I_{27}, I_3I_{17}I_{18},$
$I_2I_{33}, I_2I_{17}I_{23}, I_2I_{17}I_{22}, I_2I_{17}^3, I_1I_{20}I_{23}, I_1I_{20}I_{22}, I_1I_{19}I_{23}, I_1I_{19}I_{22},$
$I_1I_{18}I_{26}, I_1I_{18}I_{25}, I_1I_{17}I_{29}, I_1I_{17}I_{28}, I_1I_{17}^2I_{20}, I_1I_{17}^2I_{19}, I_1I_5I_{21}, I_1I_4I_{21},$
$I_1I_3I_{21}, I_1^2I_{33}, I_1^2I_{17}I_{23}, I_1^2I_{17}I_{22}, I_1^2I_{17}^3$
$\mathcal{F}_{(3,3,1)}: I_{18}I_{24}, I_2I_{27}, I_2I_{17}I_{18}, I_1I_{18}I_{20}, I_1I_{18}I_{19}, I_1I_2I_{21}, I_1^2I_{27}, I_1^2I_{17}I_{18}, I_1^2I_{27}, I_1^3I_{21}$
$\mathcal{F}_{(3,3,3)}: I_{24}I_{29}, I_{24}I_{28}, I_{20}^3, I_{19}I_{20}^2, I_{19}^2I_{20}, I_{19}^3, I_{18}I_{35}, I_{17}I_{20}I_{24}, I_{17}I_{19}I_{24}, I_{10}I_{21}, I_6I_{27},$
$I_6I_{17}I_{18}, I_5I_{18}I_{20}, I_5I_{18}I_{19}, I_4I_{18}I_{20}, I_4I_{18}I_{19}, I_3I_{18}I_{20}, I_3I_{18}I_{19}, I_2I_{20}I_{23}, I_2I_{20}I_{22},$
$I_2I_{19}I_{23}, I_2I_{19}I_{22}, I_2I_{18}I_{26}, I_2I_{18}I_{25}, I_2I_{17}I_{29}, I_2I_{17}I_{28}, I_2I_{17}^2I_{20}, I_1I_{17}^2I_{19}, I_2I_5I_{21},$

$I_2I_4I_{21}, I_2I_3I_{21}, I_1I_{23}I_{24}, I_1I_{22}I_{24}, I_1I_{20}I_{29}, I_1I_{20}I_{28}, I_1I_{19}I_{29}, I_1I_{19}I_{28}, I_1I_{18}I_{32},$
$I_1I_{18}I_{32}, I_1I_{18}I_{31}, I_1I_{18}I_{30}, I_1I_{17}I_{20}^2, I_1I_{17}I_{19}I_{20}, I_1I_{17}I_{19}^2, I_1I_{17}^2I_{24}, I_1I_6I_{21},$
$I_1I_5I_{27}, I_1I_5I_{17}I_{18}, I_1I_4I_{27}, I_1I_4I_{17}I_{18}, I_1I_3I_{27}, I_1I_3I_{17}I_{18}, I_1I_2I_{33}, I_1I_2I_{17}I_{23},$
$I_1I_2I_{17}I_{22}, I_1I_2I_{17}^3, I_1^2I_{20}I_{23}, I_1^2I_{20}I_{22}, I_1^2I_{19}I_{23}, I_1^2I_{19}I_{22}, I_1^2I_{18}I_{26}, I_1^2I_{18}I_{25}, I_1^2I_{17}I_{29},$
$I_1^2I_{17}I_{28}, I_1^2I_{17}^2I_{20}, I_1^2I_{17}^2I_{19}, I_1^2I_5I_{21}, I_1^2I_4I_{21}, I_1^2I_3I_{21}, I_1^3I_{33}, I_1^3I_{17}I_{23}, I_1^3I_{17}I_{22}, , I_1^3I_{17}^3$

We remark that the algorithms described above can be generalized for algebraic covariants in relation to a linear group of transformations of polynomial differential systems with coefficients in a field $\mathbb{k}$ of characteristic zero in n variables of degree k.

A consequence of the work of Vulpe on semi-invariants is that the polynomial relations between the covariants are equivalent to the same relations between their leading terms. One can improve the efficiency of these algorithms by replacing the elements of the minimal system of the centro-affine covariants of the differential systems by the set of their leading terms.

REFERENCES

1. A. A. Andronov, *Theory of Oscillations*, Princeton University Press, Princeton, New Jersy, 1949.
2. D. Boularas and D. Dali, Sur les bases de concomitants centro-affines des systèmes différentiels, plans, *Cahiers Math.*, **2** (1987), 25-30.
3. D. Dali, Gröbner bases of algebraic invariants in polynomial differential systems, *Le Matematiche*, to appear.
4. N. I. Danilyuk and C. S. Sibirskii, Syzygies between centro-affines invariants of a quadratic differential system, *Differentsialnyje Uravnenija*, **17(2)** (1981), 210-219.
5. J. Dieudonné and J. Carrell, Invariant theory, old and new, *Adv. Math.*, **4** (1970), 1-80.
6. G. B. Gurevich, *Foundation of the Theory of Algebraic Invariants*, Moscow & Leningrad, GITTL, 1948.
7. D. Hilbert, *Invariant Theory*, Cambridge University Press, 1993.
8. A. M. Liapunov, *Problème général de la stabilité du mouvement*, J. Math. Studies, Vol. 17, Princeton University Press, 1947.
9. M. Voldman, Iu. T. Calin and N. I. Vulpe, Affine invariant condition for the topological distinction of quadratic systems with critical point of 4th multiplicity, *Publicacions Matemàtiques*, **40** (1996), 431-441.
10. H. Poincaré, Sur les courbes définies par les équations différentielles, *J. Math. Pures Appl.* **1** (1985), 167-244.
11. C. S. Sibirskii, *Introduction to the Algebraic Theory of Invariants of Differential Equations*, Nonlinear Science, Theory and Applications, Manchester University Press, 1988.

12. J. L. Mott, A. Kandel and T. P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Prentice Hall, 1986.

13. N. I. Vulpe, Polynomial bases of comitants of differential systems and their applications in qualitative theory (Russian, English and French summary), in: *Shtintsa*, Kishinev, 1986.

Dahira Dali
Faculty of Mathematics,
USTHB, P. O. Box 32,
El Alia Bab Ezzouar Algiers 16111,
Algeria
E-mail: dddahira@gmail.com

Sui Sun Cheng
Department of Mathematics,
Tsing Hua University,
Hsinchu 30043,
Taiwan
E-mail: sscheng@math.nthu.edu.tw