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STEADY STATES AND STANDING PULSES OF A SKEW-GRADIENT SYSTEM

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Abstract. We study a reaction-diffusion system of activator-inhibitor type. Variational and ordered methods are used to obtain the existence of steady states and standing pulses. It will be seen that the diffusion rates seem to play important roles in the existence of standing pulses.

1. Introduction

This paper deals with the following system of reaction diffusion equations:

$$\begin{aligned} u_t &= d\Delta u + f(u) - k_1 v - k_2 w, \\ \tau_1 v_t &= d_1 \Delta v + u - \gamma_1 v, \\ \tau_2 w_t &= d_2 \Delta w + u - \gamma_2 w, \end{aligned} \tag{1.1}$$

where $d_1,\ d_2,\ d,\ k_1,\ k_2,\ \tau_1,\ \tau_2,\ \gamma_1,\ \gamma_2 \in (0,\infty)$ and $f(u) = u(u-\beta)(1-u),\ 0 < \beta < \frac{1}{2}$. System (1.1) has been studied as a model [4] for gas-discharge systems. By rescaling if necessary, we may assume that $k_1 = k_2 = 1$. Notice that (1.1) can be written in the form

$$\begin{aligned} u_t &= d\Delta u + \frac{\partial H(u,v,w)}{\partial u}, \\ \tau_1 v_t &= d_1 \Delta v - \frac{\partial H(u,v,w)}{\partial v}, \\ \tau_2 w_t &= d_2 \Delta w - \frac{\partial H(u,v,w)}{\partial w}, \end{aligned} \tag{1.2}$$

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where

$$H(u,v,w) = -uv - uw + \frac{\gamma_1}{2}v^2 + \frac{\gamma_2}{2}w^2 - F(u).$$

As being proposed by Yanagida [23], a reaction-diffusion system with such a structure is referred as a *skew-gradient* system. It is easily seen that a steady state of (1.2) is a critical point of Φ defined by

$$\text{(1.3)}\quad \Phi(u,v,w)=\int_\Omega \frac{1}{2}(d\nabla u,\nabla u)-\frac{1}{2}(d_1\nabla v,\nabla v)-\frac{1}{2}(d_2\nabla w,\nabla w)-H(u,v,w)dx.$$

In the past twenty years, there have been many works [6-9, 12, 13, 14, 16, 20, 23, 24] on the system of FitzHugh-Nagumo type equations:

$$\text{(1.4)}\qquad \begin{aligned} u_t &= d\Delta u + f(u) - v, \\ \tau v_t &= d_1\Delta v + u - \gamma_1 v. \end{aligned}$$

Here u can be viewed as an activator and v acts to be an inhibitor. Clearly (1.4) is a skew-gradient system. In case $d_1 = 0$, (1.4) has been considered as a model for the Hodgkin-Huxley system[15, 21] to describe the behavior of electrical impluses in the axon of the squid.

Partially motivated by the works related to (1.4), we try to get better understanding on the effect of adding more inhibitors with possibly different diffusion rates. Consider the case where

$$(\mathrm{f}_1)\quad \frac{1}{\gamma_1} + \frac{1}{\gamma_2} < \frac{2\beta^2 - 5\beta + 2}{9}.$$

It is easy to see that $(0,\ 0,\ 0)$, $(\theta_-,\ \frac{\theta_-}{\gamma_1},\ \frac{\theta_-}{\gamma_2})$ and $(\theta_+,\ \frac{\theta_+}{\gamma_1},\ \frac{\theta_+}{\gamma_2})$ are constant equilibria of (1.2), where $0 < \theta_- < \theta_+$. Let Ω be a smooth bounded domain in $\mathbb{R}^N$. We are interested in the existence of non-trivial solutions of

$$\text{(1.5)}\qquad \begin{aligned} &-d\Delta u = f(u) - v - w, \\ &-d_1\Delta v = u - \gamma_1 v, x \in \Omega, \\ &-d_2\Delta w = u - \gamma_2 w, \\ &u = v = w = 0 \text{ on } \partial\Omega. \end{aligned}$$

For each $u \in H_0^1(\Omega)$, denoted by $\mathcal{L}_i u$ the unique solution of

$$\text{(1.6)}\qquad \begin{aligned} &-d_i\Delta z + \gamma_i z = u, \quad x \in \Omega, \\ &z = 0 \qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned}$$

Define

$$I(u)=\int_{\Omega}\frac{d}{2}|\nabla u|^2+\frac{1}{2}u\mathcal{L}_1(u)+\frac{1}{2}u\mathcal{L}_2(u)+F(u)dx, \tag{1.7}$$

where $F(u)=-\int_0^u f(s)ds$. If u is a critical point of I over $H_0^1(\Omega)$ then standard regularity theory [11] shows that $(u,\ \mathcal{L}_1u,\ \mathcal{L}_2u)$ is a classical solution of (1.5). Using variational arguments, we obtain the following result.

Theorem 1.1. *Assume that* (f_1) *is satisfied. If* Ω *contains a large ball, then* (1.5) *has at least two non-trivial solutions.*

Observe that $u\equiv 0$ is a local minimizer of I. We will show that there exists a global minimizer u^* of I with $I(u^*)<0$, and thus $(u^*,\ \mathcal{L}_1u^*,\ \mathcal{L}_2u^*)$ is a solution of (1.5). Then applying the Mountain Pass Lemma yields another critical point u_* of I from which the second solution follows.

In [23] Yanagida introduced the notation of mini-maximizer to study the stability of steady states of skew-gradient system. A steady state (u,v,w) is called a mini-maximizer of Φ if u is a local minimizer of $\Phi(\cdot,v,w)$ and (v,w) is a local maximizer of $\Phi(u,\cdot)$. It has been shown [23] that non-degenerate mini-maximizers of Φ are linearly stable. In particular, Yanagida's result [23] tells that for any $\tau_1,\tau_2>0$, $(0,0,0)$ is a stable steady state of (1.2). More recently, Chen and Hu [6] extended Yanagida's result by making use of relative Morse index of critical points of Φ. Suppose u^* and u_* are non-degenerated critical points of I, we will see that the results of [6] imply that, for any $\tau_1,\ \tau_2>0$, $(u_*,\ \mathcal{L}_1u_*,\ \mathcal{L}_2u_*)$ is always unstable, while $(u^*,\ \mathcal{L}_1u^*,\ \mathcal{L}_2u^*)$ is stable if τ_1 and τ_2 are small.

Theorem 1.2. *Suppose* u^* *is a non-degenerate minimizer of* I. *If* $\min\{\frac{\gamma_1}{\tau_1},\frac{\gamma_2}{\tau_2}\}>\frac{\beta^2-\beta+1}{3}$, *then* $(u^*,\mathcal{L}_1u^*,\mathcal{L}_2u^*)$ *is a stable steady state of (1.1).*

With the action of one more inhibitor added in (1.1), the diffusion rates seem to play important roles in the existence of standing pulses of (1.1). For $i=1,\ 2$, set $\tilde{d}_i=\frac{d}{d_i}$ and $\sigma_i=(\gamma_i\tilde{d}_i-(M+\frac{2\beta^2-5\beta+2}{9}))^{-1}$ where $M=\max_{0\le s\le 1}-f'(s)$. A standing pulse of (1.2) will be obtained under the following conditions:

(f_2) $\tilde{d}_i>\gamma_i^{-1}(M+\frac{2\beta^2-5\beta+2}{9})$ for $i=1,\ 2$.

(f_3) $\tilde{d}_1\sigma_1+\tilde{d}_2\sigma_2<\frac{2\beta^2-5\beta+2}{9}$.

(f_4) $\sqrt{\tilde{d}_1}\,\sigma_1+\sqrt{\tilde{d}_2}\,\sigma_2<1$.

Theorem 1.3. *If* (f_1)-(f_4) *are satisfied, there exists a standing pulse to* (1.2).

Through out the paper, we denoted by $\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$ and $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ respectively. Let B_R be a ball in $\mathbb{R}^N$ centered at the origin with radius R and $|\Omega| = \int_\Omega dx$.

2. Existence and Stability of Steady States

In this section, variational methods are used to obtain non-trivial solutions of (1.5). Note that F has minima at 0 and 1; $F(0) = 0$ and $F(1) = \dfrac{2\beta - 1}{12} < 0$. Multiplying equation (1.6) by z and integrating by parts yield

$$\int_\Omega u\mathcal{L}_i u dx = \int_\Omega d_i|\nabla z|^2 + \gamma_i z^2 dx. \tag{2.1}$$

Hence $\mathcal{L}_i : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator with $\|\mathcal{L}_i\| \leq \dfrac{1}{\gamma_i}$. Indeed, $\mathcal{L}_i$ is also a bounded operator from $H_0^1(\Omega)$ to itself.

To prove Theorem 1.1, we need some estimates as stated in the next two lemmas.

Lemma 2.1. *If R is large enough, there exists a $\psi \in H_0^1(B_R)$ such that $I(\psi) < 0$.*

Lemma 2.2. *There are positive numbers r and α such that $I(u) \geq \alpha$ for $\|u\| = r$.*

Proof. [Proof of theorem 1.1]. Since $\mathcal{L}_i$ is a compact operator from $H_0^1(\Omega)$ to itself, I satisfies the Palais-Smale condition (see e.g. [19]). By (2.1) we know that I is bounded from below. Moreover, Lemma 2.1 implies that $\inf_{u \in H_0^1(\Omega)} I(u) < 0 = I(0)$. Thus a minimizer u^* of I gives a non-trivial solution of (1.5). Invoking Lemma 2.2 and Mountain Pass Lemma, we yield the second solution u_* of (1.5) through the following minimax framework:

$$I(u_*) = \inf_{h \in \Gamma} \max_{\theta \in [0,1]} I(h(\theta)) > 0,$$

where $\Gamma = \{h \in C([0,1], H_0^1(\Omega)) |\ h(0) = 0, h(1) = u^*\}$. ∎

Proof. [Proof of Lemma 2.1]. Let $A_R = B_R \setminus B_{R-1}$ and

$$\psi_R = \begin{cases} \theta^+ & 0 \le |x| \le R-1 \\ \theta^+(R-|x|) & R-1 \le |x| \le R. \end{cases}$$

By straightforward calculation

$$\begin{aligned} I(\psi_R) &= \int_{A_R} \frac{d}{2}|\nabla\psi_R|^2 dx + \int_{B_R} \frac{1}{2}\psi_R\mathcal{L}_1(\psi_R) + \frac{1}{2}\psi_R\mathcal{L}_2(\psi_R) + F(\psi_R)dx \\ &\le \int_{A_R} \frac{d}{2}|\nabla\psi_R|^2 dx + \int_{B_R} \frac{1}{2}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})\psi_R^2 + F(\psi_R)dx. \end{aligned}$$

Since (f_1) implies that $\beta + (\frac{1}{\gamma_1} + \frac{1}{\gamma_2}) < \frac{2(\beta+1)^2}{9}$ and $\frac{1}{2}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})(\theta^+)^2 + F(\theta^+) < 0$, we see that $I(\psi_R) \le I_1 + I_2$, where

$$I_1 = \int_{A_R} \frac{d}{2}(\theta^+)^2 + \psi_R^2(\frac{1}{2}\psi_R - \frac{\beta+1}{3})^2 dx$$

and

$$I_2 = \int_{B_{R-1}} \frac{1}{2}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})(\theta^+)^2 + F(\theta^+)dx.$$

Hence there are positive numbers C_1 and C_2, not depending on R, such that

$$I_1 \le C_1 R^{N-1} \quad \text{and} \quad I_2 \le -C_2(R-1)^N,$$

from which we know $I(\psi_R) < 0$ if R is large enough. ■

Proof. [Proof of Lemma 2.2]. Since $\int_\Omega u\mathcal{L}_i u \ge 0$,

$$I(u) \;\ge\; \int_\Omega \frac{d}{2}|\nabla u|^2 + F(u)dx.$$

This together with Sobolev inequality implies that if r is small enough and $\|u\| = r$ then $I(u) \ge \alpha > 0$. ■

To study the stability of a steady state (u, v, w) of (1.2), we analyze the spectrum of

$$\lambda JT\xi = D\Delta\xi - \nabla^2 H(u,v,w)\xi, \quad \xi|_{\partial\mathbf{\Omega}} = 0, \tag{2.2}$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_1 & 0 \\ 0 & 0 & \tau_2 \end{pmatrix}, D = \begin{pmatrix} -d & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix}, J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A steady state (u, v, w) is stable if all the eigenvalues of (2.2) have negative real part and unstable if at least one of the eigenvalues has positive real part. For a critical point (u, v, w) of Φ, we let $\Phi''(u, v, w)$ denote the second Frechet derivative of Φ at (u, v, w). A critical point (u, v, w) is non-degenerate if the null space of $\Phi''(u, v, w)$ is trivial.

Let $E = H_0^1(\Omega)$. If A is a self-adjoint Fredholm operator on E, there is a unique A-invariant orthogonal splitting

$$E = E_+(A) \oplus E_-(A) \oplus E_0(A)$$

with $E_+(A)$, $E_-(A)$ and $E_0(A)$ being respectively the subspaces on which A is positive definite, negative definite and null. For a pair of self-adjoint Fredholm operators A and $\bar{A}$, a relative Morse index $i(A, \bar{A})$ is defined by

$$\text{(2.3)} \quad i(A, \bar{A}) = \dim(E_-(\bar{A}) \cap E_-(A)^\perp) - \dim(E_-(\bar{A})^\perp \cap E_-(A)).$$

Let Q^+ and Q^- be the orthogonal projections from E to $E_+(J)$ and $E_-(J)$ respectively. Define $\Psi_0 = T^{-\frac{1}{2}}(D\Delta - \nabla^2 H(u, v, w))T^{-\frac{1}{2}}$, $\psi_1 = Q^- \Psi_0 Q^-$ and $\psi_2 = Q^+ \Psi_0 Q^+$. Set $\mathfrak{D} = H^2(\mathbf{\Omega}, \mathbb{R}^3) \cap H_0^1(\mathbf{\Omega}, \mathbb{R}^3)$,

$$\rho_i(\psi_1) = \inf_{z \in \mathfrak{D}} \frac{\langle \psi_1 z, z \rangle_{L^2}}{\|Q^- z\|_{L^2}^2}$$

and

$$\rho_s(\psi_2) = \sup_{z \in \mathfrak{D}} \frac{\langle \psi_2 z, z \rangle_{L^2}}{\|Q^+ z\|_{L^2}^2}.$$

Theorem 2.1. *Suppose* $\dim E_0(\Phi''(\bar{u}, \bar{v}, \bar{w})) = 0$. (i)*If* $i(-J, \Phi''(\bar{u}, \bar{v}, \bar{w})) \neq 0$, *then for any positive* τ_1 *and* τ_2, $(\bar{u}, \bar{v}, \bar{w})$ *is an unstable steady state of* (1.2). (ii)*If* $i(-J, \Phi''(\bar{u}, \bar{v}, \bar{w})) = 0$ *and* $\rho_i(\psi_1) > \rho_s(\psi_2)$, *then* $(\bar{u}, \bar{v}, \bar{w})$ *is stable.*

We refer to [6] for a proof of Theorem 2.1.

Proof. [Proof of Theorem 1.2]. Let u^* be a non-degenerate minimizer of I, $v^* = \mathcal{L}_1 u^*$ and $w^* = \mathcal{L}_2 u^*$. By an argument used in [6], we know that $i(J, \Phi''(u^*, v^*, w^*)) = 0$. Direct calculation gives

$$\Psi_0 = \begin{pmatrix} -d\Delta - f'(u^*) & \tau_1^{-\frac{1}{2}} & \tau_2^{-\frac{1}{2}} \\ \tau_1^{-\frac{1}{2}} & (d_1\Delta - \gamma_1)\tau_1^{-1} & 0 \\ \tau_2^{-\frac{1}{2}} & 0 & (d_2\Delta - \gamma_2)\tau_2^{-1} \end{pmatrix}$$

Note that $f'(u) = -3u^2 + 2(\beta+1)u - \beta \le \dfrac{\beta^2-\beta+1}{3}$. It is easy to check that $\rho_i(\psi_1) = \rho_i(-d\Delta - f'(u^*)) \ge d\lambda_1 - \frac{\beta^2-\beta+1}{3} \ge -\frac{\beta^2-\beta+1}{3}$, where $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \lambda_k \le \cdots$ are eigenvalues of $-\Delta$. Since $\rho_s(\psi_2) = \max\{\rho_s((d_1\Delta - \gamma_1)\tau_1^{-1}), \rho_s((d_2\Delta-\gamma_2)\tau_2^{-1})\} = \max\left\{-\dfrac{(d_1\lambda_1+\gamma_1)}{\tau_1}, -\dfrac{(d_2\lambda_2+\gamma_2)}{\tau_2}\right\} < \max\left\{-\dfrac{\gamma_1}{\tau_1}, -\frac{\gamma_2}{\tau_2}\right\}$. Applying Theorem 2.1 completes the proof. ■

3. Quasi-monotone Systems

In this section, an ordered method is used to study the existence of steady states of (1.1). Let $U = (u_1,\ u_2, ...,\ u_m)$ and $V = (v_1,\ v_2, ...,\ v_m)$ be continuous functions on Ω. Denoted by $U \le V$ if for each $i = 1, ..., m$, $u_i(x) \le v_i(x)$ on Ω. Furthermore, $U < V$ if $U \le V$ and $U \ne V$, and $U \ll V$ if $u_i(x) < v_i(x)$ for all $i = 1, ..., m$, $x \in \Omega$. If $U \le V$, let $[U,\ V]$ be an ordered interval defined by $[U,\ V] = \{W : U \le W \le V\}$.

Let $\mathcal{F}(U) = (\mathcal{F}_1(U), ..., \mathcal{F}_m(U))$ and consider an elliptic system

$$\begin{aligned} -\Delta U &= \mathcal{F}(U) \qquad x \in \Omega, \\ U &= \mathbf{0} \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

System (3.1) is *quasi-monotone* if $\dfrac{\partial \mathcal{F}_i}{\partial u_j} \ge 0$ for $i \ne j$ and there is a $K > 0$ such that $|\dfrac{\partial \mathcal{F}_i}{\partial u_i}| \le K$ for $i = 1, ..., m$. Let $D(\Omega) = \{\ U = (u_1,\ u_2, ...,\ u_m) \mid 0 \le U$ and $u_i \in C_0^\infty(\Omega)\ \}$. A function W is a *supersolution* of (3.1) if $W \ge \mathbf{0}$ on $\partial\Omega$ and $\int_\Omega W(-\Delta\Psi)dx \ge \int_\Omega \mathcal{F}(W)\Psi dx$ for every $\Psi \in D(\Omega)$. A *supersolution* W is said to be *strict* if $W > \mathbf{0}$ on $\partial\Omega$ and there exists a $\Psi \in D(\Omega)$ such that $\int_\Omega W(-\Delta\Psi)dx > \int_\Omega \mathcal{F}(W)\Psi dx$. A *subsolution* can be defined in the same manner with reversed inequalities.

Notice that if (f_1)-(f_3) are satisfied, (1.5) can be converted to a quasi-monotone system through the transformation

$$z = \tilde{d}_1 u - \frac{1}{\sigma_1}v, \quad y = \tilde{d}_2 u - \frac{1}{\sigma_2}w.$$

Straightforward calculation shows that (u, v, w) is a solution of (1.5) if and only if (u, z, y) satisfies

$$\begin{aligned} -d\Delta u &= f(u) - (\tilde{d}_1\sigma_1 + \tilde{d}_2\sigma_2)u + \sigma_1 z + \sigma_2 y, \\ -d_1\Delta z &= f(u) + M^* u - (\gamma_1 - \sigma_1)z + \sigma_2 y, \\ -d_2\Delta y &= f(u) + M^* u + \sigma_1 z - (\gamma_2 - \sigma_2)y, \\ u|_{\partial\Omega} &= z|_{\partial\Omega} = y|_{\partial\Omega} = 0, \end{aligned} \tag{3.2}$$

where

$$M^* = M + \frac{2\beta^2 - 5\beta + 2}{9} - (\tilde{d}_1\sigma_1 + \tilde{d}_2\sigma_2).$$

Define

$$\tilde{f}(u) = \begin{cases} -\beta u & \text{if} \quad u < 0 \\ f(u) & \text{if} \quad 0 \le u \le 1 \\ (\beta - 1)u & \text{if} \quad u \ge 1. \end{cases}$$

We remark that $\tilde{f} \in C^{1,1}(\mathbb{R})$ and the function $\tilde{f}(s) - (\tilde{d}_1\sigma_1 + \tilde{d}_2\sigma_2)s$ has zeros at $0,\ \theta_*$ and θ^*. It is easy to check that $0 < \theta^- < \theta_* < \theta^* < \theta^+$ and $\int_0^{\theta^*} (\tilde{f}(s) - (\tilde{d}_1\sigma_1 + \tilde{d}_2\sigma_2)s)ds > 0$. As to be seen, the solutions we are looking for actually satisfy $0 \le u(x) \le 1$; in what follows we may take $\tilde{f}$ to replace f in (3.2).

By constructing two pairs of subsolutions and supersolutions, it will be shown that (1.5) possesses two positive solutions.

Theorem 3.1. *Assume that* (f_1)-(f_3) *are satisfied. There exists a* $\rho > 0$ *such that if* $\Omega \supset B_R$ *and* $R \ge \rho$ *then* (1.5) *possesses two solutions* (u, v, w) *and* $(\tilde{u}, \tilde{v}, \tilde{w})$ *in the order*

$$(0, 0, 0) < (u, v, w) < (\tilde{u}, \tilde{v}, \tilde{w}) < (\theta_+,\ \frac{\theta_+}{\gamma_1},\ \frac{\theta_+}{\gamma_2}).$$

In case $\Omega = B_R$*,* $u,\ v,\ w,\ \tilde{u},\ \tilde{v},\ \tilde{w}$ *are strictly decreasing functions of* $|x|$*,*

$$\theta^- < u(0) < \theta^+ \ , \quad v(0) < \frac{1}{\gamma_1}\theta^+ \quad \textit{and} \quad w(0) < \frac{1}{\gamma_2}\theta^+, \tag{3.3}$$

and

$$u(x) < \theta^* \quad \textit{if} \quad |x| \ge \rho. \tag{3.4}$$

We now state a theorem which will be used to prove Theorem 3.1.

Theorem 3.2. *Let* $(\overline{V},\ \overline{W})$ *and* $(\widetilde{V},\ \widetilde{W})$ *be two pairs of subsolutions and supersolutions of* (3.1) *with* $\overline{W},\ \widetilde{V}$ *being strict and in the order of*

$$\overline{V} < \overline{W} < \widetilde{W}, \quad \overline{V} < \widetilde{V} < \widetilde{W}, \quad \overline{W} \not\ge \widetilde{V}.$$

Then (3.1) *has two solutions* $\overline{U},\ \widetilde{U}$ *which are in the order of* $\overline{V} \le \overline{U} < \overline{W}$ *and* $\widetilde{V} < \widetilde{U} \le \widetilde{W}$*. Moreover, if* $\overline{U} \ll \overline{W}$ *and* $\widetilde{U} \gg \widetilde{V}$ *then there exists a solution* U *such that*

$$V_{\min} < U < W_{\max}, \quad U \not\le \overline{W}, \quad \textit{and} \quad U \not\ge \widetilde{V},$$

where $V_{\min}$ *and* $W_{\max}$ *are respectively the minimal and maximal solutions in* $[\overline{V},\ \widetilde{W}]$.

Theorem 3.2 is a direct application of a fixed point theorem due to Amann [1, 2]. In case $\Omega = B_R$ the next theorem, obtained in [22], gives the radial symmetry of solutions.

Theorem 3.3. *Let* $U = (u_1, ..., u_m)$ *bs a positive solution of* (3.1) *on* B_R. *Then, for each* i, u_i *is radially symmetric and* $\dfrac{\partial u_i}{\partial r} < 0$ *if* $r = |x|$ *and* $r \in (0,\ R)$.

In the proof of Theorem 3.1, two pairs of subsolutions and supersolutions will be selected to satisfy the hypotheses of Theorem 3.2.

Proof. [Proof of Theorem 3.1]. Consider the boundary value problem

$$\begin{aligned} &-d\Delta\varphi = f(\varphi) - (\tilde{d}_1\sigma_1 + \tilde{d}_2\sigma_2)\varphi, \quad x \in B_\rho, \\ &\varphi = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial B_\rho. \end{aligned} \tag{3.5}$$

If ρ is sufficiently large then (3.5) has a positive solution φ satisfying $\theta_* < \|\varphi\|_\infty < \theta^*$(see e.g. [7]). By taking a fixed ρ, this function φ will be used to construct a subsolution of (3.2). Let $(\overline{V},\ \overline{W})$ be the first pair and $(\widetilde{V},\ \widetilde{W})$ be the second pair of subsolutions and supersolutions of (3.2):

$$\overline{V} = (0,0,0), \overline{W} = (\ \theta^-\ ,\ (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^-\ ,\ (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^-\ )$$

and

$$\widetilde{V} = (\hat{\varphi},0,0), \widetilde{W} = (\ \theta^+\ ,\ (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^+\ ,\ (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^+\ ),$$

where

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in B_\rho \\ 0 & \text{if } x \in \Omega \setminus B_\rho. \end{cases}$$

Furthermore, it is easily seen that $\overline{V} < \overline{W} < \widetilde{W}$, $\overline{V} < \widetilde{V} < \widetilde{W}$ and $\widetilde{V} \not\leq \overline{W}$. As a consequence of Theorem 3.2, we obtain two solutions $\overline{U} = (\bar{u}, \bar{z}, \bar{y})$ and $\widetilde{U} = (\tilde{u}, \tilde{z}, \tilde{y})$ of (3.2) which in the order $\overline{V} \leq \overline{U} < \overline{W}$ and $\widetilde{V} < \widetilde{U} \leq \widetilde{W}$. Also, as stated in Theorem 3.2, if $\overline{U} \ll \overline{W}$ and $\widetilde{U} \gg \widetilde{V}$, then there is a solution $U = (u, z, y)$ of (3.2) which satisfying

$$V_{\min} < U < W_{\max}\,, \quad U \not\leq \overline{W} \quad \text{and} \quad U \not\geq \widetilde{V}, \tag{3.6}$$

where $V_{\min}$ and $W_{\max}$ are respectively the minimal and maximal solutions in $[\overline{V}, \widetilde{W}]$.

We now verify that $\overline{U} \ll \overline{W}$ and $\widetilde{U} \gg \widetilde{V}$. Clearly,

$$0 \leq \bar{u}(x) \leq \theta^-, \quad 0 \leq \bar{z}(x) \leq (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^- \ \text{ and } \ 0 \leq \bar{y}(x) \leq (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^-.$$

We are going to apply the maximum principle to show that $\overline{U} \ll \overline{W}$; that is,

$$\bar{u}(x) < \theta^- \quad , \quad \bar{z}(x) < (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^- \quad \text{and} \quad \bar{y}(x) < (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^-.$$

Straightforward calculation yields

$$\begin{aligned}(-d\Delta + \delta)(\bar{u} - \theta^-) &\le f(\bar{u}) + \sigma_1(\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^- + \sigma_2(\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^- - \delta\theta^- \\ &= f(\bar{u}) - \tau\theta^- \le 0.\end{aligned}$$

By the strong maximum principle, we know that $\bar{u}(x) < \theta^-$ for all $x \in \Omega$. Likewise,

$$(-d_1\Delta + (\gamma_1 - \sigma_1))(\bar{z} - (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^-) \le f(\bar{u}) - \tau\theta^- + M^*(\bar{u} - \theta^-) \le 0 \tag{3.7}$$

and

$$(-d_2\Delta + (\gamma_2 - \sigma_2))(\bar{y} - (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^-) \le f(\bar{u}) - \tau\theta^- + M^*(\bar{u} - \theta^-) \le 0. \tag{3.8}$$

Again, with the aid of maximum principle, we have $\bar{z}(x) < (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^-$ and $\bar{y}(x) < (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^-$ for all $x \in \Omega$. Also, $\widetilde{U} \gg \widetilde{V}$ can be verified in a similar way. Indeed, if $\delta = (\tilde{d}_1\sigma_1 + \tilde{d}_2\sigma_2)$, by the Mean Value Theorem

$$\begin{aligned}(-d\Delta + \omega)(\tilde{u} - \varphi) &= f(\tilde{u}) - \delta\tilde{u} + \sigma_1\tilde{z} + \sigma_2\tilde{y} + \omega\tilde{u} - f(\varphi) + \delta\varphi - \omega\varphi \\ &\ge (f'(\nu\tilde{u} + (1-\nu)\varphi) - \delta + \omega)(\tilde{u} - \varphi) \quad \text{for some } \nu \in [0,\ 1].\end{aligned}$$

If ω is sufficiently large, $(-d\Delta + \omega)(\tilde{u} - \varphi) \ge 0$ on B_ρ and $(-d\Delta + \omega)(\tilde{u}) \ge 0$ on Ω. It follows from the strong maximum principle that $\tilde{u}(x) > \hat{\varphi}(x)$ in Ω. A direct calculation gives $(-d_1\Delta + (\gamma_1 - \sigma_1))\tilde{z} = f(\tilde{u}) + M^*\tilde{u} + \sigma_2\tilde{y} \ge 0$ and $(-d_2\Delta + (\gamma_2 - \sigma_2))\tilde{y} = f(\tilde{u}) + M^*\tilde{u} + \sigma_1\tilde{z} \ge 0$. Then the strong maximum principle implies $\tilde{z} > 0$ and $\tilde{y} > 0$ in Ω.

Recall from (3.6) that

$$(0,\ 0,\ 0) < U = (u, z, y) < W_{\max} < \widetilde{W}, \tag{3.9}$$

$$U \not\le \overline{W} = (\ \theta^-\ ,\ (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^-\ ,\ (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^-\ ) \tag{3.10}$$

and

$$U \not\ge \widetilde{V} = (\hat{\varphi}, 0, 0). \tag{3.11}$$

We claim that

(3.12) $$\text{there is a } \xi \in \Omega \text{ such that } u(\xi) > \theta^-;$$

for otherwise (3.7)-(3.8)would imply $z \le (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^-$ and $y \le (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^-$, which would be contrary to (3.10). In view of (3.11) there exists a $\zeta \in B_\rho$ such that

(3.13) $$u(\zeta) < \hat{\varphi}(\zeta) < \theta^*.$$

Next, we claim $U \gg (0,\ 0,\ 0)$ in Ω. It is clear from (3.9) that $0 \le u(x) \le \theta^+,\ 0 \le z(x) \le (\tilde{d}_1 - \frac{1}{\gamma_1\sigma_1})\theta^+$ and $0 \le y(x) \le (\tilde{d}_2 - \frac{1}{\gamma_2\sigma_2})\theta^+$ for all $x \in \Omega$. A direct calculation shows that $(-d\Delta + \omega)u \ge 0$ if ω is sufficiently large. The strong maximum principle implies that $u > 0, z > 0$ and $y > 0$ in Ω. Let $v = \sigma_1(\tilde{d}_1 u - z)$ and $w = \sigma_2(\tilde{d}_2 u - y)$. Since $(-d_1\Delta + \gamma_1)v = u$ and $(-d_2\Delta + \gamma_2)w = u$, the strong maximum principle implies that $v > 0$ and $w > 0$ in Ω. Likewise, $u < \theta^+$ and $(-d_1\Delta + \gamma_1)(v - \frac{1}{\gamma_1}\theta^+) = u - \theta^+$ imply that $v < \frac{1}{\gamma_1}\theta^+$. The same lines of reasoning as above shows that $w < \frac{1}{\gamma_2}\theta^+$.

Before proving (3.3) and (3.4), we state a Proposition.

Proposition 3.1. *Suppose* $(u,\ z,\ y)$ *is a radially symmetric positive solution of* (3.2) *and* $\frac{\partial u}{\partial r},\ \frac{\partial z}{\partial r},\ \frac{\partial y}{\partial r} < 0$ *on* $(0,\ R]$. *If* $v = \sigma_1(\tilde{d}_1 u - z)$ *and* $w = \sigma_2(\tilde{d}_2 u - y)$ *then* $(u,\ v,\ w)$ *is a positive solution of* (1.5) *and* $\frac{\partial v}{\partial r},\ \frac{\partial w}{\partial r} < 0$ *on* $(0,\ R]$.

If $\Omega = B_R$ and $R \ge \rho$, we know from Proposition 3.1 that u, v, w are radially symmetric solutions and strictly decreasing in r. This together with (3.13) gives (3.4).

From the proof of (3.12), we know (3.3) holds. Now the proof of Theorem 3.1 is complete. ∎

Proof. [Proof of Proposition 3.1]. It is clear that v and w are radially symmetric. Since $(-d_1\Delta+\gamma_1)v = u$ and $(-d_2\Delta+\gamma_2)w = u$, the strong maximum principle implies that $v > 0$ and $w > 0$ in B_R.

To show that $\frac{\partial v}{\partial r} < 0$ on $(0, R]$, we note that v satisfies

(3.14) $$\begin{aligned} &-d_1(\frac{\partial^2 v}{\partial r^2} + \frac{N-1}{r}\frac{\partial v}{\partial r}) + \gamma_1 v = u, \quad r \in (0,\ R), \\ &\frac{\partial v}{\partial r}(R) < 0. \end{aligned}$$

Letting $\bar{v} = \dfrac{\partial v}{\partial r}$ and differentiating (3.14) with respect to r, we have

(3.15)
$$-d_1\left(\frac{\partial^2\bar{v}}{\partial r^2}+\frac{N-1}{r}\frac{\partial\bar{v}}{\partial r}\right)+\left(\frac{d_1(N-1)}{r^2}+\gamma_1\right)\bar{v}=\tfrac{\partial u}{\partial r}<0,\ r\in(0,\ R),$$
$$\bar{v}(R)<0.$$

We claim $\bar{v} < 0$ on $(0,\ R]$; for otherwise there is an $r_0 \in (0, R)$ such that $\bar{v}(r_0) = \max\limits_{r\in(0,R]} \bar{v}(r) \geq 0$. Simple calculation shows that the left hand side of (3.15) is non-negative at r_0, which is contrary to $\dfrac{\partial u}{\partial r}(r_0) < 0$. Similarly, $\dfrac{\partial w}{\partial r} < 0$ on $(0,\ R]$. The proof is complete. ■

Remark 3.1. In the proof of Theorem 3.1, we cannot rule out the possibility of $\bar{U}$ being the constant solution $(0, 0, 0)$.

4. Standing Pulses

Recall from Theorem 3.1, there is a solution $U = (u, z, y)$ of (3.2) which satisfying (3.6). If $v = \sigma_1(\tilde{d}_1 u - z)$ and $w = \sigma_2(\tilde{d}_2 u - y)$ then $(u,\ v,\ w)$ is a positive solution of (1.5). In case $\Omega = B_R$, this solution will be denoted by $(u_R(r),\ v_R(r),\ w_R(r))$. A standing pulse will be obtained as the limit of a sequence of solutions $\{(u_R, v_R, w_R)\}$ as $R \to \infty$. We start with a technical Lemma.

Lemma 4.1. *If* (f_1)-(f_4) *are satisfied, then for all* $r \in [0,\ R]$

$$du_R'(r)^2 - d_1 v_R'(r)^2 - d_2 w_R'(r)^2 \geq 0. \tag{4.1}$$

Proof. For simplicity in notation, the subscript R will be suppressed from $(v_R,\ w_R,\ u_R)$. Note that

$$du'(r)^2 - d_1 v'(r)^2 - d_2 w'(r)^2 = d[u'(r)^2 - \frac{1}{\tilde{d}_1}v'(r)^2 - \frac{1}{\tilde{d}_2}w'(r)^2].$$

Since $u'(r) < 0,\ v'(r) < 0$ and $w'(r) < 0$, it follows that

$$\begin{aligned}
&u'(r)^2 - \frac{1}{\tilde{d}_1}v'(r)^2 - \frac{1}{\tilde{d}_2}w'(r)^2 \\
&\geq u'(r)^2 - \left(\frac{1}{\sqrt{\tilde{d}_1}}v'(r) + \frac{1}{\sqrt{\tilde{d}_2}}w'(r)\right)^2 \\
&= \left[u'(r) - \frac{1}{\sqrt{\tilde{d}_1}}v'(r) - \frac{1}{\sqrt{\tilde{d}_2}}w'(r)\right]\left[u'(r) + \frac{1}{\sqrt{\tilde{d}_1}}v'(r) + \frac{1}{\sqrt{\tilde{d}_2}}w'(r)\right].
\end{aligned}$$

Moreover, $z'(r) < 0,\ y'(r) < 0$ and (f_4) imply that

$$
\begin{aligned}
& u'(r) - \frac{1}{\sqrt{\tilde{d}_1}} v'(r) - \frac{1}{\sqrt{\tilde{d}_2}} w'(r) \\
= \ & \left[1 - (\sigma_1\sqrt{\tilde{d}_1} + \sigma_2\sqrt{\tilde{d}_2})\right] u'(r) + \frac{\sigma_1}{\sqrt{\tilde{d}_1}} z'(r) + \frac{\sigma_2}{\sqrt{\tilde{d}_2}} y'(r) \le 0.
\end{aligned}
$$

The proof is complete. ∎

Proof. [Proof of Theorem 1.3]. In view of (3.3), for every $m \in \mathbb{N}$ if $R \ge m\rho$,

$$
\|u_R\|_{L^\infty(B_{m\rho})} \le \theta^+ \ , \quad \|v_R\|_{L^\infty(B_{m\rho})} \le \frac{1}{\gamma_1}\theta^+ \ , \quad \|w_R\|_{L^\infty(B_{m\rho})} \le \frac{1}{\gamma_2}\theta^+
$$

and

$$
\|f(u_R)\|_{L^\infty(B_{m\rho})} \le \sup_{0 \le x \le \rho_0^+} |f(x)|.
$$

By the standard elliptic estimates [11], $\{u_R : R \ge m\rho\}$ are bounded in $C^{2,1}(\bar{B}_{m\rho})$ and thus precompact in $C^2(\bar{B}_{m\rho})$. Then through a diagonal process, a subsequence of $\{(u_R,\ v_R,\ w_R)\}$ converges to $(u,\ v,\ w)$ in C^2 on compact subsets of $\mathbb{R}^N$. Clearly, $(u,\ v,\ w)$ is a radially symmetric solution of (1.5) on $\mathbb{R}^N$, $u'(r) \le 0,\ v'(r) \le 0$ and $w'(r) \le 0$ for all $r > 0$. Furthermore,

$$
u(0) = \max_{x \in \mathbb{R}^N} u(x),\ v(0) = \max_{x \in \mathbb{R}^N} v(x),\ w(0) = \max_{x \in \mathbb{R}^N} w(x)
$$

and (3.3) implies that

$$
\theta^- \le u(0) < \theta^+ \ , \ 0 < v(0) < \frac{1}{\gamma_1}\,\theta^+ \ , \ 0 < w(0) < \frac{1}{\gamma_2}\,\theta^+ \ .
$$

Therefore $(u,\ v,\ w)$ is not the constant solution $(0,\ 0,\ 0)$. Also, (3.4) rules out the possibility of $(u,\ v,\ w)$ being the constant solution $(\theta^+,\ \frac{1}{\gamma_1}\theta^+, \frac{1}{\gamma_2}\theta^+)$.

Next, we are going to prove that $u(x) \to 0,\ v(x) \to 0,\ w(x) \to 0$ as $|x| \to \infty$. Let $l_u = \lim_{r\to\infty} u(r) = \inf_{r>0} u(r)$, $l_v = \lim_{r\to\infty} v(r) = \inf_{r>0} v(r)$ and $l_w = \lim_{r\to\infty} w(r) = \inf_{r>0} w(r)$. We claim

$$
\text{(4.2)} \qquad l_u \in \{\ 0\ ,\ \theta^-\ ,\ \theta^+\ \}, \quad l_v = \frac{1}{\gamma_1} l_u, \quad \text{and} \quad l_w = \frac{1}{\gamma_2} l_u.
$$

We will also show that the case of $l_u = \theta^-$ or $l_u = \theta^+$ cannot be true, and thus $l_u = l_v = l_w = 0$ must hold.

Since u is non-increasing, (3.3) implies $l_u \le u(R_0) \le \theta^* < \theta^+$, from which we know $l_u \ne \theta^+$. We now prove (4.2). Let prime denotes differentiation with respect to r. Then

$$-du'' - \frac{d(N-1)}{r}u' \;=\; f(u) - v - w, \tag{4.3}$$

$$-d_1v'' - \frac{d_1(N-1)}{r}v' \;=\; u - \gamma_1 v, \tag{4.4}$$

$$-d_2w'' - \frac{d_2(N-1)}{r}w' \;=\; u - \gamma_2 w, \tag{4.5}$$

Let $u_0 = u(0),\ v_0 = v(0),\ w_0 = w(0)$ and $F(u) = \int_0^u f(s)ds$. Multiplying (4.3) by $(-u')$, (4.4) by $(-v')$ and (4.5) by $(-w')$, and integrating from 0 to R, we get

$$\begin{aligned} &\frac{d}{2}u'(R)^2 + d(N-1)\int_0^R \frac{u'^2}{r}dr - \int_0^R u'vdr - \int_0^R u'wdr \\ &= \; F(u_0) - F(u(R)), \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\frac{d_1}{2}v'(R)^2 + d_1(N-1)\int_0^R \frac{v'^2}{r}dr - \int_0^R u'vdr \\ &= \; -(u(R)v(R) - u_0v_0) + \frac{\gamma_1}{2}(v(R)^2 - v_0^2) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\frac{d_2}{2}w'(R)^2 + d_2(N-1)\int_0^R \frac{w'^2}{r}dr - \int_0^R u'wdr \\ &= \; -(u(R)w(R) - u_0w_0) + \frac{\gamma_2}{2}(w(R)^2 - w_0^2). \end{aligned} \tag{4.8}$$

Subtracting (4.7) and (4.8) from (4.6) yields

$$\begin{aligned} &\frac{1}{2}(du'(R)^2 - d_1v'(R)^2 - d_2w'(R)^2) + (N-1)\int_0^R \frac{du'^2 - d_1v'^2 - d_2w'^2}{r}dr \\ &= \; F(u_0) - F(u(R)) + (u(R)v(R) - u_0v_0) + (u(R)w(R) - u_0w_0) \\ &-\frac{\gamma_1}{2}(v(R)^2 - v_0^2) - \frac{\gamma_2}{2}(w(R)^2 - w_0^2). \end{aligned} \tag{4.9}$$

Note that $u'(r) \le 0,\ v'(r) \le 0$ and $w'(r) \le 0$. Then the boundedness of u implies that the left hand side of (4.6) is bounded and positive. Hence $\lim\limits_{r\to\infty} u'(r)$ exists. This together with $\lim\limits_{r\to\infty} u(r) = l_u$ implies that $u'(r) \to 0$ as $r \to \infty$. By the same lines of reasoning, $v'(r)\to 0$ and $w'(r)\to 0$ as $r\to\infty$. Then we see from (4.3)-(4.5) that $-du''(r)\to f(l_u)-l_v-l_w$, $-d_1v''(r)\to l_u-\gamma_1 l_v$ and $-d_2w''(r)\to l_u-\gamma_2 l_w$ as $r\to\infty$. Consequently $f(l_u)-l_v-l_w=0$, $l_u-\gamma_1 l_v=0$ and $l_u-\gamma_2 l_w=0$. This completes the proof of (4.2).

Having known that

$$\int_0^{\theta^-}\left(f(s)-\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)s\right)ds<0,$$

We next show that

$$\int_0^{l_u}\left(f(s)-\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)s\right)ds\ge 0 \tag{4.10}$$

to rule out the possibility of $l_u=\theta^-$. If $N>1$, letting $R\to\infty$, we see from (4.2) and (4.6) that

$$\begin{aligned}&(N-1)\int_0^\infty\frac{du'^2-d_1v'^2-d_2w'^2}{r}dr\\ =\;& F(u_0)-F(l_u)-u_0v_0-u_0w_0+\frac{1}{2}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)l_u^2+\frac{\gamma_1}{2}v_0^2+\frac{\gamma_2}{2}w_0^2.\end{aligned}\tag{4.11}$$

On the other hand, since (4.6) also holds if $(u,\ v,\ w)$ is replaced by $(u_R,\ v_R,\ w_R)$, it follows from $u_R(R)=0,\ v_R(R)=0$ and $w_R(R)=0$ that

$$\begin{aligned}&\frac{1}{2}(du_R'(R)^2-d_1v_R'(R)^2-d_2w_R'(R)^2)\\ &+(N-1)\int_0^R\frac{du_R'^2-d_1v_R'^2-d_2w_R'^2}{r}dr\\ =\;& F(u_R(0))-u_R(0)v_R(0)-u_R(0)w_R(0)+\frac{\gamma_1}{2}v_R(0)^2+\frac{\gamma_2}{2}w_R(0)^2.\end{aligned}\tag{4.12}$$

If $\rho\le R$, (4.12) and (4.1) imply that

$$\begin{aligned}(N-1)\int_0^\rho\frac{du_R'^2-d_1v_R'^2-d_2w_R'^2}{r}dr\;&\le\;(N-1)\int_0^R\frac{du_R'^2-d_1v_R'^2-d_2w_R'^2}{r}dr\\ &\le\; F(u_R(0))-u_R(0)v_R(0)-u_R(0)w_R(0)+\frac{\gamma_1}{2}v_R(0)^2+\frac{\gamma_2}{2}w_R(0)^2.\end{aligned}$$

Since $u_R\to u,\ v_R\to v$ and $w_R\to w$ on compact subsets of $\mathbb{R}^N$,

$$(N-1)\int_0^\rho\frac{du'^2-d_1v'^2-d_2w'^2}{r}dr\le F(u_0)-u_0v_0-u_0w_0+\frac{\gamma_1}{2}v_0^2+\frac{\gamma_2}{2}w_0^2. \tag{4.13}$$

Note that (4.13) holds for every $\rho > 0$. Letting $\rho \to \infty$ yields

$$(4.14)\quad (N-1)\int_0^\infty \frac{du'^2 - d_1 v'^2 - d_2 w'^2}{r} dr \le F(u_0) - u_0 v_0 - u_0 w_0 + \frac{\gamma_1}{2} v_0^2 + \frac{\gamma_2}{2} w_0^2.$$

Combining (4.11) with (4.14) gives (4.10).

For $N = 1$, (4.10) easily follows from a simpler argument. The proof is complete. ∎

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