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RIESZ MEANS ASSOCIATED WITH HOMOGENEOUS FUNCTIONS ON HARDY SPACES

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Abstract. In this note we prove sharp weak type estimates for lacunary maximal operators of Riesz means associated with homogeneous functions on H^p spaces, $0 < p \leq 1$.

1. Introduction

We suppose that $\rho \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous function of degree one. For a Schwartz function $f \in \Im(\mathbb{R}^n)$, we denote $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i<x,\xi>}\, dx$ by the Fourier transform and $f^\vee(x) = \int_{\mathbb{R}^n} f(\xi)\, e^{i<x,\xi>}\, d\xi$ by the inverse Fourier transform. For $f \in \Im(\mathbb{R}^n)$, we are interested in Riesz means $\mathcal{S}_k^\delta$ defined by

$$\widehat{\mathcal{S}_k^\delta f}(\xi) = \left(1 - \frac{\rho(\xi)}{2^k}\right)_+^\delta \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n,$$

and the corresponding lacunary maximal function

$$\mathcal{S}_*^\delta f(x) = \sup_{k \in \mathbb{Z}} |\mathcal{S}_k^\delta f(x)|.$$

Let $\delta_p = n(1/p - 1/2) - 1/2$ be the critical index. In the case where $\rho(\xi) = |\xi|^2$, E. M. Stein [10] showed that the maximal operator of $\mathcal{S}_k^\delta$ is of weak type $(1,1)$ for $\delta > \delta_1$, and also proved it for any isotropic distance function ρ which is real analytic. This is still true even though $|\xi|^2$ is replaced by an arbitrary distance function $\rho \in C^{n+1}(\mathbb{R}^n \setminus \{0\})$ in A. Seeger [9]. At the critical index $\delta_1 = (n-1)/2$, S. Sato in [8] proved that the lacunary Bochner-Riesz operator on $H^1(\mathbb{R}^n)$ converges almost everywhere.

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In this article we obtain a sharp weak type (p, p) estimate on $H^p(\mathbb{R}^n)$ $(0 < p \leq 1)$ of the lacunary maximal Riesz operator $\mathcal{S}_*^\delta$, where we only assume homogeneity and smoothness of ρ without any finite type condition. We note that H^p are the standard real Hardy spaces as defined (see [12]).

Theorem 1. *Suppose* $0 < p \leq 1$ *and* $\delta = \delta_p = n(1/p - 1/2) - 1/2$*. Then* $\mathcal{S}_*^\delta$ *maps* $H^p(\mathbb{R}^n)$ *boundedly into weak-*$L^p(\mathbb{R}^n)$*; that is, there exists a constant* C *such that for all* $f \in H^p(\mathbb{R}^n)$

$$\left|\left\{x \in \mathbb{R}^n : \mathcal{S}_*^\delta f(x) > \alpha\right\}\right| \leq C\left(\frac{\|f\|_{H^p(\mathbb{R}^n)}}{\alpha}\right)^p \tag{1.1}$$

for all $\alpha > 0$*. The constant* C *does not depend on* f *or* α*.*

By a standard argument the theorem implies:

Corollary 1. *For all* $f \in H^p(\mathbb{R}^n)$*,* $\delta = \delta_p$ *and* $0 < p \leq 1$*, the operator* $\mathcal{S}_k^\delta f$ *converges to* f *a.e. as* $k \to \infty$*.*

Remark 1.

(i) It is sharp in the sense that $\mathcal{S}_*^\delta$ is a bounded operator of $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $\delta > \delta_p$, but it fails to be of weak type (p, p) on $H^p(\mathbb{R}^n)$ for all $\delta < \delta_p$ (see [4]).

(ii) When $\delta_1 = (n-1)/2$, E. M. Stein [11] proved the existence of an $f \in H^1$ such that almost everywhere convergence of the Bochner-Riesz means fails. For the case $0 < p < 1$, it is well known that the maximal Bochner-Riesz operator maps $H^p(\mathbb{R}^n)$ to weak $L^p(\mathbb{R}^n)$ if $\delta = \delta_p$ in Stein, Taibleson and Weiss [13].

(iii) In this problem, we do not have any finite type condition, and thus the techniques in [8, 13] do not work. For the proof of Theorem 1, we shall use Littlewood-Paley square-functions for $H^p(\mathbb{R}^n)$ $(0 < p \leq 1)$ and adapt some ideas of Christ and Sogge [4].

In what follows, the letter C denote some positive constant that may not be the same at each occurrence.

2. Preliminaries

2.1. Hardy spaces

We shall use the equivalent characterizations of the Hardy spaces in terms of atomic decompositions, and Littlewood-Paley square-functions ([6, 7, 12]).

Definition 1. Let $\chi \in C_0^\infty(\mathbb{R})$ be non negative, have support in $(1/2, 4)$ and be equal to 1 on $(1, 2)$ such that $\sum_m \chi(2^{-m}s) = 1$. Set $\chi_m(s) = \chi(2^{-m}s)$. We define Littlewood-Paley operators in $\mathbb{R}^n$ by $\widehat{L_m f}(\xi) = \chi_m(|\xi|)\,\widehat{f}(\xi)$. For $0 < p \le 1$ we define the Hardy spaces H^p as the space of all tempered distribution for which the quantity

$$\|f\|_{H^p} = \left\| \left(\sum_{m\in\mathbb{Z}} |L_m f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} < \infty.$$

Definition 2. Let $0 < p \le 1$ and d be an integer that satisfies $d \ge n(1/p - 1)$. Let Q be a cube in $\mathbb{R}^n$. We say that a is a (p, d)-atom associated with Q if a is supported on $Q \subset \mathbb{R}^n$ and satisfies

$$(i)\ \|a\|_{L^\infty(\mathbb{R}^n)} \le |Q|^{-1/p}, \qquad (ii)\ \int_{\mathbb{R}^n} a(x)x^\beta\,dx = 0$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ is an n-tuple of non-negative integers satisfying $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n \le d$, and $x^\beta = x^{\beta_1}x^{\beta_2}\cdots x^{\beta_n}$.

If $\{a_j\}$ is a collection of (p, d)-atoms and $\{\lambda_j\}$ is a sequence of complex numbers with $\sum_{j=1}^\infty |\lambda_j|^p < \infty$, then the series $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in the sense of distributions, and its sum belongs to $H^p(\mathbb{R}^n)$ with the quasinorm

$$\|f\|_{H^p(\mathbb{R}^n)} = \inf_{\sum_{j=1}^\infty \lambda_j a_j = f} \left(\sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p}.$$

2.2. Kernel estimates

We adapt a decomposition of the Bochner-Riesz multiplier $(1-\rho)_+^\delta$ as in [2]. Let $\varphi \in C_0^\infty(\mathbb{R})$ be supported in $(1/2, 2)$ such that $\sum_{j\ge1} \varphi(2^j s) = 1$ for $0 < s < 1$. For $j \in \mathbb{N}$, let $\Psi_j = \varphi(2^{j+1}(1-\rho))(1-\rho)_+^\delta$ and $\Psi_0 = (1-\rho)_+^\delta - \sum_{j\in N} \Psi_j$. For each $j \in \mathbb{Z}$, we now introduce a partition of unity $\Xi_{j\nu}$, $\nu = 1, 2, \cdots, N_j$, on the unit sphere Σ_ρ which extends to $\mathbb{R}^n$ by way of $\Pi_{j\nu}(A_{2^k}\zeta) = \Xi_{j\nu}(\zeta)$, $k \in \mathbb{Z}$, $\zeta \in \Sigma_\rho := \{\xi \in \mathbb{R}^n : \rho(\xi) = 1\}$, and which satisfies the following properties ; there are finite number of points $\zeta_{j1}, \zeta_{j2}, \cdots, \zeta_{jN_j} \in \Sigma_\rho$ such that for $\nu = 1, 2, \cdots, N_j$,

(i) $\displaystyle\sum_{\nu=1}^{N_j} \Pi_{j\nu}(\zeta) \equiv 1$ for all $\zeta \in \Sigma_\rho$,

(ii) $\Xi_{j\nu}(\zeta) = 1$ for all $\zeta \in \Sigma_\rho \cap B(\zeta_{j\nu}, 2^{-j/2})$,

(iii) $\Xi_{j\nu}$ is supported in $\Sigma_\rho \cap B(\zeta_{j\nu}, c_1 2^{-j/2})$,

(iv) $|\mathcal{D}^\gamma \Pi_{j\nu}(\xi)| \le c_2\, 2^{|\gamma| j/2}$ for any multi index γ, if $1/2 \le \rho(\xi) \le 2$,

(v) $N_j \le c_3 2^{j(n-1)/2}$ for fixed j, where $B(\zeta_0, r)$ denotes the ball in $\mathbb{R}^n$ with center $\zeta_0 \in \Sigma_\rho$ and radius $r > 0$ and the positive constants c_1, c_2, c_3 do not depend on j. For each j, let $G_0^{j,\nu} = [\Psi_j \, \Pi_{j\nu}]^\vee$ and $G_0 = [\Psi_0]^\vee$. In view of [5], the kernel G_0 has a nice decay, and thus its corresponding maximal operator satisfies Theorem 1. Thus we treat the estimates for $G_0^{j,\nu}$.

Set $\mathcal{S}_k^{j,\nu} f = G_k^{j,\nu} * f$ where

$$G_k^{j,\nu}(x) = 2^{kn}\, G_0^{j,\nu}(2^k x). \tag{2.1}$$

Denote $G_k = \sum_j \sum_\nu G_k^{j,\nu} = \sum_j G_k^j$, and $\mathcal{S}_k^\delta f = G_k * f$.

Lemma 1. *For fixed $j \in \mathbb{N}$ and for $\nu = 1, 2, \cdots, N_j$, let $T_{\zeta_{j\nu}}(\Sigma_\rho)$ be the tangent space of Σ_ρ at $\zeta_{j\nu} \in \Sigma_\rho$, $\{e_{j\nu}^\ell\}_{\ell=1}^{n-1}$ be an orthonormal basis of $T_{\zeta_{j\nu}}(\Sigma_\rho)$, and $e_{j\nu}^0$ be the outer unit normal vector to Σ_ρ at $\zeta_{j\nu} \in \Sigma_\rho$. Then there are estimates as follows : for any $N \in \mathbb{N}$*

$$\begin{aligned} &\left|G_0^{j,\nu}(x)\right| \\ &\le C\, 2^{-j\delta} \frac{2^{-j}}{(1 + 2^{-j}\,| < x, e_{j\nu}^0 > |)^N} \frac{2^{-j(n-1)/2}}{\prod_{\ell=1}^{n-1}(1 + 2^{-j/2}\,| < x, e_{j\nu}^\ell > |)^N}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &|(\widetilde{G}_0^{j,\nu} * G_0^{j,\nu})(x)| \le C\, 2^{-j\delta}\, 2^{-j(\delta + \frac{(n+1)}{2})} \\ &\frac{1}{(1{+}2^{-j}\,| < x, e_{j\nu}^0 > |)^N} \frac{1}{\prod_{\ell=1}^{n-1}(1{+}2^{-j/2}| < x, e_{j\nu}^\ell > |)^N}. \end{aligned} \tag{2.3}$$

Sketch of Proof. For (2.2), see [9] for details. Consider the case (2.3). Fix j and ν. The multiplier for $\mathcal{S}_0^{j,\nu *} \mathcal{S}_0^{j,\nu}$ is $|\Psi_j \, \Pi_{j\nu}|^2$, which has the same size and smoothness properties as $2^{-j(n-1)/2} \Psi_j \, \Pi_{j\nu}$. Thus the same argument used for (2.2) establishes the desired estimate. ∎

Immediately, from (2.2) we obtain

Lemma 2. *The inequality*

$$\left\|G_0^{j,\nu}\right\|_{L^1(\mathbb{R}^n)} \le C\, 2^{-j\delta}$$

holds for all j and ν.

3. Weak Type (p,p) Estimates on $H^p(\mathbb{R}^n)$, $0 < p \le 1$

To prove Theorem 1 we shall need Lemmas 3 and 4 due to M. Christ [3] for $p = 1$ and Stein, Taibleson and Weiss [13] for $0 < p < 1$, respectively.

Lemma 3. *For any $\alpha > 0$ and any finite collection of dyadic cubes Q and associated positive scalars λ_Q, there exists a collection of pairwise disjoint dyadic cubes S such that*

$$(1) \qquad \sum_{Q\subset S} \lambda_Q \le 8\,\alpha|S| \quad \text{for all } S,$$

$$(2) \qquad \sum |S| \le \alpha^{-1} \sum |\lambda_Q|,$$

$$(3) \qquad \left\| \sum_{Q\not\subset\, any\, S} \lambda_Q |Q|^{-1}\chi_Q \right\|_\infty \le \alpha.$$

Lemma 4. *Suppose $0 < p < 1$ and $\{f_i\}$ is a sequence of measurable functions such that*

$$(3.1) \qquad |\{x : |f_i(x)| > \alpha > 0\}| \le \alpha^{-p}$$

for $i = 1, 2, 3, \cdots$. If $\sum_{i=1}^{\infty} |\lambda_i|^p \le 1$, then

$$\left| \left\{ x : \left| \sum_{i=1}^{\infty} \lambda_i f_i(x) \right| > \alpha \right\} \right| \le \frac{2-p}{1-p}\,\alpha^{-p}.$$

We first consider the case $0 < p < 1$. In view of Lemma 4, it is enough to show (3.1) for $\mathcal{S}_*^\delta f$ in order to prove Theorem 1.

Proposition 1. *Let $0 < p < 1$. Suppose f is a (p, N)-atom $(N \ge n(1/p - 1))$ on $\mathbb{R}^n$ and $\delta = \delta_p = n(1/p - 1/2) - 1/2$. Then there exists a constant $C = C(n, p)$ such that*

$$\left|\{x \in \mathbb{R}^n : \mathcal{S}_*^\delta f(x) > \alpha\}\right| \le C\,\alpha^{-p}$$

for all $\alpha > 0$.

Proof. Since $\mathcal{S}_k^\delta$ is translation invariant, we can assume that f is supported in a cube Q centered at the origin. We write $f = f^j$ if $d(Q)$ is the side length of Q and $d(Q) = 2^j, j > 0$, and $f = f^0$ where $d(Q) \le 1$ if $j = 0$. Then

$$\left\{x \in \mathbb{R}^n : \sup_{k\in\mathbb{Z}} |\mathcal{S}_k^\delta f(x)| > \alpha\right\} \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$$

where

$$\begin{aligned}\mathcal{A}_1 &= \Big\{x \in Q^* : \sup_{k\in\mathbb{Z}} |\mathcal{S}_k^\delta f(x)| > \frac{\alpha}{4}\Big\},\\ \mathcal{A}_2 &= \Big\{x \in \mathbb{R}^n \setminus Q^* : \sup_{k\in\mathbb{Z}} \sum_{s\geq 0}\sum_{j>0} |\mathcal{S}_k^{j+s} f^j(x)| > \frac{\alpha}{4}\Big\},\\ \mathcal{A}_3 &= \Big\{x \in \mathbb{R}^n \setminus Q^* : \sup_{k\in\mathbb{Z}} \sum_{s<0}\sum_{j>0} |\mathcal{S}_k^{j+s} f^j(x)| > \frac{\alpha}{4}\Big\},\\ \mathcal{A}_4 &= \Big\{x \in \mathbb{R}^n \setminus Q^* : \sup_{k\in\mathbb{Z}} \sum_{s\geq 0} |\mathcal{S}_k^{s} f^0(x)| > \frac{\alpha}{4}\Big\},\end{aligned}$$

where Q^* is the cube concentric with Q and with sides of twice the length.

The L^1-boundedness of G_k (see Lemma 2 and (2.1)) implies

$$|\mathcal{S}_k^\delta f(x)| \leq C\, \|G_k\|_1\, \|f\|_\infty \leq C\, \|G_k\|_1\, |Q|^{-1/p}.$$

Consequently,

$$\sup_{k\in\mathbb{Z}} |\mathcal{S}_k^\delta f(x)| \leq C\, |Q|^{-1/p}$$

for all $x \in Q^*$. From Chebyshev's inequality we get

$$|\mathcal{A}_1| \leq C\, \alpha^{-p}.$$

Next, we concentrate on the estimates of the measures $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$. For this we claim that the following holds with $\epsilon = \frac{n}{p} - n$:

$$\Big\|\sup_{k\in\mathbb{Z}} \sum_{j>0} |\mathcal{S}_k^{j+s} f^j|\Big\|^2_{L^2(\mathbb{R}^n)} \leq C\, 2^{-\epsilon s}\, \alpha^{2-p} \qquad \text{for} \quad s \geq 0, \tag{3.2}$$

$$\Big\|\sup_{k\in\mathbb{Z}} \sum_{j>0} |\mathcal{S}_k^{j+s} f^j|\Big\|^p_{L^p(\mathbb{R}^n\setminus Q^*)} \leq C\, 2^{s} \qquad \text{for} \quad s < 0, \tag{3.3}$$

and

$$\|\sup_{k\in\mathbb{Z}} |\mathcal{S}_k^{s} f^0|\|^2_{L^2(\mathbb{R}^n)} \leq C\, 2^{-\epsilon s}\, \alpha^{2-p} \qquad \text{for} \quad s \geq 0. \tag{3.4}$$

We first consider (3.3). In view of a result of Baernstein and Sawyer ([1], p.6), it is enough to show

$$\sum_{j>0} 2^{jn(1-p)} \Big\|\sum_{k} |\mathcal{S}_k^{j+s} f^j|\Big\|^p_{L^1(D)} \leq C\, 2^{s}$$

where $D = \{x : 2^{j-s} \le |x| < 2^{j-s+1}\}$. For this we use Littlewood-Paley theory in H^p, $p \le 1$. For convenience we write $\sum_l L_l f^j = \sum_l f_l^j$. Now we set $f^j = \sum_l L_l f^j = \sum_l f_l^j$ and also note that $\mathcal{S}_k^{j+s} = \sum_\nu \mathcal{S}_k^{j+s,\nu}$. From a Littlewood-Paley decomposition of f^j, it then follows that

$$\mathcal{S}_k^{j+s,\nu} f^j = \mathcal{S}_k^{j+s,\nu}\big(\sum_{|k-l|\le 10} f_l^j\big).$$

Using this

$$\sum_k |\mathcal{S}_k^{j+s,\nu} f^j| \;=\; \sum_k |\mathcal{S}_k^{j+s,\nu}\big(\sum_{|k-l|\le 10} f_l^j\big)|.$$

Thus, it suffices to show that

$$\sum_{j>0}\sum_\nu 2^{jn(1-p)} \Big\| \sum_k |\mathcal{S}_k^{j+s,\nu} f_k^j| \Big\|_{L^1(D)}^p \le C\, 2^s. \tag{3.5}$$

Fix $k,\ j,\ \nu$ and $s<0$. Then, by Lemma 2 and (2.1)

$$\begin{aligned}
&\Big\| \sum_k |\mathcal{S}_k^{j+s,\nu} f_k^j| \Big\|_{L^1(D)}^p \\
&\le C \left| \sum_k \int_{\{2^{j-s}\le|x|<2^{j-s+1}\}} \int_{|y|<2^j} |G_k^{j,\nu}(x-y)\, f_k^j(y)|\, dy\, dx \right|^p \\
&\le C\, 2^{-j\delta p}\, 2^{sNp}\, \| \sum_k |f_k^j| \|_{L^1(D)}^p \\
&\le C\, 2^{-j(\frac{n}{p}-\frac{n+1}{2})p}\, 2^{sNp}\, \big(\| \sum_{|k-j|\le 10} |f_k^j| \|_{L^1(D)}^p + \| \sum_{|k-j|>10} |f_k^j| \|_{L^1(D)}^p\big).
\end{aligned} \tag{3.6}$$

For the case $|k-j|\le 10$ we use Schwartz's inequality and Littlewood-Paley theory in H^p, and thus

$$\begin{aligned}
\Big\| \textstyle\sum_{|k-j|\le 10} |f_k^j| \Big\|_{L^1(D)}^p &\le C \Big\| (\sum_k |f_k^j|^2)^{1/2} \Big\|_{L^1(\mathbb{R}^n)}^p \\
&\le C\, \|f^j\|_{H^1(\mathbb{R}^n)}^p \le\ C\, 2^{jn(p-1)}.
\end{aligned} \tag{3.7}$$

We now consider the case $|k-j|>10$.
If $k-j>10$, then

$$\begin{aligned}
\|f_k^j\|_{L^1(D)} &\le 2^{kn} \int_{|x|>2^{j+1}} \int_{|y|<2^j} |\chi^\vee(2^k(x-y))|\, |f^j(y)|\, dy\, dx \\
&\le C\, 2^{jn(1-1/p)}\, 2^{kn} \int_{2^k|x|>2^{k-j+1}} \frac{1}{(1+2^k|x|)^N}\, dx \\
&\le C\, 2^{jn(1-1/p)}\, 2^{-(k-j)(N-n)}
\end{aligned} \tag{3.8}$$

If we assume that $k-j<-10$, we use the moment condition in Definition 2. Let P_k denote the N-th order Taylor polynomial of the function $\chi_k^\vee(x-\cdot)$ expanded near the origin. Then we have that $P_k(y) = 2^{kn}\sum_{|\gamma|\le N} C_{\gamma,N}\,[\chi^{\vee(\gamma)}(2^k x)](2^k y)^\gamma$ for fixed k.
Thus we have

$$f_k^j(x) = 2^{kn}\int_{|y|<2^j} [\chi^\vee(2^k(x-y)) - P_k(2^k y)]\, f^j(y)\, dy,$$

and

$$\begin{aligned}
&\|f_k^j\|_{L^1(D)}\\
&\le 2^{kn}\int_{|x|>2^{j+1}}\int_{|y|<2^j}\sum_{|\gamma|=N+1}\frac{1}{n!}\,|\chi^{\vee(\gamma)}(2^k x)|\,|2^k y|^{N+1}\,|f^j(y)|\,dy\,dx\\
&\le C\,2^{jn(1-1/p)}\,2^{(k-j)(N+1)}\int_{2^k|x|>2^{k-j+1}}\frac{2^{kn}}{(1+2^k|x|)^N}\,dx\\
&\le C\,2^{jn(1-1/p)}\,2^{(k-j)(n+1)}.
\end{aligned}\tag{3.9}$$

From (3.8) and (3.9), we have

$$\begin{aligned}
\|\sum_{|k-j|>10}|f_k^j|\|_{L^1(D)}^p \le& \sum_{|k-j|>10}\|f_k^j\|_{L^1(D)}^p\\
\le& C\,2^{jn(p-1)}\max\{1, 2^{-10p(N-n)}, 2^{-10p(n+1)}\}\\
\le& C\,2^{jn(p-1)},
\end{aligned}\tag{3.10}$$

by taking $N\ge\max\{n, 1/p, n(1/p-1)\}$.

Putting together with (3.6), (3.7), and (3.10), we get

$$\begin{aligned}
&\sum_{\nu=1}^{c2^{j(n-1)/2}} 2^{jn(1-p)}\left\|\sum_k |\mathcal{S}_k^{j+s,\nu} f^j|\right\|_{L^1(D)}^p\\
&\le C\,2^{-\frac{j(n+1)}{2}(1-p)}\,2^s.
\end{aligned}$$

After summing over j, the inequality (3.5) follows at once.

We turn to the case (3.2). The orthogonality property of $\mathcal{S}_k^{j+s,\nu}$ yields

$$\begin{aligned}
\left\|\sup_{k\in\mathbb{Z}}\sum_{j>0}|\mathcal{S}_k^{j+s} f^j|\right\|_{L^2(\mathbb{R}^n)}^2 &\le \sum_{j>0}\left\|(\sum_k |\mathcal{S}_k^{j+s} f^j|^2)^{1/2}\right\|_2^2\\
&= \sum_{j>0}\sum_\nu\sum_k \|\mathcal{S}_k^{j+s,\nu} f^j\|_2^2.
\end{aligned}$$

Since there are about $c\,2^{j(n-1)/2}$ values of ν for each j, it suffices to show that

$$\sum_k \left\|\mathcal{S}_k^{j,\nu} f^{j-s}\right\|_2^2 \;\le\; C\,\alpha^{2-p}\,2^{-j(n/p-n)}\,2^{-j(n-1)/2}\,2^{-s(n/p-n)} \tag{3.11}$$

for all $j > s \ge 0$ and all ν.

Expanding the left-hand side of (3.11), we reduce to

$$\begin{aligned} &\sum_k |< \widetilde{G}_k^{j,\nu} * G_k^{j,\nu} * f^{j-s}, f^{j-s} >| \\ \le &\sum_k \sum_{|k-l|\le 10} |< \widetilde{G}_k^{j,\nu} * G_k^{j,\nu} * f_l^{j-s}, f_l^{j-s} >| \\ \le &\; C \sum_k |< \widetilde{G}_k^{j,\nu} * G_k^{j,\nu} * f_k^{j-s}, f_k^{j-s} >|. \end{aligned} \tag{3.12}$$

On the other hand, in view of (2.3) and (2.1), it is easy to see that $\|\widetilde{G}_k^{j,\nu} * G_k^{j,\nu}\|_\infty \le C\,2^{-j\delta}\,2^{-jn/p}$, where C is independent of k. With this and by the Schwartz and Minkowski inequalities, (3.12) is bounded by

$$\begin{aligned} &C \int_{\mathbb{R}^n} \left(\sum_k |\widetilde{G}_k^{j,\nu} * G_k^{j,\nu} * f_k^{j-s}(x)|^2\right)^{\frac12} \left(\sum_k |f_k^{j-s}(x)|^2\right)^{\frac12} dx \\ \le &\; C\,2^{-j\delta}\,2^{-jn/p} \left\| \left(\sum_k |f_k^{j-s}|^2\right)^{\frac12} \right\|_{L^1(\mathbb{R}^n)}^2. \end{aligned} \tag{3.13}$$

Here we note that for $\{f_k^{j-s}\} \subset L^1(\ell^2) \cap L^2(\ell^2)$ and $\alpha > 0$, by the Calderón-Zygmund theory we have

$$\left\| \left(\sum_k |f_k^{j-s}|^2\right)^{\frac12} \right\|_{L^1(\mathbb{R}^n)} \le C\,\alpha|Q^*|,$$

where the side length of Q is 2^{j-s}.

With $\delta = \delta_p = n(1/p - 1/2) - 1/2$, the above (3.13) is bounded by

$$\begin{aligned} &C\,2^{-j\delta}\,2^{-jn/p} \left\| \left(\sum_k |f_k^{j-s}|^2\right)^{\frac12} \right\|_{L^1(\mathbb{R}^n)}^{2-p} \left\| \left(\sum_k |f_k^{j-s}|^2\right)^{\frac12} \right\|_{L^1(\mathbb{R}^n)}^{p} \\ \le &\; C\,\alpha^{2-p}\,2^{-j\delta}\,2^{-jn/p}\,2^{in} \\ \le &\; C\,\alpha^{2-p}\,2^{-j(n/p-n)}\,2^{-j(n-1)/2}\,2^{-\epsilon s}, \end{aligned}$$

where $i = j - s > 0$ and $\epsilon = n/p - n$.

Thus, summing over $\nu = 1, 2, \cdots, c2^{j(n-1)/2}$ and j, we get (3.2) as desired. As for (3.4), we follow the arguments used for (3.2). Therefore, from (3.2) through (3.4), it follows by the application of Chebyshev's inequality that

$$|\mathcal{A}_2| \ + \ |\mathcal{A}_3| \ + \ |\mathcal{A}_4| \ \leq \ C \ \alpha^{-p}.$$

This completes the proof. ∎

We turn to the case $p = 1$.

Proposition 2. *If* $\delta = \delta_1 = (n-1)/2$, *there exists a constant* $C = C(n)$ *such that*

$$\big|\{x \in \mathbb{R}^n : \mathcal{S}_*^\delta f(x) > \alpha\}\big| \leq C\, \alpha^{-1} \, \|f\|_{H^1(\mathbb{R}^n)}$$

for all $\alpha > 0$.

Proof. Let $f(x) = \sum \lambda_Q \, a_Q(x)$ be an element of $H^1(\mathbb{R}^n)$, chosen arbitrarily except that the sum has finitely many terms, that $\sum \lambda_Q \leq 2 \, \|f\|_{H^1}$ and that $\alpha > 0$ is given. Let $d(S)$ be the side length of S. Applying Lemma 3, set $B = \sum_j B_j$ where $B_j = \sum_{Q \subset S, \, d(S) = 2^j} \lambda_Q b_Q$ if $d(S) = 2^j, j > 0$, and $B_0 = \sum_{Q \subset S, \, d(S) \leq 1} \lambda_Q b_Q$ where $d(S) \leq 1$ if $j = 0$. Now $g = f - B$ and $\|g\|_\infty \leq C\, \alpha$.

Consider

$$\big\{x \in \mathbb{R}^n : |\mathcal{S}_*^\delta f(x)| > \alpha\big\} \subset \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$$

where Ω_1 is the union of the double cubes S^* and

$$\begin{aligned}
\Omega_2 &= \left\{x \in \mathbb{R}^n : \sup_{k \in \mathbb{Z}} |\mathcal{S}_k^\delta g(x)| > \frac{\alpha}{5}\right\}, \\
\Omega_3 &= \left\{x \in \mathbb{R}^n : \sup_{k \in \mathbb{Z}} \sum_{s \geq 0} \sum_{j > 0} |\mathcal{S}_k^{j+s} B_j(x)| > \frac{\alpha}{5}\right\}, \\
\Omega_4 &= \left\{x \in \mathbb{R}^n \setminus \Omega_1 : \sup_{k \in \mathbb{Z}} \sum_{s < 0} \sum_{j > 0} |\mathcal{S}_k^{j+s} B_j(x)| > \frac{\alpha}{5}\right\}, \\
\Omega_5 &= \left\{x \in \mathbb{R}^n : \sup_{k \in \mathbb{Z}} \sum_{s \geq 0} |\mathcal{S}_k^{s} B_0(x)| > \frac{\alpha}{5}\right\}.
\end{aligned}$$

By the disjointness of the cubes S and Lemma 3-(2) we have

$$|\Omega_1| \ \leq \ \sum |S^*| \ \leq \ \frac{C}{\alpha} \sum |\lambda_Q|.$$

In order to estimate the measure of Ω_2, we consider

$$\begin{aligned}\|\sup_{k\in\mathbb{Z}}|\mathcal{S}_k^\delta g|\|_{L^2}^2 &\le \Big\|\Big(\sum_k|\mathcal{S}_k^\delta g|^2\Big)^{\frac{1}{2}}\Big\|_2^2\\ &= \sum_k\|\mathcal{S}_k^\delta g\|_2^2.\end{aligned}$$

Using the Plancherel theorem and Chebyshev's inequality imply

$$|\Omega_2| \le C\,\frac{\|g\|_2^2}{\alpha^2} \le \frac{C}{\alpha}\sum|\lambda_Q|.$$

Next in order to estimate the measure of Ω_3, Ω_4 and Ω_5, we shall show that the following holds with $\epsilon = \frac{(n-1)}{4}$:

$$\Big\|\sup_{k\in\mathbb{Z}}\sum_{j>0}|\mathcal{S}_k^{j+s}B_j|\Big\|_{L^2(\mathbb{R}^n)}^2 \le C\,2^{-\epsilon s}\,\alpha\,\|B\|_{H^1(\mathbb{R}^n)} \qquad \text{for} \quad s\ge 0, \tag{3.14}$$

$$\Big\|\sup_{k\in\mathbb{Z}}\sum_{j>0}|\mathcal{S}_k^{j+s}B_j|\Big\|_{L^1(\mathbb{R}^n\setminus\Omega_1)} \le C\,2^{s}\,\|B\|_{H^1(\mathbb{R}^n)} \qquad \text{for} \quad s<0, \tag{3.15}$$

and

$$\|\sup_{k\in\mathbb{Z}}|\mathcal{S}_k^{s}B_0|\|_{L^2(\mathbb{R}^n)}^2 \le C\,2^{-\epsilon s}\,\alpha\,\|B_0\|_{H^1(\mathbb{R}^n)} \qquad \text{for} \quad s\ge 0. \tag{3.16}$$

In (3.15), for each s

$$\Big\|\sup_{k\in\mathbb{Z}}\sum_{j>0}|\mathcal{S}_k^{j+s}B_j|\Big\|_{L^1(\mathbb{R}^n\setminus\Omega_1)} \le \sum_{j>0}\|\sum_k|\mathcal{S}_k^{j+s}B_j|\|_{L^1(\mathbb{R}^n\setminus\Omega_1)},$$

and thus claim that

$$\|\sum_k|\mathcal{S}_k^{j+s}B_j|\|_{L^1(\mathbb{R}^n\setminus\Omega_1)} \le C\,2^s\sum_{Q\subset S,\,d(S)=2^j}|\lambda_Q|.$$

Set $b_Q=\sum_m L_m b_Q=\sum_m b_Q^m$. Since $\mathcal{S}_k^{j+s,\nu}(\sum_{|k-m|>10}b_Q^m)=0$, it suffices to show

$$\Big\|\sum_k|\mathcal{S}_k^{j+s,\nu}b_Q^k|\Big\|_{L^1(\mathbb{R}^n\setminus\Omega_1)} \le C\,2^{-j(n-1)/2}\,2^s.$$

Fix $k,\ j,\ \nu$ and $s<0$. Likewise (3.6)-(3.10), we have

$$\begin{aligned}\Big\|\sum_k|\mathcal{S}_k^{j+s,\nu}b_Q^k|\Big\|_{L^1(\mathbb{R}^n\setminus\Omega_1)} &\le C\,2^{-j(n-1)/2}\;2^{Ns}\,\|\sum_k|b_Q^k|\|_{L^1(\mathbb{R}^n\setminus\Omega_1)}\\ &\le C\,2^{-j(n-1)/2}\;2^{s}.\end{aligned}$$

After summing over $\nu = 1, 2, \cdots, c2^{j(n-1)/2}$ and j, (3.15) follows at once.

We proceed to the case (3.14). Then by the orthogonality property of $\mathcal{S}_k^{j+s,\nu}$

$$\Big\| \sup_{k\in\mathbb{Z}} \sum_{j>0} |\mathcal{S}_k^{j+s} B_j| \Big\|_{L^2(\mathbb{R}^n)}^2 \le \sum_{j>0} \Big\| \big(\sum_k |\mathcal{S}_k^{j+s} B_j|^2 \big)^{\frac{1}{2}} \Big\|_2^2 = \sum_{j>0} \sum_{\nu} \sum_k \|\mathcal{S}_k^{j+s,\nu} B_j\|_2^2.$$

Since there are about $c\, 2^{j(n-1)/2}$ values of ν for each j, it suffices to show that

$$\sum_k \|\mathcal{S}_k^{j,\nu} B_{j-s}\|_2^2 \le C\ \alpha\ 2^{-\epsilon s}\, 2^{-j(n-1)/2} \sum_{Q\subset S,\, d(S)=2^{j-s}} |\lambda_Q| \tag{3.17}$$

for all $j > s \ge 0$ and all ν.
Now set $B_{j-s} = \sum_m B_{j-s,m}$ where $B_{j-s,m} = \sum_{Q\subset S,\, d(S)=2^{j-s}} \lambda_Q\, b_Q^m$. Expanding the left-hand side of (3.17), we have

$$\begin{aligned} &\sum_k | < \widetilde{G}_k^{j,\nu} * G_k^{j,\nu} * B_{j-s}, B_{j-s} > | \\ &\le \sum_k \sum_{|k-m|\le 10} | < \widetilde{G}_k^{j,\nu} * G_k^{j,\nu} * B_{j-s,m}, B_{j-s,m} > | \\ &\le C \sum_k | < \widetilde{G}_k^{j,\nu} * G_k^{j,\nu} * B_{j-s,k}, B_{j-s,k} > |. \end{aligned}$$

By the Schwartz, Minkowski inequalities, and Littlewood-Paley theory on H^1, the above is bounded by

$$\begin{aligned} &C \int_{\mathbb{R}^n} \left(\sum_k |\widetilde{G}_k^{j,\nu} * G_k^{j,\nu} * B_{j-s,k}(x)|^2 \right)^{\frac{1}{2}} \big(\sum_k |B_{j-s,k}(x)|^2\big)^{\frac{1}{2}}\ dx \\ &\le C \int_{\mathbb{R}^n} \left[\sum_k \|\widetilde{G}_k^{j,\nu} * G_k^{j,\nu}\|_\infty^2 \Big(\int_{\mathbb{R}^n} |B_{j-s,k}(x)|\, dx \Big)^2 \right]^{\frac{1}{2}} \left[\sum_k |B_{j-s,k}(x)|^2 \right]^{\frac{1}{2}}\ dx \\ &\le C\, 2^{-j(n-1)/2}\, 2^{-jn} \left\| \left(\sum_k |B_{j-s,k}|^2 \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{R}^n)}^2 \\ &\le C\, 2^{-j(n-1)/2} 2^{-jn} \left(\sum_{Q\subset S,\, d(S)=2^{j-s}} |\lambda_Q| \right)^2, \end{aligned}$$

where $\|\widetilde{G}_k^{j,\nu} * G_k^{j,\nu}\|_\infty \le C\, 2^{-j(n-1)/2}\, 2^{-jn}$.

Thus from Lemma 3-(1), it follows that

$$\begin{aligned}
&C\ \alpha\, 2^{-j(n-1)/2}\, 2^{-jn}\, |S| \sum_{Q\subset S,\, d(S)=2^{j-s}} |\lambda_Q| \\
&\leq C\ \alpha\ 2^{-j(n-1)/2}\ 2^{-j(n-1)/2}\, 2^{-i(n+1)/2}\, 2^{in} \sum_{Q\subset S,\, d(S)=2^{j-s}} |\lambda_Q| \\
&\leq C\ \alpha\ 2^{-j(n-1)/2}\ 2^{-j(n-1)/2}\, 2^{j(n-1)/4} \sum_{Q\subset S,\, d(S)=2^{j-s}} |\lambda_Q| \\
&\leq C\, \alpha\ 2^{-j(n-1)/2}\ 2^{-\epsilon s} \sum_{Q\subset S,\, d(S)=2^{j-s}} |\lambda_Q|
\end{aligned}$$

for all $0 < i = j - s \leq j/2$ and $\epsilon = (n-1)/4$. Hence, we obtain the desired estimate (3.17). Thus summing over $\nu = 1, 2, \cdots, c2^{j(n-1)/2}$ and j, we get (3.14). Finally, it follows (3.16) by the same method used for (3.14). ■

Proof of Theorem 1. The case $p = 1$ is proved in Proposition 2. Suppose now that $0 < p < 1$. Let $f = \sum_{i=1}^{\infty} \lambda_i f_i \in H^p(\mathbb{R}^n)$. Then we see that $\mathcal{S}_*^{\delta} f$ is well-defined on $H^p(\mathbb{R}^n)$, since each $\mathcal{S}_k^{\delta} f_i$ is the convolution of the atom f_i with the kernel G_k. Furthermore, we obtain that $\mathcal{S}_*^{\delta} f$ satisfies a uniform weak type estimate when f_i is a (p, N)-atom $(N \geq n(1/p - 1))$ in Proposition 1. Since $|\mathcal{S}_*^{\delta} f(x)| \leq \sum_{i=1}^{\infty} |\lambda_i||\mathcal{S}_*^{\delta} f_i(x)|$ and $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$, the inequality (1.1) for $p < 1$ follows from Stein, Taibleson, and Weiss's lemma (see [13]) on adding up weak type functions. This completes the proof. ■

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