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STRONG CONVERGENCE THEOREM FOR A GENERALIZED EQUILIBRIUM PROBLEM AND A PSEUDOCONTRACTIVE MAPPING IN A HILBERT SPACE

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Abstract. Very recently, Takahashi and Takahashi [S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Analysis 69 (2008) 1025-1033] suggested and analyzed an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In this paper, we introduce an implicit viscosity approximation method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a pseudocontractive mapping in a Hilbert space. Then, we prove a strong convergence theorem which is the improvements and development of Takahashi and Takahashi's (2008) corresponding result. Using this theorem, we prove three new strong convergence theorems in fixed point problems, variational inequalities and equilibrium problems.

1. INTRODUCTION

Let X be a real Banach space with the dual X^*. Let C be a nonempty closed convex subset of X. Recall that a self-mapping $f : C \to C$ is said to be k-Lipschitz on C if $k \in \mathbb{R}_+$ and

$$\|f(x) - f(y)\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

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If f is k-Lipschitz with $k < 1$, then f is called a k-contraction mapping or a contraction mapping with coefficient k. Note that each contraction $f : C \to C$ has a unique fixed point in C. A self-mapping $f : C \to C$ is said to be nonexpansive if it is Lipschitz with $k = 1$. We use $F(f)$ to denote the set of fixed points of f; i.e., $F(f) = \{x \in C : x = f(x)\}$. Also, recall that a self-mapping $T : C \to C$ is called pseudocontractive if for each $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2,$$

where $J : X \to 2^{X^*}$ denotes the normalized duality mapping defined on X, by

$$J(x) = \{\varphi \in X^* : \langle x, \varphi \rangle = \|x\|^2 = \|\varphi\|^2\}, \quad \forall x \in X.$$

It is well known that the class of pseudocontractive mappings is an important and significant generalization of nonexpansive mappings.

Within the past 30 years or so, a great deal of effort has gone into the existence of fixed points of pseudocontractive mappings (including nonexpansive mappings) and iterative construction of fixed points of pseudocontractive mappings (including nonexpansive mappings); see, e.g., [3-4, 6-26]. In particular, in 2000, Moudafi [6] established the following strong convergence theorem.

Theorem 1.1. (see [6]). *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive self-mapping on C such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \frac{1}{1+\varepsilon_n} Tx_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n)$$

for all $n \geq 1$, where $\{\varepsilon_n\} \subset (0,1)$ satisfies

$$\lim_{n\to\infty} \varepsilon_n = 0, \ \sum_{n=1}^{\infty} \varepsilon_n = \infty \text{ and } \lim_{n\to\infty} |\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}| = 0.$$

Then, $\{x_n\}$ converges strongly to $z \in F(T)$, where $z = P_{F(T)} f(z)$ and $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Such a method for approximation of fixed points is called the viscosity approximation method. Subsequently, Xu [21] further considered the viscosity approximation methods for nonexpansive self-mappings on a nonempty closed convex subset C of a uniformly smooth Banach space X. Moreover, the above strong convergence theorem for the viscosity approximation method is extended to the setting of uniformly smooth Banach spaces.

Theorem 1.2. (cf. [21, Theorem 4.2]). *Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f : C \to C$ a fixed contraction. For arbitrary initial value $x_0 \in C$ the sequence $\{x_n\}$ is defined by*

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n f(x_n), \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ satisfies

$(H1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(H2)$ $\sum_{n=0}^{\infty} \alpha_n = \infty$;

$(H3)$ *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then $\{x_n\}$ converges strongly to a fixed point $\hat{p} \in F(T)$, which solves the variational inequality

$$\langle (I-f)(\hat{p}), J(\hat{p}-p)\rangle \leq 0, \quad \forall p \in F(T).$$

Very recently, Rafiq [20] introduced a Mann type implicit iterative method, and proved the following theorem.

Theorem 1.3. *Let K be a compact convex subset of a real Hilbert space H, $T : K \to K$ a hemicontractive mapping. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$, the sequence $\{x_n\}$ is defined by*

$$x_0 \in K, \qquad x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Furthermore, Ceng, Petruşel and Yao [22, Theorem 3.2] introduced the following iterative method for finding a fixed point of a pseudocontractive mapping in a Banach space. Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and C be a nonempty closed convex subset of X. Let $f : C \to C$ be a fixed contraction mapping, $S : C \to C$ be a nonexpansive mapping and $T : C \to C$ be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$. For $x_0 \in C$, let x_n be given by the following iterative scheme:

$$\begin{cases} y_n = \beta_n f(x_{n-1}) + \gamma_n S x_{n-1} + (1-\beta_n-\gamma_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ with $\beta_n + \gamma_n \leq 1$. They proved that under appropriate restrictions on $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$, the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$, which solves the following variational inequality:

$$\langle (I - f)(p), J(p - u)\rangle \leq 0, \quad \forall u \in F(T).$$

On the other hand, Let C be a nonempty closed convex subset of H, let $F : C \times C \to R$ be a bifunction and let $A : C \to H$ be a nonlinear mapping. Very recently, Takahashi and Takahashi [17] introduced and considered the following equilibrium problem:

$$\text{(1.1)} \qquad \text{Find } z \in C \text{ such that } F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C.$$

The set of such $z \in C$ is denoted by EP, i.e.,

$$EP = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0,\ \forall y \in C\}.$$

If $A \equiv 0$, EP is denoted by $EP(F)$. If $F \equiv 0$, EP is also denoted by $VI(C, A)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inrqualities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, e.g., [5, 1].

In 2007, motivated by Combettes and Hirsoaga [2], Moudafi [6], and Tada and Takahashi [8], Takahashi and Takahashi [9] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of (1.1) with $A \equiv 0$ and the set of fixed points of a nonexpansive mapping in a Hilbert space, and proved the following strong convergence theorem which is connected with the results in [2, 11].

Theorem 1.4. (cf. [9, Theorem 3.2]). *Let C be a nonempty closed convex subset of H. Let $F : C \times C \to R$ be a bifunction satisfying the assumptions $(A1), (A2), (A3)$ and $(A4)$:*

$(A1)$ $F(x, x) = 0$ *for all* $x \in C$;

$(A2)$ F *is monotone, i.e.,* $F(x, y) + F(y, x) \leq 0$ *for all* $x, y \in C$;

$(A3)$ $\lim\limits_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y),\ \forall x, y, z \in C$;

$(A4)$ *for each* $x \in C,\ y \mapsto F(x, y)$ *is convex and lower semicontinuous.*
Let $T : C \to H$ be a nonexpansive mapping such that $F(T) \cap EP(F) \neq \emptyset$, let $f : H \to H$ be contraction and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \dfrac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$
$$\liminf_{n\to\infty} \lambda_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then, $\{x_n\}$ *and* $\{y_n\}$ *converges strongly to* $z \in F(T) \cap EP(F)$, *where* $z = P_{F(T)\cap EP(F)} f(z)$.

Very recently, Theorem 1.4 has been extended to develop several more general results in [24-26]. Furthermore, Takahashi and Takahashi [17] introduced an iterative method for finding a common element of the set of solutions of (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space, and then proved that the sequence generated by the method converges strongly to a common element of two sets.

Theorem 1.5. (cf. [17, Theorem 3.1]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $F : C \times C \to R$ *be a bifunction satisfying* $(A1), (A2), (A3)$ *and* $(A4)$. *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping and let* $S : C \to C$ *be a nonexpansive mapping such that* $F(S) \cap EP \neq \emptyset$. *Let* $u \in C$ *and* $x_1 \in C$ *and let* $\{z_n\} \subset C$ *and* $\{x_n\} \subset C$ *be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) z_n], \quad \forall n \in N, \end{cases}$$

where $\{\alpha_n\} \subset [0,1], \{\beta_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset [0, 2\alpha]$ *satisfy*

$$0 < c \leq \beta_n \leq d < 1, \qquad 0 < a \leq \lambda_n \leq b < 2\alpha,$$
$$\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0, \ \lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ *converges strongly to* $z = P_{F(S)\cap EP} u$.

All of above motivate us to construct an implicit viscosity approximation method for finding a common element of the set EP of solutions of (1.1) and the set $F(T)$ of fixed points of a pseudocontractive mapping T without compactness assumption (on T) in a Hilbert space H. Let C be a nonempty closed convex subset of H and let $F : C \times C \to R$ be a bifunction satisfying (A1), (A2), (A3) and (A4). Let $f : C \to C$ be a fixed contraction mapping, $S : C \to C$ be a nonexpansive mapping,

$T : C \to C$ be a continuous pseudocontractive mapping, and $A : C \to H$ be an α-inverse-strongly monotone mapping such that $EP \cap F(T) \neq \emptyset$. Inspired by the work stated as above, we introduce the following implicit viscosity approximation method for finding a common element of two sets EP and $F(T)$: $\{x_n\} \subset C$ is a sequence generated by $x_0 \in C$ and

$$\begin{cases} F(z_{n-1}, y) + \langle Ax_{n-1}, y - z_{n-1}\rangle + \dfrac{1}{\lambda_n}\langle y - z_{n-1}, z_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n f(x_{n-1}) + \gamma_n S z_{n-1} + (1 - \beta_n - \gamma_n) x_{n-1}, \\ x_n = \alpha_n y_n + (1 - \alpha_n) T x_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$ with $\beta_n + \gamma_n \leq 1, \ \forall n \geq 1$.

It is proven that under some appropriate conditions, $\{x_n\}$ converges strongly to $p^* = P_{EP \cap F(T)} f(p^*)$. Compared with Takahashi and Takahashi's Theorem 1.5, our result is novel and new in the following aspects:

(i) the class of nonexpansive mappings is extended to the more general class of pseudocontractive mappings;

(ii) the fixed element u in Theorem 1.5 is extended to the more general contraction mapping $f : C \to C$, and the nonexpansive mapping $S : C \to C$ in Theorem 1.5 is viewed as a perturbed mapping in our result;

(iii) the explicit iterative method in Theorem 1.5 is extended to develop the implicit viscosity approximation method in our result;

(iv) under the lack of the restrictions that $0 < a \leq \lambda_n \leq b < 2\alpha$ and $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$, our result (i.e., Theorem 3.1 in Section 3) guarantees that $\{x_n\}$ is strongly convergent;

(v) we present the method of proof, which is very different from Takahashi and Takahashi's one of Theorem 1.5. Indeed, we apply Xu's idea in the proof of Theorem 1.2 and the technique of resolvent operators to prove our result.

2. Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strong to x. We denote by N and R the sets of all positive integers and all real numbers, respectively. For every point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. We know that P_C is a firmly nonexpansive mapping from H onto C, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C.$$

A mapping S of C into itself is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We know that the set $F(S)$ of fixed points of S is closed and convex. Further, if C is bounded, closed and convex, then $F(S)$ is nonempty. A mapping $A : C \to H$ is called inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Such a mapping A is also called α-inverse-strongly monotone. We know that if $S : C \to C$ is nonexpansive, then $A = I - S$ is $\frac{1}{2}$-inverse-strongly monotone; see [10] for more details.

We need the following lemmas and proposition for the proof of our main result.

Lemma 2.1. (cf. [14, Lemma 1.1]). *Let K be a nonempty closed convex subset of a real Banach space X and $T : K \to K$ be a continuous pseudocontractive mapping. Then*

(i) *(see [7, Theorem 6]) $V = (2I - T)^{-1}$ is a nonexpansive self-mapping on K, and the fixed point set $F(V) = F(T)$, where I is the identity mapping of X;*

(ii) *if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $\lim_{n\to\infty} \|x_n - Vx_n\| = 0$.*

Let C be a nonempty closed convex subset of H. Throughout this paper, let us assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

$(A1)$ $F(x, x) = 0$, $\forall x \in C$;

$(A2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;

$(A3)$ $\lim_{t\to 0^+} F(tz + (1 - t)x, y) \leq F(x, y)$, $\forall x, y, z \in C$;

$(A4)$ for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We know the following Lemmas 2.2 and 2.3; see, e.g., [1, 2].

Lemma 2.2. (cf. [17, Lemma 2.2]). *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ into R satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if $T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$, then the following hold:

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H;$$

(3) $F(T_r) = EP(F)$*;*

(4) $EP(F)$ *is closed and convex.*

Lemma 2.3. (cf. [17, Lemma 2.3]). *Let* C, H, F *and* $T_r x$ *be as in Lemma 2.2. Then the following holds:*

$$\|T_s x - T_t x\|^2 \le \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle$$

for all $s, t > 0$ *and* $x \in H$.

Proposition 2.1. (cf. [21, Theorem 4.1]). *Let* X *be a uniformly smooth Banach space,* C *a nonempty closed convex subset of* X, $T : C \to C$ *a nonexpansive mapping with* $F(T) \neq \emptyset$, *and* f *a fixed contraction mapping. For each* $t \in (0, 1)$, *let* $z_t \in C$ *be the unique fixed point of the contraction* $C \ni z \mapsto tf(z) + (1-t)Tz$ *on* C, *that is,*

$$z_t = tf(z_t) + (1-t)Tz_t.$$

Then $\{z_t\}$ *converges strongly to a fixed point* $p^* \in F(T)$, *which solves the variational inequality*

$$\langle (I-f)(p^*), J(p^* - p)\rangle \le 0, \quad \forall p \in F(T).$$

Corollary 2.1. (see [23]). *Let* C *be a nonempty closed convex subset of a uniformly smooth Banach space* X *and let* $A : C \to C$ *be a nonexpansive mapping with a fixed point. For each fixed* $u \in C$ *and every* $t \in (0, 1)$, *the unique fixed point* $x_t \in C$ *of the contraction* $C \ni z \mapsto tu + (1-t)Az$ *converges strongly to a fixed point of* A *as* $t \to 0^+$.

Proof. Putting $f(x) = u$, $\forall x \in C$, from Proposition 2.1 we obtain the desired conclusion. ∎

Lemma 2.4. (cf. [21, Lemma 2.1]). *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 - b_n)a_n + b_n c_n, \quad \forall n \ge 1,$$

where $\{b_n\}_{n=1}^{\infty}$ *and* $\{c_n\}_{n=1}^{\infty}$ *are real sequences satisfying*

(i) $\{b_n\}_{n=1}^{\infty} \subset (0,1),\ \sum_{n=1}^{\infty} b_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n c_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

We are now in a position to state and prove our main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying (A1), (A2), (A3) and (A4). Let $f : C \to C$ be a fixed contraction mapping, $S : C \to C$ be a nonexpansive mapping, $T : C \to C$ be a continuous pseudocontractive mapping, and $A : C \to H$ be an α-inverse-strongly monotone mapping such that $F(T) \cap EP \neq \emptyset$. For $x_0 \in C$, let $\{x_n\} \subset C$ be defined by*

$$(3.1)\qquad \begin{cases} F(z_{n-1}, y) + \langle Ax_{n-1}, y - z_{n-1}\rangle \\ \quad + \dfrac{1}{\lambda_n}\langle y - z_{n-1}, z_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n f(x_{n-1}) + \gamma_n S z_{n-1} + (1 - \beta_n - \gamma_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1 - \alpha_n)Tx_n, \quad \forall n \in N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are sequences of nonnegative real numbers satisfying the conditions:

(i) $\{\alpha_n\} \subset (0,1],\ \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\{\beta_n\} \subset (0,1],\ \sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0,\ \beta_n + \gamma_n \leq 1,\ \forall n \in N$;

(iv) $0 < \lambda_n \leq 2\alpha,\ \forall n \in N$.

Then $\{x_n\}$ converges strongly to $p^* = P_{F(T)} f(p^*)$. Assume additionally that $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and that the existence of $\lim_{n\to\infty} \|x_n - z_n\|$ implies $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Then $p^* = P_{F(T)\cap EP} f(p^*)$.

Proof. Let $f : C \to C$ be a fixed contraction mapping with coefficient $k \in [0,1)$. Next, we divide the proof into several steps.

Step 1. We claim that $\{x_n\}, \{z_n\}, \{Ax_n\}, \{f(x_n)\}, \{Sz_n\}, \{y_n\}$, and $\{Tx_n\}$ are bounded. Indeed, note that z_{n-1} can be rewritten as $z_{n-1} = T_{\lambda_n}(x_{n-1} - \lambda_n Ax_{n-1})$ for each $n \in N$. Take $z \in F(T) \cap EP$. Since $z = T_{\lambda_n}(z - \lambda_n Az),\ A$ is α-inverse-strongly monotone and $0 < \lambda_n \leq 2\alpha$, we know that, for any $n \in N$,

$$
\begin{aligned}
&\|z_{n-1} - z\|^2 \\
&= \|T_{\lambda_n}(x_{n-1} - \lambda_n Ax_{n-1}) - T_{\lambda_n}(z - \lambda_n Az)\|^2 \\
&\leq \|(x_{n-1} - \lambda_n Ax_{n-1}) - (z - \lambda_n Az)\|^2 \\
&= \|(x_{n-1} - z) - \lambda_n(Ax_{n-1} - Az)\|^2 \\
&= \|x_{n-1}-z\|^2 - 2\lambda_n\langle x_{n-1}-z, Ax_{n-1}-Az\rangle + \lambda_n^2\|Ax_{n-1}-Az\|^2 \\
&\leq \|x_{n-1}-z\|^2 - 2\lambda_n\alpha\|Ax_{n-1}-Az\|^2 + \lambda_n^2\|Ax_{n-1}-Az\|^2 \\
&= \|x_{n-1} - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_{n-1} - Az\|^2 \\
&\leq \|x_{n-1} - z\|^2.
\end{aligned}
\tag{3.2}
$$

Also, it follows from (3.1) that

$$
\begin{aligned}
\|x_n - z\|^2 &= \langle \alpha_n y_n + (1 - \alpha_n)Tx_n - z, x_n - z\rangle \\
&= (1 - \alpha_n)\langle Tx_n - z, x_n - z\rangle + \alpha_n\langle y_n - z, x_n - z\rangle \\
&\leq (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n\|y_n - z\|\|x_n - z\|,
\end{aligned}
$$

and hence

$$\|x_n - z\|^2 \leq \|y_n - z\|\|x_n - z\|.$$

So, $\|x_n - z\| \leq \|y_n - z\|,\ \forall n \in N$. Thus we have from (3.2)

$$
\begin{aligned}
\|x_n - z\| &\leq \|y_n - z\| \\
&\leq \beta_n\|f(x_{n-1}) - z\| + \gamma_n\|Sz_{n-1} - z\| + (1 - \beta_n - \gamma_n)\|x_{n-1} - z\| \\
&\leq \beta_n(\|f(x_{n-1}) - f(z)\| + \|f(z) - z\|) + \gamma_n(\|Sz_{n-1} - Sz\| \\
&\quad + \|Sz - z\|) + (1 - \beta_n - \gamma_n)\|x_{n-1} - z\| \\
&\leq \beta_n(k\|x_{n-1} - z\| + \|f(z) - z\|) + \gamma_n(\|z_{n-1} - z\| \\
&\quad + \|Sz - z\|) + (1 - \beta_n - \gamma_n)\|x_{n-1} - z\| \\
&\leq \beta_n(k\|x_{n-1} - z\| + \|f(z) - z\|) + \gamma_n(\|x_{n-1} - z\| \\
&\quad + \|Sz - z\|) + (1 - \beta_n - \gamma_n)\|x_{n-1} - z\| \\
&= (1 - (1 - k)\beta_n)\|x_{n-1} - z\| + \beta_n\|f(z) - z\| + \gamma_n\|Sz - z\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0$, we may assume, without loss of generality, that $\gamma_n \leq \beta_n,\ \forall n \in N$. This implies that

$$\begin{aligned}
&\|x_n - z\| \\
&\leq (1-(1-k)\beta_n)\|x_{n-1} - z\| + \beta_n(\|f(z) - z\| + \|Sz - z\|) \\
&= (1-(1-k)\beta_n)\|x_{n-1} - z\| + (1-k)\beta_n \cdot \frac{1}{1-k}(\|f(z) - z\| + \|Sz - z\|) \\
&\leq \max\{\|x_{n-1} - z\|, \frac{1}{1-k}(\|f(z) - z\| + \|Sz - z\|)\}.
\end{aligned}$$

By induction, we derive

$$\|x_n - z\| \leq \max\{\|x_0 - z\|, \frac{1}{1-k}(\|f(z) - z\| + \|Sz - z\|)\}, \quad \text{for all } n \geq 0.$$

Thus, we know that $\{x_n\}$ is bounded and so is $\{z_n\}$ due to (3.2). Observe that

$$\|f(x_n)\| \leq \|f(x_n) - f(z)\| + \|f(z)\| \leq k\|x_n - z\| + \|f(z)\|,$$

and

$$\|Sz_n\| \leq \|Sz_n - Sz\| + \|Sz\| \leq \|z_n - z\| + \|Sz\|.$$

Hence it follows from the boundedness of $\{x_n\}$ and $\{z_n\}$ that $\{f(x_n)\}$ and $\{Sz_n\}$ are bounded. Meantime, utilizing the Lipschitz continuity of A we can also deduce that $\{Ax_n\}$ is bounded. Since $y_n = \beta_n f(x_{n-1}) + \gamma_n S z_{n-1} + (1-\beta_n-\gamma_n)x_{n-1},\ \forall n \in N$, we obtain that $\{y_n\}$ is bounded. Also, since $\lim_{n\to\infty}\alpha_n = 0$, there exist $n_0 \geq 1$ and $a \in (0,1)$, such that $\alpha_n \leq a,\ \forall n \geq n_0$. Note that $x_n = \alpha_n y_n + (1-\alpha_n)Tx_n,\ \forall n \in N$. Consequently, we have

$$Tx_n = \frac{1}{1-\alpha_n}x_n - \frac{\alpha_n}{1-\alpha_n}y_n,$$

and so

$$\begin{aligned}
\|Tx_n\| &\leq \frac{1}{1-\alpha_n}\|x_n\| + \frac{\alpha_n}{1-\alpha_n}\|y_n\| \\
&\leq \frac{1}{1-a}\|x_n\| + \frac{a}{1-a}\|y_n\|.
\end{aligned}$$

This shows that $\{Tx_n\}$ is also bounded.

Step 2. We claim that if we define $V = (2I - T)^{-1}$, where I is the identity mapping of X, then $\lim_{n\to\infty}\|x_n - Vx_n\| = 0$ and $F(V) = F(T)$ is nonempty, closed and convex. Indeed, suppose $V := (2I - T)^{-1}$. From condition (i), we obtain

$$\|x_n - Tx_n\| = \alpha_n\|y_n - Tx_n\| \to 0 \quad (n \to \infty).$$

Then, according to Lemma 2.1 it is known that V is a nonexpansive self-mapping on C, the fixed point set $F(V) = F(T)$ and $\lim_{n\to\infty} \|x_n - Vx_n\| = 0$. Since V is a nonexpansive self-mapping on C and $F(T)$ is nonempty, we deduce that $F(V) = F(T)$ is nonempty, closed and convex.

Step 3. We claim that

$$\limsup_{n\to\infty} \langle p^* - f(p^*), p^* - x_n\rangle \leq 0, \quad p^* \in F(T), \tag{3.3}$$

where $p^* = \lim_{t\to 0^+} z_t$ with z_t being the fixed point of the mapping $z \mapsto tf(z) + (1-t)Vz$. Indeed, z_t solves the fixed point equation

$$z_t = tf(z_t) + (1-t)Vz_t.$$

Then we have

$$z_t - x_n = (1-t)(Vz_t - x_n) + t(f(z_t) - x_n). \tag{3.4}$$

Thus, utilizing (3.4) and the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H,$$

we obtain

$$\begin{aligned}
\|z_t - x_n\|^2 &\leq (1-t)^2\|Vz_t - x_n\|^2 + 2t\langle f(z_t) - x_n, z_t - x_n\rangle \\
&\leq (1-t)^2[\|Vz_t - Vx_n\| + \|Vx_n - x_n\|]^2 + 2t\langle f(z_t) - x_n, z_t - x_n\rangle \\
&\leq (1-t)^2[\|z_t - x_n\| + \|Vx_n - x_n\|]^2 + 2t\langle f(z_t) - x_n, z_t - x_n\rangle \\
&= (1-t)^2[\|z_t - x_n\|^2 + 2\|z_t - x_n\|\|Vx_n - x_n\| + \|Vx_n - x_n\|^2] \\
&\quad +2t\langle f(z_t) - x_n, z_t - x_n\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
\|z_t - x_n\|^2 &\leq (1-t)^2\|z_t - x_n\|^2 + \|Vx_n - x_n\| \times [2\|z_t - x_n\| + \|Vx_n - x_n\|] \\
&\quad +2t\langle f(z_t) - z_t, z_t - x_n\rangle + 2t\|z_t - x_n\|^2 \\
&= (1+t^2)\|z_t - x_n\|^2 + \|Vx_n - x_n\| \times [2\|z_t - x_n\| + \|Vx_n - x_n\|] \\
&\quad +2t\langle f(z_t) - z_t, z_t - x_n\rangle.
\end{aligned}$$

It follows that

$$\langle z_t - f(z_t), z_t - x_n\rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}\|Vx_n - x_n\|[2\|z_t - x_n\| + \|Vx_n - x_n\|]. \tag{3.5}$$

Letting $n \to \infty$ in (3.5) and noting that $\lim_{n\to\infty} \|x_n - Vx_n\| = 0$, we have

$$\limsup_{n\to\infty}\langle z_t - f(z_t), z_t - x_n\rangle \le \frac{t}{2}M, \tag{3.6}$$

where M is a constant such that $\|z_t - x_n\|^2 \le M$ for all $n \ge 0$ and $t \in (0, 1)$. Utilizing Proposition 2.1 we deduce that z_t converges strongly to a fixed point $p^* \in F(V)$ $(= F(T))$, which solves the variational inequality

$$\langle (I - f)(p^*), p^* - p\rangle \le 0, \quad \forall p \in F(V). \tag{3.7}$$

Further, by letting $t \to 0^+$ in (3.6), we can readily know that (3.3) holds.

Step 4. We claim that $x_n \to p^*$ as $n \to \infty$. Indeed, using inequalities: $\|Sx - Sy\| \le \|x - y\|$, $\|f(x) - f(y)\| \le k\|x - y\|$ and

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2,$$

from (3.1) we conclude that

$$\begin{aligned}
\|x_n - p^*\|^2 &= (1-\alpha_n)\langle Tx_n - p^*, x_n - p^*\rangle + \alpha_n\langle y_n - p^*, x_n - p^*\rangle\\
&\le (1-\alpha_n)\|x_n - p^*\|^2 + \alpha_n\langle y_n - p^*, x_n - p^*\rangle\\
&= (1-\alpha_n)\|x_n - p^*\|^2 + \alpha_n(1-\beta_n-\gamma_n)\langle x_{n-1} - p^*, x_n - p^*\rangle\\
&\quad +\alpha_n\beta_n\langle f(x_{n-1}) - p^*, x_n - p^*\rangle + \alpha_n\gamma_n\langle Sz_{n-1} - p^*, x_n - p^*\rangle.
\end{aligned}$$

Thus, from (3.2) we obtain that

$$\begin{aligned}
&\|x_n - p^*\|^2\\
&\le (1-\beta_n-\gamma_n)\langle x_{n-1} - p^*, x_n - p^*\rangle + \beta_n\langle f(x_{n-1}) - p^*, x_n - p^*\rangle\\
&\quad +\gamma_n\langle Sz_{n-1} - p^*, x_n - p^*\rangle\\
&\le (1-\beta_n-\gamma_n)\|x_{n-1} - p^*\|\|x_n - p^*\| + \beta_n(k\|x_{n-1} - p^*\|\|x_n - p^*\|\\
&\quad +\langle f(p^*) - p^*, x_n - p^*\rangle) + \gamma_n\|Sz_{n-1} - p^*\|\|x_n - p^*\|\\
&= (1-(1-k)\beta_n-\gamma_n)\|x_{n-1} - p^*\|\|x_n - p^*\| + \beta_n\langle f(p^*) - p^*, x_n - p^*\rangle\\
&\quad +\gamma_n\|Sz_{n-1} - p^*\|\|x_n - p^*\|\\
&\le (1-(1-k)\beta_n)\|x_{n-1} - p^*\|\|x_n - p^*\| + \beta_n\langle f(p^*) - p^*, x_n - p^*\rangle\\
&\quad +\gamma_n\|Sz_{n-1} - p^*\|\|x_n - p^*\|\\
&\le (1-(1-k)\beta_n)\frac{\|x_{n-1} - p^*\|^2 + \|x_n - p^*\|^2}{2} + \beta_n\langle f(p^*) - p^*, x_n - p^*\rangle\\
&\quad +\gamma_n\|Sz_{n-1} - p^*\|\|x_n - p^*\|,
\end{aligned}$$

which hence implies that

$$\begin{aligned}\|x_n - p^*\|^2 &\le \frac{1-(1-k)\beta_n}{1+(1-k)\beta_n}\|x_{n-1}-p^*\|^2 + \frac{2\beta_n}{1+(1-k)\beta_n}\langle f(p^*)-p^*, x_n-p^*\rangle \\ &\quad + \frac{2\gamma_n}{1+(1-k)\beta_n}\|Sz_{n-1}-p^*\|\|x_n-p^*\| \\ &= (1-\frac{2(1-k)\beta_n}{1+(1-k)\beta_n})\|x_{n-1}-p^*\|^2 \\ &\quad + \frac{2(1-k)\beta_n}{1+(1-k)\beta_n}\cdot\frac{1}{1-k}[\langle f(p^*)-p^*, x_n-p^*\rangle \\ &\quad + \frac{\gamma_n}{\beta_n}\|Sz_{n-1}-p^*\|\|x_n-p^*\|].\end{aligned}$$

Put $b_n = \frac{2(1-k)\beta_n}{1+(1-k)\beta_n}$ and

$$c_n = \frac{1}{1-k}[\langle f(p^*)-p^*, x_n-p^*\rangle + \frac{\gamma_n}{\beta_n}\|Sz_{n-1}-p^*\|\|x_n-p^*\|].$$

Then the last inequality can be rewritten as

$$\|x_n-p^*\|^2 \le (1-b_n)\|x_{n-1}-p^*\|^2 + b_n c_n. \tag{3.8}$$

Since $\sum_{n=1}^{\infty}\beta_n = \infty$, we have $\sum_{n=1}^{\infty}\frac{\beta_n}{1+(1-k)\beta_n} = \infty$ and hence $\sum_{n=1}^{\infty} b_n = \infty$. Note that $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0$ and $\limsup_{n\to\infty}\langle f(p^*)-p^*, x_n-p^*\rangle \le 0$ due to (3.3). Thus, according to the boundedness of $\{\|Sz_{n-1}-p^*\|\|x_n-p^*\|\}$, we have $\limsup_{n\to\infty} c_n \le 0$. Therefore, applying Lemma 2.4 to (3.8), we infer that $\lim_{n\to\infty}\|x_n-p^*\| = 0$.

Step 5. We claim that if $\lim_{n\to\infty}\lambda_n = \lambda > 0$ and the existence of $\lim_{n\to\infty}\|x_n-z_n\|$ implies $\lim_{n\to\infty}\|x_n-z_n\| = 0$, then $p^* = P_{F(T)\cap EP}f(p^*)$. Indeed, let us show that $p^* \in EP$. First, utilizing Lemma 2.2 we have

$$\begin{aligned}&\|T_{\lambda_n}(p^*-\lambda_n Ap^*)-T_\lambda(p^*-\lambda Ap^*)\| \\ &= \|T_{\lambda_n}(p^*-\lambda_n Ap^*)-T_{\lambda_n}(p^*-\lambda Ap^*)+T_{\lambda_n}(p^*-\lambda Ap^*)-T_\lambda(p^*-\lambda Ap^*)\| \\ &\le \|T_{\lambda_n}(p^*-\lambda_n Ap^*)-T_{\lambda_n}(p^*-\lambda Ap^*)\| \\ &\quad + \|T_{\lambda_n}(p^*-\lambda Ap^*)-T_\lambda(p^*-\lambda Ap^*)\| \\ &\le \|(p^*-\lambda_n Ap^*)-(p^*-\lambda Ap^*)\| + \|T_{\lambda_n}(p^*-\lambda Ap^*)-T_\lambda(p^*-\lambda Ap^*)\| \\ &= |\lambda_n-\lambda|\|Ap^*\| + \|T_{\lambda_n}(p^*-\lambda Ap^*)-T_\lambda(p^*-\lambda Ap^*)\|,\end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
&\|z_{n-1} - T_{\lambda_n}(p^* - \lambda_n Ap^*)\|^2 \\
&= \|T_{\lambda_n}(x_{n-1} - \lambda_n Ax_{n-1}) - T_{\lambda_n}(p^* - \lambda_n Ap^*)\|^2 \\
&\leq \|(x_{n-1} - \lambda_n Ax_{n-1}) - (p^* - \lambda_n Ap^*)\|^2 \\
&= \|(x_{n-1} - p^*) - \lambda_n(Ax_{n-1} - Ap^*)\|^2 \\
&= \|x_{n-1} - p^*\|^2 - 2\lambda_n\langle x_{n-1} - p^*, Ax_{n-1} - Ap^*\rangle + \lambda_n^2\|Ax_{n-1} - Ap^*\|^2 \\
&\leq \|x_{n-1} - p^*\|^2 - 2\lambda_n\alpha\|Ax_{n-1} - Ap^*\|^2 + \lambda_n^2\|Ax_{n-1} - Ap^*\|^2 \\
&= \|x_{n-1} - p^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_{n-1} - Ap^*\|^2 \\
&\leq \|x_{n-1} - p^*\|^2.
\end{aligned} \tag{3.10}
$$

Since $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and $\lim_{n\to\infty} \|x_n - p^*\| = 0$, it follows from (3.9) and (3.10) and Lemma 2.3 that

$$\lim_{n\to\infty} \|T_{\lambda_n}(p^* - \lambda_n Ap^*) - T_\lambda(p^* - \lambda Ap^*)\| = 0,$$

and

$$\lim_{n\to\infty} \|z_{n-1} - T_{\lambda_n}(p^* - \lambda_n Ap^*)\| = 0.$$

Thus, we have that

$$
\begin{aligned}
&\|z_{n-1} - T_\lambda(p^* - \lambda Ap^*)\| \\
&= \|z_{n-1} - T_{\lambda_n}(p^* - \lambda_n Ap^*) + T_{\lambda_n}(p^* - \lambda_n Ap^*) - T_\lambda(p^* - \lambda Ap^*)\| \\
&\leq \|z_{n-1} - T_{\lambda_n}(p^* - \lambda_n Ap^*)\| + \|T_{\lambda_n}(p^* - \lambda_n Ap^*) \\
&\quad -T_\lambda(p^* - \lambda Ap^*)\| \to 0 \quad (n \to \infty),
\end{aligned}
$$

that is,

$$\lim_{n\to\infty} \|z_n - T_\lambda(p^* - \lambda Ap^*)\| = 0.$$

Consequently, we conclude that

$$
\begin{aligned}
&\|(x_n - z_n) - (p^* - T_\lambda(p^* - \lambda Ap^*))\| \\
&\leq \|x_n - p^*\| + \|z_n - T_\lambda(p^* - \lambda Ap^*)\| \to 0 \quad (n \to \infty),
\end{aligned}
$$

that is,

$$\lim_{n\to\infty} \|(x_n - z_n) - (p^* - T_\lambda(p^* - \lambda Ap^*))\| = 0.$$

This implies that $\lim_{n\to\infty} \|x_n - z_n\|$ exists. Therefore, utilizing the assumption that the existence of $\lim_{n\to\infty} \|x_n - z_n\|$ implies $\lim_{n\to\infty} \|x_n - z_n\| = 0$, we know that

$$\begin{aligned}
&\|p^* - T_\lambda(p^* - \lambda A p^*)\| \\
&= \|p^* - x_n + x_n - z_n + z_n - T_\lambda(p^* - \lambda A p^*)\| \\
&\le \|p^* - x_n\| + \|x_n - z_n\| + \|z_n - T_\lambda(p^* - \lambda A p^*)\| \to 0 \quad (n \to \infty),
\end{aligned}$$

that is,

$$p^* = T_\lambda(p^* - \lambda A p^*),$$

which is equivalent to the following

$$F(p^*, y) + \langle A p^*, y - p^* \rangle \ge 0, \quad \forall y \in C,$$

and hence $p^* \in EP$. Thus, this immediately implies that $p^* \in F(T) \cap EP$. Furthermore, from (3.7) it follows that

$$\langle (I - f)(p^*), p^* - p \rangle \le 0, \quad \forall p \in F(V) \cap EP \subset F(V),$$

which is equivalent to the following

$$\langle f(p^*) - p^*, p - p^* \rangle \le 0, \quad \forall p \in F(T) \cap EP.$$

This shows that $p^* = P_{F(T)\cap EP} f(p^*)$. This completes the proof. ∎

4. Applications

Using our main theorem, we obtain several strong convergence theorems in a Hilbert space (see also [17]).

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying (A1), (A2), (A3) and (A4). Let $f : C \to C$ be a fixed contraction mapping, $S : C \to C$ be a nonexpansive mapping, and $T : C \to C$ be a continuous pseudocontractive mapping such that $EP(F) \cap F(T) \ne \emptyset$. For $x_0 \in C$, let $\{x_n\} \subset C$ be defined by*

$$\begin{cases}
F(z_{n-1}, y) + \dfrac{1}{\lambda_n}\langle y - z_{n-1}, z_{n-1} - x_{n-1} \rangle \ge 0, \quad \forall y \in C, \\
y_n = \beta_n f(x_{n-1}) + \gamma_n S z_{n-1} + (1 - \beta_n - \gamma_n) x_{n-1}, \\
x_n = \alpha_n y_n + (1 - \alpha_n) T x_n, \quad \forall n \in N,
\end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are sequences of nonnegative real numbers satisfying the conditions:

(i) $\{\alpha_n\} \subset (0,1],\ \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\{\beta_n\} \subset (0,1],\ \sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n\to\infty} (\gamma_n/\beta_n) = 0,\ \beta_n + \gamma_n \le 1,\ \forall n \in N$;

(iv) $0 < \lambda_n \le \lambda,\ \forall n \in N$, *for some* $\lambda \in (0,\infty)$.

Then $\{x_n\}$ *converges strongly to* $p^* = P_{F(T)}f(p^*)$. *Assume additionally that* $\lim_{n\to\infty} \lambda_n = \lambda > 0$ *and that the existence of* $\lim_{n\to\infty} \|x_n - z_n\|$ *implies* $\lim_{n\to\infty} \|x_n - z_n\| = 0$. *Then* $p^* = P_{EP(F)\cap F(T)}f(p^*)$.

Proof. In Theorem 3.1, put $A \equiv 0$. Then for all $\alpha \in (0,\infty)$, we have that

$$\langle x - y, Ax - Ay\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Take $\alpha = \lambda/2$. Then, $0 < \lambda_n \le 2\alpha,\ \forall n \in N$. Thus, utilizing Theorem 3.1 we obtain the desired result. ∎

Theorem 4.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping,* $f : C \to C$ *be a fixed contraction mapping,* $S : C \to C$ *be a nonexpansive mapping, and* $T : C \to C$ *be a continuous pseudocontractive mapping such that* $VI(C, A) \cap F(T) \ne \emptyset$. *For* $x_0 \in C$, *let* $\{x_n\} \subset C$ *be defined by*

$$\begin{cases} y_n = \beta_n f(x_{n-1}) + \gamma_n SP_C(x_{n-1} - \lambda_n Ax_{n-1}) + (1 - \beta_n - \gamma_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1 - \alpha_n)Tx_n, \quad \forall n \in N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\lambda_n\}$ *are sequences of nonnegative real numbers satisfying the conditions:*

(i) $\{\alpha_n\} \subset (0,1],\ \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\{\beta_n\} \subset (0,1],\ \sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n\to\infty} (\gamma_n/\beta_n) = 0,\ \beta_n + \gamma_n \le 1,\ \forall n \in N$;

(iv) $0 < \lambda_n \le 2\alpha,\ \forall n \in N$.

Then $\{x_n\}$ *converges strongly to* $p^* = P_{F(T)}f(p^*)$. *Assume additionally that* $\lim_{n\to\infty} \lambda_n = \lambda > 0$ *and that the existence of* $\lim_{n\to\infty} \|x_n - z_n\|$ *implies* $\lim_{n\to\infty} \|x_n - z_n\| = 0$. *Then* $p^* = P_{VI(C,A)\cap F(T)}f(p^*)$.

Proof. In Theorem 3.1, put $F \equiv 0$. Then, we obtain that

$$\langle Ax_{n-1}, y - z_{n-1}\rangle + \frac{1}{\lambda_n}\langle y - z_{n-1}, z_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C,\ \forall n \in N.$$

This implies that

$$\langle y - z_{n-1}, x_{n-1} - \lambda_n Ax_{n-1} - z_{n-1}\rangle \geq 0, \quad \forall y \in C.$$

So, we get that $P_C(x_{n-1} - \lambda_n Ax_{n-1}) = z_{n-1}$ for all $n \in N$. Then, we obtain the desired result from Theorem 3.1. ∎

A mapping $V : C \to C$ is called strictly pseudocontractive if there exists κ with $0 \leq \kappa < 1$ such that

$$\|Vx - Vy\|^2 \leq \|x - y\|^2 + \kappa\|(I - V)x - (I - V)y\|^2, \quad \forall x, y \in C.$$

Such a mapping V is called strictly κ-pseudocontractive. Putting $A = I - V$, we know that

$$\langle x - y, Vx - Vy\rangle \geq \frac{1-\kappa}{2}\|Vx - Vy\|^2, \quad \forall x, y \in C;$$

see, e.g., [3]. So, we have the following theorem.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying (A1), (A2), (A3) and (A4). Let $f : C \to C$ be a fixed contraction mapping, $S : C \to C$ be a nonexpansive mapping, $T : C \to C$ be a continuous pseudocontractive mapping, and $V : C \to C$ be a strictly κ-pseudocontractive mapping such that $EP \cap F(T) \neq \emptyset$, where $A = I - V$. For $x_0 \in C$, let $\{x_n\} \subset C$ be defined by*

$$\begin{cases} F(z_{n-1}, y) + \langle (I-V)x_{n-1}, y - z_{n-1}\rangle + \dfrac{1}{\lambda_n}\langle y - z_{n-1}, z_{n-1} - x_{n-1}\rangle \geq 0, \ \forall y \in C, \\ y_n = \beta_n f(x_{n-1}) + \gamma_n S z_{n-1} + (1 - \beta_n - \gamma_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1 - \alpha_n)Tx_n, \quad \forall n \in N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are sequences of nonnegative real numbers satisfying the conditions:

(*i*) $\{\alpha_n\} \subset (0, 1],\ \lim_{n\to\infty} \alpha_n = 0$;

(*ii*) $\{\beta_n\} \subset (0, 1],\ \sum_{n=1}^{\infty} \beta_n = \infty$;

(*iii*) $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0,\ \beta_n + \gamma_n \leq 1,\ \forall n \in N$;

(*iv*) $0 < \lambda_n \leq 1 - \kappa,\ \forall n \in N$.

Then $\{x_n\}$ *converges strongly to* $p^* = P_{F(T)}f(p^*)$. *Assume additionally that* $\lim_{n\to\infty}\lambda_n = \lambda > 0$ *and that the existence of* $\lim_{n\to\infty}\|x_n - z_n\|$ *implies* $\lim_{n\to\infty}\|x_n - z_n\| = 0$. *Then* $p^* = P_{EP\cap F(T)}f(p^*)$.

Proof. Since $V : C \to C$ is a strictly κ-pseudocontractive mapping, the mapping $A = I - V$ is $\frac{1-\kappa}{2}$-inverse-strongly monotone. So, from Theorem 3.1, we obtain the desired result. ∎

References

1. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63** (1994), 123-145.
2. P. L. Combettes, A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6** (2005), 117-136.
3. H. Iiduka and W. Takahashi, Weak convergence theorem by Cesàro means for nonexpansive mappings and inverse-strongly monotone mappings, *J. Nonlinear Convex Anal.*, **7** (2006), 105-113.
4. A. Moudafi, Weak convergence theorems for nonexpansive mappings and equilibrium problems, *J. Nonlinear Convex Anal.*, (in press).
5. A. Moudafi and M. Théra, Proximal and Dynamical Approaches to Equilibrium Problems, in: *Lecture Notes in Economics and Mathematical Systems*, Vol. 477, Springer, 1999, pp. 187-201.
6. A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241** (2000), 46-55.
7. R. H. Martin, Differential equations on closed subsets of a Banach space, *Trans. Amer. Math. Soc.*, **179** (1973), 399-414.
8. A. Tada and W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, *J. Optim. Theory. Appl.*, **133** (2007), 359-370.
9. S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.*, **331** (2007), 506-515.
10. W. Takahashi, *Nonlinear Functional Analysis, Yokohama Publishers*, Yokohama, 2000.
11. R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.*, **58** (1992), 486-491.

12. L. C. Ceng, P. Cubiotti and J. C. Yao, Strong convergence theorems for finite many nonexpansive mappings and applications, *Nonlinear Anal.*, **67** (2007), 1464-1473.

13. W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.*, **118** (2003), 417-428.

14. Y. S. Song and R. D. Chen, Convergence theorems of iterative algorithms for continuous pseudocontractive mappings, *Nonlinear Anal.*, **67** (2007), 486-497.

15. L. C. Ceng and H. K. Xu, Strong convergence of a hybrid viscosity approximation method with perturbed mappings for nonexpansive and accretive operators, *Taiwanese J. Math.*, **11** (2007), 661-682.

16. L. C. Zeng and J. C. Yao, Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings, *Nonlinear Anal.*, **64** (2006), 2507-2515.

17. S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, *Nonlinear Anal.*, **69** (2008), 1025-1033.

18. L. C. Ceng, H. K. Xu and J. C. Yao, Strong convergence of an iterative method with perturbed mappings for nonexpansive and accretive operators, *Numer. Funct. Anal. Optim.*, **29** (2008), 324-345.

19. Y. Yao, Y. C. Liou and R. Chen, Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces, *Nonlinear Anal.*, **67** (2007), 3311-3317.

20. A. Rafiq, On Mann iteration in Hilbert spaces, *Nonlinear Anal.*, **66** (2007), 2230-2236.

21. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004), 279-291.

22. L. C. Ceng, A. Petruşel and J. C. Yao, Strong convergence of modified implicit iterative algorithms with perturbed mappings for continuous pseudocontractive mappings, *Appl. Math. Comput.*, **209** (2009), 162-176.

23. S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.*, **75** (1980), 287-292.

24. L. C. Ceng and J. C. Yao, Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings, *Appl. Math. Comput.*, **198** (2008), 729-741.

25. L. C. Ceng and J. C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, *J. Comput. Appl. Math.*, **214** (2008), 186-201.

26. L. C. Ceng, S. Schaible and J. C. Yao, Implicit iteration scheme with perturbed mapping for equilibrium problems and fixed point problems of finitely many nonexpansive mappings, *J. Optim. Theory Appl.*, **139** (2008), 403-418.

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