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QUASILINEARITY OF THE CLASSICAL SETS OF SEQUENCES OF FUZZY NUMBERS AND SOME RELATED RESULTS

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Abstract. In the present paper, we prove that the classical sets $\ell_\infty(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$ of sequences of fuzzy numbers are normed quasilinear spaces and the $\beta-$, $\alpha-$duals of the set $\ell_1(F)$ is the set $\ell_\infty(F)$. Besides this, we show that $\ell_\infty(F)$ and $c(F)$ are normed quasialgebras and an operator defined by an infinite matrix belonging to the class $(\ell_\infty(F) : \ell_\infty(F))$ is bounded and quasilinear. Finally, as an application, we characterize the class $(\ell_1(F) : \ell_p(F))$ of infinite matrices of fuzzy numbers and establish the perfectness of the spaces $\ell_\infty(F)$ and $\ell_1(F)$.

1. Introduction

Zadeh introduced the concepts of fuzzy sets and fuzzy set operations, in his significant article [16]. Subsequently several authors discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Especially, El Naschie [11] studied the E infinity theory which has very important applications in quantum particle physics.

By $w(F)$, we denote the set of all sequences of fuzzy numbers. Throughout the text, we suppose that $1 \le p < \infty$ with $p^{-1} + q^{-1} = 1$. We define the classical sets $\ell_\infty(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$ consisting of the bounded, convergent, null and absolutely p-summable sequences of fuzzy numbers, as follows:

$$\ell_\infty(F) = \left\{ (x_k) \in w(F) : \sup_{k\in\mathbb{N}} D(x_k, \overline{0}) < \infty \right\},$$

$$c(F) = \left\{ (x_k) \in w(F) : \exists l \in E^1 \ni \lim_{k\to\infty} D(x_k, l) = 0 \right\},$$

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$$c_0(F) = \left\{(x_k) \in w(F) : \lim_{k\to\infty} D(x_k, \overline{0}) = 0\right\},$$
$$\ell_p(F) = \left\{(x_k) \in w(F) : \sum_k D(x_k, \overline{0})^p < \infty\right\};$$

where the metric D is defined by (2.2) and E^1 denotes the set of fuzzy numbers. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. In [10], it was shown that $c(F)$ and $\ell_\infty(F)$ are complete metric spaces with the Haussdorff metric D_∞ defined by

$$D_\infty(x, y) = \sup_{k\in\mathbb{N}} D(x_k, y_k),$$

where $x = (x_k)$, $y = (y_k)$ are elements of the sets $c(F)$ or $\ell_\infty(F)$. Of course, $c_0(F)$ is also a complete metric space with respect to the Haussdorff metric D_∞. Further, Nanda [10] has introduced and proved that the set $\ell_p(F)$ is a complete metric space with the Haussdorff metric D_p defined by

$$D_p(x, y) = \left\{\sum_k [D(x_k, y_k)]^p\right\}^{1/p},$$

where $x = (x_k)$, $y = (y_k)$ are in $\ell_p(F)$.

Let $\mu_1(F)$, $\mu_2(F) \subset w(F)$ and $A = (a_{nk})$ be an infinite matrix of fuzzy numbers. Then, we say that A defines a matrix mapping from $\mu_1(F)$ into $\mu_2(F)$, and denote it by writing $A : \mu_1(F) \to \mu_2(F)$, if for every sequence $u = (u_k) \in \mu_1(F)$ the sequence $Au = \{(Au)_n\}$, the A-transform of u, exists and is in $\mu_2(F)$; where

$$(Au)_n := \sum_k a_{nk} u_k\ ,\ (n \in \mathbb{N}). \tag{1.1}$$

By $(\mu_1(F) : \mu_2(F))$, we denote the class of all matrices A such that $A : \mu_1(F) \to \mu_2(F)$. Thus, $A \in (\mu_1(F) : \mu_2(F))$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $u \in \mu_1(F)$, and we have $Au = \{(Au)_n\}_{n\in\mathbb{N}} \in \mu_2(F)$ for all $u \in \mu_1(F)$. A sequence u is said to be A-summable to α if Au converges to α which is called as the A-limit of u. We denote the n^{th} row of a matrix $A = (a_{nk})$ by A_n for all $n \in \mathbb{N}$, i.e. $A_n := \{a_{nk}\}_{k=0}^\infty$ for all $n \in \mathbb{N}$.

Mursaleen and Başarır [9] have recently introduced some new sets of sequences of fuzzy numbers generated by a non-negative regular matrix A some of which reduced to the Maddox spaces $\ell_\infty(p, F)$, $c(p, F)$, $c_0(p, F)$ and $\ell(p, F)$ of sequences of fuzzy numbers for the special cases of that matrix A. Quite recently; Talo and Başar [13] have extended the main results of Başar and Altay [3] to the fuzzy numbers. Finally, Talo and Başar [14] have introduced the spaces $\ell_\infty(F,f)$, $c(F,f)$, $c_0(F,f)$ and $\ell_p(F,f)$ of sequences of fuzzy numbers defined by a modulus function and given some topological properties of the spaces together with some inclusion relations.

In [2], Aseev introduced the concepts of quasilinear spaces and quasilinear operators which enable us to consider both linear and nonlinear spaces of subsets and multivalued mappings from a single point of view. Following Aseev [2], Rojas-Medar et al. have recently extended some results of linear functional analysis to the fuzzy context, in [8]. Their work has motivated us to study the quasilinearity of the classical sets of sequences of fuzzy numbers. The main emphasis of this paper is to study the quasilinearity of the classical sets $\ell_\infty(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$ of sequences of fuzzy numbers and to obtain the $\beta-$, $\alpha-$duals of the set $\ell_1(F)$, and to characterize the class of infinite matrices of fuzzy numbers from $\ell_1(F)$ to $\ell_p(F)$. Additionally, it is proved that $\ell_\infty(F)$ and $c(F)$ are normed quasialgebras and an operator defined by an infinite matrix belonging to the class $(\ell_\infty(F):\ell_\infty(F))$ is bounded and quasilinear.

The rest of this paper is organized, as follows:

Section 2 comprises some required definitions and results related with the quasilinear spaces, fuzzy numbers, and sequences and series of fuzzy numbers. Section 3 is devoted to the normed quasilinearity of the classical sets of sequences of fuzzy numbers. Furthermore, it is also proved in Section 3 that the normed quasilinear spaces $\ell_\infty(F)$ and $c(F)$ are normed quasialgebras and an operator defined by an infinite matrix belonging to the class $(\ell_\infty(F):\ell_\infty(F))$ is bounded and quasilinear. In Section 4, the $\beta-$, $\alpha-$duals of the set $\ell_1(F)$ are determined and the perfectness of the spaces $\ell_\infty(F)$, $\ell_1(F)$ is showed, and the characterization of the class $(\ell_1(F):\ell_p(F))$ of infinite matrices of fuzzy numbers is obtained. In the final section of the paper, the results are summarized, open problems and further suggestions are recorded. In order to give a full knowledge to the readers on the sets of sequences of fuzzy numbers, in addition to the references some new documents are listed at the end of the paper.

2. Preliminaries, Background and Notation

Following Aseev [2], we begin with defining the concepts of quasilinear space, normed quasilinear space and quasilinear operator:

Definition 2.1. A set X is called a *quasilinear space* if a partial order relation $\preceq$, an algebraic sum operation $(+)$ and an operation of multiplication by real numbers $(\cdot)$ are defined on it and satisfy the following properties for any elements $x,y,z,v\in X$ and any real numbers α,β:

(q.1) $x \preceq x$.

(q.2) $x \preceq y$, $y \preceq z \Rightarrow x \preceq z$.

(q.3) $x \preceq y$, $y \preceq x \Rightarrow x = y$.

(q.4) $x + y = y + x$.

(q.5) $x + (y + z) = (x + y) + z$.

(q.6) There exists an element $\theta \in X$, called neutral element, such that $x + \theta = x$.

(q.7) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$.

(q.8) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$.

(q.9) $1 \cdot x = x$.

(q.10) $0 \cdot x = \theta$.

(q.11) $(\alpha + \beta) \cdot x \preceq \alpha \cdot x + \beta \cdot x$.

(q.12) $x \preceq y$ and $z \preceq v \Rightarrow x + z \preceq y + v$.

(q.13) $x \preceq y \Rightarrow \alpha \cdot x \preceq \alpha \cdot y$.

An element $x' \in X$ is called an *inverse* of $x \in X$, if $x + x' = \theta$. Obviously, if an element x has an inverse x', then it is unique. If any element x in the quasilinear space X has an inverse element $x' \in X$, then the partial order on X is determined by equality and consequently X is a linear space with scalars in $\mathbb{R}$.

Definition 2.2. Let X be a quasilinear space. A real function $\|\cdot\|_X : X \to \mathbb{R}$ is called a norm if the following conditions hold for any $x, y \in X$ and any $\alpha \in \mathbb{R}$:

(n.1) $\|x\|_X > 0$ if $x \neq \theta$.

(n.2) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$.

(n.3) $\|\alpha \cdot x\|_X = |\alpha| \cdot \|x\|_X$.

(n.4) If $x \preceq y$, then $\|x\|_X \leq \|y\|_X$.

(n.5) If for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \preceq y + x_\varepsilon$ and $\|x_\varepsilon\|_X \leq \varepsilon$ then $x \preceq y$.

A quasilinear space X with a norm defined on it, is called a *normed quasilinear space*.

If any $x \in X$ has an inverse element $x' \in X$, then the concept of a normed quasilinear space coincides with the concept of a real normed linear space.

The Haussdorff metric H on a normed quasilinear space X is defined by

$$H(x,y) = \inf\left\{r \geq 0 : x \preceq y + a_1^r,\ y \preceq x + a_2^r,\ a_i^r \in X,\ \|a_i^r\| \leq r,\ i = 1,2\right\}.$$

Lemma 2.3. *The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is a continuous function with respect to the Hausdorff metric.*

Lemma 2.4. *The following statements hold:*

(i) *Suppose that $x_k \to x$ and $y_k \to y$ and that $x_k \preceq y_k$ for any positive integer k. Then $x \preceq y$.*

(ii) *Suppose that $x_k \to x$ and $z_k \to x$. If $x_k \preceq y_k \preceq z_k$ for any positive integer k, then $y_k \to x$.*

(iii) *If $x_k + y_k \to x$ and $y_k \to \theta$, then $x_k \to x$.*

Let W be the set of all closed bounded intervals A of real numbers with end-points $\underline{A}$ and $\overline{A}$, i.e. $A = [\underline{A}, \overline{A}]$. The operations addition and multiplication by a real number on W are defined, as follows:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} = [\underline{A} + \underline{B}, \overline{A} + \overline{B}] \\ \alpha A &= \{\alpha a : a \in A\}. \end{aligned}$$

Since W is a partially ordered set with respect to the relation subset or equal $(\subseteq)$, that is to say that

$$A \subseteq B \text{ if and only if } \underline{B} \leq \underline{A} \text{ and } \overline{A} \leq \overline{B},$$

W is a quasilinear space with the algebraic operations, above. Define the function $\|\cdot\|$ on W by

$$\|A\| = \max\{|\underline{A}|, |\overline{A}|\}.$$

Then W is a normed quasilinear space with this norm. The Hausdorff metric d obtained from this norm, on W is defined by (cf. Diamond and Kloeden [4])

$$d(A, B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can easily be observed that (W, d) is a complete metric space, (cf. Nanda [10]). Now, we define the quasilinearity of an operator.

Definition 2.5. Let X and Y be quasilinear spaces. If the operator $T : X \to Y$ satisfies the following conditions:

(o.1) $T(\alpha \cdot x) = \alpha \cdot Tx$;

(o.2) $T(x + y) \preceq Tx + Ty$;

(o.3) If $x \preceq y$, then $Tx \preceq Ty$;

for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$, then T is called a quasilinear operator. By $L(X : Y)$, we denote the set of all quasilinear operators from X to Y. Let $T_1, T_2 \in L(X : Y)$. Then we write $T_1 \preceq T_2$ whenever $T_1x \preceq T_2x$ for all $x \in X$. It is easy to see that $L(X : Y)$ is a quasilinear space with the usual algebraic operations addition and scalar multiplication of operators.

A quasilinear operator $T : X \to Y$ is said to be bounded if and only if there exists a $K > 0$ such that $\|Tx\|_Y \leq K\|x\|_X$ for all $x \in X$. By $B(X : Y)$, we denote the set of all bounded quasilinear operators from X to Y. It is natural that the order on the set $B(X : Y)$ can be defined by the similar way used in Definition 2.5, above. It is not hard to show that $B(X : Y)$ is a normed quasilinear space with the usual algebraic operations addition and scalar multiplication of operators, normed by

$$\|T\| = \sup_{\|x\|_X=1} \|Tx\|_Y.$$

Now, we give four known propositions concerned some properties of quasilinear operators of which the final one is analogous to the Banach-Steinhaus theorem.

Proposition 2.6. *Let $T \in L(X : Y)$. Then, we have:*

(*i*) $T\theta = \theta$.

(*ii*) *T is bounded if and only if T is continuous on $\theta \in X$.*

(*iii*) *If T is continuous on $\theta \in X$ then T is uniformly continuous on X.*

Proposition 2.7. *The following statements hold:*

(a) *Let $T \in B(X : Y)$. Then, the Lipschitz condition $\|Tx\|_Y \preceq \|T\|_B\|x\|_X$ holds for all $x \in X$.*

(b) *The composition $T_2 \circ T_1$ is in the quasilinear space $B(X : Z)$ if $T_1 \in B(X : Y)$ and $T_2 \in B(Y : Z)$.*

Proposition 2.8. *If the sequence $\{T_k\} \subset B(X : Y)$ is convergent for each $x \in X$ then the operator T defined by $Tx = \lim_{k\to\infty} T_kx$ is quasilinear.*

The operator T does not need to be bounded. But, if the space X is complete with respect to the metric defined on it, then $T \in B(X : Y)$.

Proposition 2.9. *Suppose that X is a complete normed quasilinear space, and Y a normed quasilinear space. Assume that the sequence $\{T_k\}$ of elements of the space $B(X : Y)$ is bounded at each point $x \in X$. Then the sequence $\{\|T_k\|\}$ of norms is also bounded.*

We continue by giving some required definitions. A *fuzzy number* is a fuzzy set on the real axis, i.e. a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following four conditions:

(i) u is normal, i.e. there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e. $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, (cf. Zadeh [16]).
We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and call it as *the space of fuzzy numbers*. λ*-level set* $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \lambda\} & , \quad (0 < \lambda \leq 1), \\ \overline{\{x \in \mathbb{R} : u(x) > \lambda\}} & , \quad (\lambda = 0), \end{cases}$$

where $\overline{\{x \in \mathbb{R} : u(x) > \lambda\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > \lambda\}$ in the usual topology of $\mathbb{R}$. The set $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) = \begin{cases} 1 & , \quad (x = r) \\ 0 & , \quad (x \neq r) \end{cases} .$$

Representation Theorem. [5]. *Let* $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ *for* $u \in E^1$ *and for each* $\lambda \in [0,1]$. *Then the following statements hold:*

(*i*) u^- *is a bounded and non-decreasing left continuous function on* $(0,1]$.

(*ii*) u^+ *is a bounded and non-increasing left continuous function on* $(0,1]$.

(*iii*) *The functions* u^- *and* u^+ *are right continuous at the point* $\lambda = 0$.

(*iv*) $u^-(1) \leq u^+(1)$.

Conversely, if the pair of functions α *and* β *satisfies the conditions (i)-(iv), then there exists a unique* $u \in E^1$ *such that* $[u]_\lambda = [\alpha(\lambda), \beta(\lambda)]$ *for each* $\lambda \in [0,1]$. *The fuzzy number* u *corresponding to the pair of functions* α *and* β *is defined by* $u : \mathbb{R} \to [0,1]$, $u(x) = \sup\{\lambda : \alpha(\lambda) \leq x \leq \beta(\lambda)\}$.

Let $u, v, w \in E^1$ and $\alpha \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on E^1 by

$$\begin{aligned} u + v = w \Leftrightarrow & \ [w]_\lambda = [u]_\lambda + [v]_\lambda \ \text{ for all } \lambda \in [0,1] \\ \Leftrightarrow & \ w^-(\lambda) = u^-(\lambda) + v^-(\lambda) \ \text{ and } w^+(\lambda) \\ = & \ u^+(\lambda) + v^+(\lambda) \ \text{ for all } \lambda \in [0,1], \end{aligned}$$

$$[\alpha u]_\lambda = \alpha[u]_\lambda \text{ for all } \lambda \in [0,1]$$

and

$$uv = w \Leftrightarrow [w]_\lambda = [u]_\lambda[v]_\lambda \text{ for all } \lambda \in [0,1],$$

where it is immediate that

$$w^-(\lambda) = \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

and

$$w^+(\lambda) = \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

for all $\lambda \in [0,1]$.

The partial ordering relation $\subseteq$ on E^1 is defined as follows:

$$u \subseteq v \Leftrightarrow u(x) \le v(x) \text{ for all } x \in \mathbb{R} \Leftrightarrow [u]_\lambda \subseteq [v]_\lambda \text{ for all } \lambda \in [0,1].$$

In spite of the product being commutative and associative, it is not distributive on addition in E^1. But the following relation holds

$$u \cdot (v + w) \subseteq u \cdot v + u \cdot w$$

for any $u, v, w \in E^1$. It is known that E^1 is a quasilinear space with the partial ordering relation and the algebraic operations, defined above, (see M. A. Rojas-Medar et al. [8]). The unit element with respect to the operation $(+)$ of the space E^1 is $\overline{0}$. We define the norm on E^1 by

$$\|u\| = \sup_{\lambda\in[0,1]} \max\{|u^-(\lambda)|, |u^+(\lambda)|\} = \max\{|u^-(0)|, |u^+(0)|\}. \tag{2.1}$$

Thus, E^1 is a normed quasilinear space with the norm defined by (2.1). The Hausdorff metric D obtained from the norm given by (2.1) is defined by

$$\begin{aligned} D(u,v) &= \sup_{\lambda\in[0,1]} d([u]_\lambda, [v]_\lambda) \\ &= \sup_{\lambda\in[0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}. \end{aligned} \tag{2.2}$$

It is immediate that $\|\cdot\| = D(\cdot, \overline{0})$.

Following Matloka [7], we now give some definitions concerned with the sequences of fuzzy numbers which are needed in the text.

Definition 2.10. A sequence $x = (x_k)$ of fuzzy numbers is a function x from the set $\mathbb{N}$ into the set E^1, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. The fuzzy number x_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the k^{th} term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers.

Definition 2.11. An infinite matrix $A = (a_{nk})$ of fuzzy numbers is a double sequence of fuzzy numbers defined by a function A from the set $\mathbb{N} \times \mathbb{N}$ into the set E^1. The fuzzy number a_{nk} denotes the value of the function at $(n, k) \in \mathbb{N} \times \mathbb{N}$ and is called as the element of the matrix which stands on the n^{th} row and k^{th} column.

Definition 2.12. A sequence $(u_k) \in w(F)$ is called convergent with limit $u \in E^1$, if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$D(u_k, u) < \varepsilon \text{ for all } k \geq n_0.$$

If the sequence $(u_k) \in w(F)$ converges to a fuzzy number u then by the definition of D, the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are uniformly convergent to $u^-(\lambda)$ and $u^+(\lambda)$ in $[0, 1]$, respectively. Indeed, by combining the definition of D and the fact $\lim_{k\to\infty} D(u_k, u) = 0$ one can observe that

$$\lim_{k\to\infty} \sup_{\lambda\in[0,1]} \max\{|u_k^-(\lambda) - u^-(\lambda)|, |u_k^+(\lambda) - u^+(\lambda)|\} = 0.$$

Therefore, we have

$$\lim_{k\to\infty} \sup_{\lambda\in[0,1]} |u_k^-(\lambda) - u^-(\lambda)| = 0 \quad \text{and} \quad \lim_{k\to\infty} \sup_{\lambda\in[0,1]} |u_k^+(\lambda) - u^+(\lambda)| = 0$$

which yield that the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are uniformly convergent to $u^-(\lambda)$ and $u^+(\lambda)$ in $[0, 1]$, and conversely.

Definition 2.13. A sequence $(u_k) \in w(F)$ is called bounded if and only if

$$\sup_{k\in\mathbb{N}} \|u_k\| = \sup_{k\in\mathbb{N}} \sup_{\lambda\in[0,1]} \max\{|u_k^-(\lambda)|, |u_k^+(\lambda)|\} < \infty.$$

Definition 2.14. Let $(u_k) \in w(F)$. Then the expression $\sum u_k$ is called a *series of fuzzy numbers*. Denote $s_n = \sum_{k=0}^n u_k$ for all $n \in \mathbb{N}$, if the sequence (s_n) converges to a fuzzy number u then we say that the series $\sum u_k$ of fuzzy numbers converges to u and write $\sum u_k = u$. We say otherwise the series of fuzzy numbers diverges. By $cs(F)$, we denote the set of all convergent series of fuzzy numbers. As this, if the sequence (s_n) is bounded then we say that the series $\sum u_k$ of fuzzy numbers is bounded.

Remark. $\sum u_k = u$ implies as $n \to \infty$ that

$$\sum_{k=0}^{n} u_k^-(\lambda) \to u^-(\lambda) \quad \text{and} \quad \sum_{k=0}^{n} u_k^+(\lambda) \to u^+(\lambda),$$

uniformly in $\lambda \in [0,1]$. Conversely, if the fuzzy numbers $u_k = \{(u_k^-(\lambda),(u_k^+(\lambda)) : \lambda \in [0,1]\}$, $\sum_{k=0}^{\infty} u_k^-(\lambda) = u^-(\lambda)$ and $\sum_{k=0}^{\infty} u_k^+(\lambda) = u^+(\lambda)$ converge uniformly in λ, then $u = \{(u^-(\lambda),(u^+(\lambda)) : \lambda \in [0,1]\}$ defines a fuzzy number such that $u = \sum u_k$.

Now, we may give the well-known theorem concerning uniform convergence of the series of functions:

Weierstrass M test. Let $u_k : [a,b] \to \mathbb{R}$ be given. If there exists an $M_k \geq 0$ such that $|u_k(x)| \leq M_k$ for all $k \in \mathbb{N}$ and the series $\sum M_k$ converges, then the series $\sum_{k=0}^{\infty} u_k(x)$ is uniformly and absolutely convergent in $[a,b]$.

3. The Quasilinearity of the Classical Sets of Sequences of Fuzzy Numbers

In this section, our purpose is to give the basic theorem on the quasilinearity of the classical sets of sequences of fuzzy numbers together with some other results.

The addition $(+)$ and scalar multiplication $(\cdot)$ are defined on $w(F)$ by $(u_k) + (v_k) = (u_k + v_k)$ and $\alpha(u_k) = (\alpha u_k)$, as usual, for (u_k), $(v_k) \in w(F)$ and $\alpha \in \mathbb{R}$. $\theta = (\overline{0})$ is the unit element of the space $w(F)$ with respect to addition.

Let us define the partial ordering relation $\preceq$ on $w(F)$, as follows: Let $u = (u_k)$, $v = (v_k) \in w(F)$. Then,

$$u \preceq v \Leftrightarrow u_k \subseteq v_k \text{ for all } k \in \mathbb{N}. \tag{3.1}$$

It is a routine verification that $w(F)$ is a quasilinear space with the partial ordering relation defined by (3.1) and the usual algebraic operations addition and scalar multiplication.

Now, we give the main theorem:

Main Theorem. *The following statements hold:*

- (i) *Let $\mu(F)$ denotes anyone of the sets $\ell_\infty(F)$, $c(F)$ and $c_0(F)$. Then the set $\mu(F)$ is a quasilinear space with the partial ordering relation $\preceq$ defined by (3.1) and usual algebraic operations addition, scalar multiplication. Additionally, the set $\mu(F)$ is a normed quasilinear space with the norm $\|\cdot\|_\infty$, defined by*

$$\|u\|_\infty = \sup_{k\in\mathbb{N}} \|u_k\|; \; (u = (u_k) \in \mu(F)).$$

- (ii) *The set $\ell_p(F)$ is a quasilinear space with the partial ordering relation defined by (3.1) and usual algebraic operations addition, scalar multiplication.*

Additionally, the set $\ell_p(F)$ is a normed quasilinear space with the norm $\|\cdot\|_p$, defined by

$$\|u\|_p = \left(\sum_k \|u_k\|^p\right)^{1/p}, \ (u = (u_k) \in \ell_p(F)). \tag{3.2}$$

Proof. Since the proofs are similar for the sets $c_0(F)$, $c(F)$, $\ell_\infty(F)$ and $\ell_p(F)$, we give the proof only for the set $\ell_p(F)$.

(ii) Firstly, we establish that the set $\ell_p(F)$ is a quasilinear space. Since the inclusion $\ell_p(F) \subseteq w(F)$ holds, the conditions (q.1)-(q.13) are directly satisfied. So, it is sufficient to show that the set $\ell_p(F)$ is closed under the coordinatewise operations addition and scalar multiplication.

Let $u = (u_k)$, $v = (v_k) \in \ell_p(F)$ and $\alpha \in \mathbb{R}$. Then, since

$$\begin{aligned}\left(\sum_k \|u_k + v_k\|^p\right)^{1/p} &\leq \left[\sum_k (\|u_k\| + \|v_k\|)^p\right]^{1/p} \\ &\leq \left(\sum_k \|u_k\|^p\right)^{1/p} + \left(\sum_k \|v_k\|^p\right)^{1/p} < \infty\end{aligned} \tag{3.3}$$

and

$$\sum_k \|\alpha u_k\|^p = |\alpha|^p \sum_k \|u_k\|^p < \infty, \tag{3.4}$$

$u + v \in \ell_p(F)$ and $\alpha u \in \ell_p(F)$. It is trivial that $\theta = (\overline{0}) \in \ell_p(F)$. Hence, $\ell_p(F)$ is a quasilinear space.

Finally, we should show that the relation (3.2) satisfies the conditions (n.1)-(n.5) on the set $\ell_p(F)$.

(a) Let $u \in \ell_p(F) \setminus \{\theta\}$. Then, there exists an $k \in \mathbb{N}$ such that $u_k \neq \overline{0}$, i.e. $\|u_k\| > 0$. Thus, one can easily see that $\|u\|_p = (\sum_k \|u_k\|^p)^{1/p} \geq \|u_k\|^{1/p} > 0$, i.e. (n.1) holds.

(b) (3.3) and (3.4) give the conditions (n.2) and (n.3).

(c) Let $u \preceq v$. Then, $u_k \subseteq v_k$ for all $k \in \mathbb{N}$ which gives us $\|u_k\| \leq \|v_k\|$. Therefore, it is immediate that $\|u\|_p \leq \|v\|_p$, i.e. (n.4) holds.

(d) Let us suppose that for an $\varepsilon > 0$, there is an u_ε in the space $\ell_p(F)$ such that $u \preceq v + u_\varepsilon$ and $\|u_\varepsilon\|_p = \left(\sum_k \|u_\varepsilon^k\|^p\right)^{1/p} \leq \varepsilon$. Then, $u_k \subseteq v_k + u_\varepsilon^k$ for all $k \in \mathbb{N}$ and $\|u_\varepsilon^k\| \leq \varepsilon$. Therefore, we have $u_k \subseteq v_k$ for each $k \in \mathbb{N}$. This means that $u \preceq v$, i.e. (n.5) holds.

This step completes the proof of case (ii). ■

Now, we may give the definition of the concepts of quasialgebra and normed quasialgebra for the quasilinear spaces.

Definition 3.15. Let $(X, \preceq, +, \cdot)$ be a quasilinear space. Then the set X is called an *quasialgebra* if the following conditions hold for the operation $\star$ defined on X:

(c.1) $(\alpha \cdot x) \star y = \alpha \cdot (x \star y)$.

(c.2) $x \star (y \star z) = (x \star y) \star z$.

(c.3) $x \star (y + z) \preceq (x \star y) + (x \star z)$ and $(y + z) \star x \preceq (y \star x) + (z \star x)$

for any $x, y, z \in X$ and any $\alpha \in \mathbb{R}$.

Definition 3.16. Let the normed quasilinear space $(X, \|\cdot\|)$ be a quasialgebra with the well-defined operation $(\cdot)$ on X. If $\|x \cdot y\| \leq \|x\|\|y\|$ and $\|1\| = 1$, for all x, $y \in X$, then the space X is called as a normed quasialgebra.

Lemma 3.17. [15, Lemma 2.6(i)]. *The inequality* $\|uv\| \leq \|u\|\|v\|$ *holds for* $u, v \in E^1$.

It is trivial that E^1 is a normed quasialgebra and the normed quasilinear space $w(F)$ is also a quasialgebra with

$$u \cdot v = (u_k) \cdot (v_k) = (u_k \cdot v_k); \; u = (u_k), v = (v_k) \in w(F).$$

The unit element with respect to the multiplication is $1 = (\overline{1})$.

Now, we may give:

Theorem 3.18. *The normed quasilinear spaces* $\ell_\infty(F)$ *and* $c(F)$ *are normed quasialgebras with* $\|\cdot\|_\infty$.

The spaces $c_0(F)$ and $\ell_p(F)$ do not have a unit element with respect to the multiplication.

Prior to giving the theorem on the quasilinearity and the boundedness of an operator defined by an infinite matrix in the class $(\ell_\infty(F) : \ell_\infty(F))$, we state the following lemma which is needed:

Lemma 3.19. [15, Basic Theorem (i)]. $A = (a_{nk}) \in (\ell_\infty(F) : \ell_\infty(F))$ *if and only if*

$$\sup_{n \in \mathbb{N}} \sum_k \|a_{nk}\| < \infty.$$

Theorem 3.20. *The operator defined by an infinite matrix $A = (a_{nk})$ from $\ell_\infty(F)$ to $\ell_\infty(F)$ is bounded and quasilinear.*

Proof. A-transform of a sequence $u \in \ell_\infty(F)$ is the sequence $Au = \{(Au)_n\}$ defined by

$$(Au)_n = \sum_k a_{nk} \cdot u_k \ , \ (n \in \mathbb{N}).$$

Let us show that this transformation satisfies the conditions (o.1)-(o.3). Let $u = (u_k)$, $v = (v_k) \in \ell_\infty(F)$ and $\alpha \in \mathbb{R}$.

(i) Since

$$\{A(\alpha u)\}_n = \sum_k a_{nk} \cdot (\alpha u_k) = \alpha \sum_k a_{nk} \cdot u_k = \alpha (Au)_n$$

for every fixed $n \in \mathbb{N}$, it is obvious that $A(\alpha u) = \alpha Au$, i.e. (o.1) is satisfied.

(ii) Because of

$$\begin{aligned} & a_{nk} \cdot (u_k + v_k) \\ \subseteq \quad & a_{nk} \cdot u_k + a_{nk} \cdot v_k \\ \Rightarrow \quad & \sum_{k=0}^{m} a_{nk} \cdot (u_k + v_k) \subseteq \sum_{k=0}^{m} a_{nk} \cdot u_k + \sum_{k=0}^{m} a_{nk} \cdot v_k \\ \Rightarrow \quad & \lim_{m \to \infty} \sum_{k=0}^{m} a_{nk} \cdot (u_k + v_k) \subseteq \lim_{m \to \infty} \sum_{k=0}^{m} a_{nk} \cdot u_k + \lim_{m \to \infty} \sum_{k=0}^{m} a_{nk} \cdot v_k \\ \Rightarrow \quad & \sum_{k=0}^{\infty} a_{nk} \cdot (u_k + v_k) \subseteq \sum_{k=0}^{\infty} a_{nk} \cdot u_k + \sum_{k=0}^{\infty} a_{nk} \cdot v_k \\ \Rightarrow \quad & \{A(u+v)\}_n \subseteq (Au)_n + (Av)_n, \end{aligned}$$

one can observe that $A(u+v) \preceq (Au + Av)$. That is (o.2) is fulfilled.

(iii) Suppose that $u \preceq v$. Then, $u_k \subseteq v_k$ and hence $a_{nk} \cdot u_k \subseteq a_{nk} \cdot v_k$ for all $k \in \mathbb{N}$ and for each fixed $n \in \mathbb{N}$. Therefore, one can easily establish by analogy to the previous case that $Au \preceq Av$, i.e. (o.3) holds.

Finally, we prove that the quasilinear operator A is bounded. By considering the fact $\sup_{n \in \mathbb{N}} \|(Au)_n\| < \infty$ by Lemma 3.19, for all $u \in \ell_\infty(F)$ one can see by Lemma 3.17 that

$$\begin{aligned}\sup_{n\in\mathbb{N}}\|(Au)_n\| &= \sup_{n\in\mathbb{N}}\left\|\sum_k a_{nk}\cdot u_k\right\| \\ &\leq \sup_{n\in\mathbb{N}}\sum_k\|a_{nk}\cdot u_k\| \\ &\leq \sup_{n\in\mathbb{N}}\sum_k\|a_{nk}\|\sup_{k\in\mathbb{N}}\|u_k\| \\ &= \|u\|_\infty\sup_{n\in\mathbb{N}}\sum_k\|a_{nk}\|.\end{aligned}$$

Thus, by taking supremum over $\|u\|_\infty = 1$ we conclude that $\|A\| \leq \sup_{n\in\mathbb{N}}\sum_k\|a_{nk}\|$ what we wished to prove. ∎

Proposition 3.21. *Let $\mu_i(F)$ denotes anyone of the spaces $\ell_\infty(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$ for $i = 1,\ 2$. Suppose that $(\mu_1(F), D)$ is a complete metric space and $\{T_n\} \subset L(\mu_1(F) : \mu_2(F))$ such that*

$$\sup_{n\in\mathbb{N}}\|T_n x\|_{\mu_2(F)} < \infty$$

for all $x \in \mu_1(F)$. Then, there exists a number $C > 0$ such that

$$\|T_n x\|_{\mu_2(F)} \leq C\|x\|_{\mu_1(F)},\ (n \in \mathbb{N})$$

for all $x \in \mu_1(F)$.

Proof. Since $\mu_1(F)$ is one of the quasilinear spaces $\ell_\infty(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$, the pointwise boundedness of the sequence $\{T_n\}$ of operators between these spaces implies its uniform boundedness, as was desired. ∎

4. The $\beta-$, $\alpha-$Duals of the Sequence Space $\ell_1(F)$ and the Characterization of the Class $(\ell_1(F) : \ell_p(F))$

In this section, we state and prove four theorems. The first two are on the $\beta-$ and $\alpha-$duals of the sequence space $\ell_1(F)$ and the third one concerns the characterization of the class $(\ell_1(F) : \ell_p(F))$ of infinite matrices of fuzzy numbers, where $1 \leq p \leq \infty$. In addition, we show that the spaces $\ell_\infty(F)$ and $\ell_1(F)$ are perfect. Finally, we prove that $A \in (\ell_p(F) : \ell_p(F))$ if $A \in (\ell_\infty(F) : \ell_\infty(F)) \cap (\ell_1(F) : \ell_1(F))$ with $1 < p < \infty$.

Firstly, we define the β-dual and α-dual of a set $\mu(F) \subset w(F)$ which are respectively denoted by $\{\mu(F)\}^\beta$ and $\{\mu(F)\}^\alpha$, as follows:

$$\{\mu(F)\}^\beta = \{(x_k) \in w(F) : (x_k y_k) \in cs(F) \text{ for all } (y_k) \in \mu(F)\}$$

and

$$\{\mu(F)\}^{\alpha} = \{(x_k) \in w(F) : (x_k y_k) \in \ell_1(F) \text{ for all } (y_k) \in \mu(F)\}.$$

It follows from here that $\{\mu(F)\}^{\alpha} \subset \{\mu(F)\}^{\beta}$.

Theorem 4.1. *The $\beta-$dual of the space $\ell_1(F)$ is the space $\ell_\infty(F)$.*

Proof. Suppose that $(u_k) \in \ell_1(F)$ and $(v_k) \in w(F)$. Then, $\sum \|u_k\| < \infty$ and the inequalities

$$|(u_k v_k)^{-}(\lambda)| \leq \|u_k v_k\| \leq \|u_k\|\|v_k\|$$

and

$$|(u_k v_k)^{+}(\lambda)| \leq \|u_k v_k\| \leq \|u_k\|\|v_k\|$$

hold for each $\lambda \in [0,1]$. If $(v_k) \in \ell_\infty(F)$, then since $\|v\|_\infty = \sup_{k\in\mathbb{N}} \|v_k\| < \infty$ one can immediately see that

$$\sum \|u_k\|\|v_k\| \leq \|v\|_\infty \sum \|u_k\| < \infty.$$

Since the series $\sum_k (u_k v_k)^{-}(\lambda)$ and $\sum_k (u_k v_k)^{+}(\lambda)$ are uniformly convergent by the Weierstrass M test, the series $\sum u_k v_k$ is also convergent. Hence, the inclusion

$$\ell_\infty(F) \subseteq \{\ell_1(F)\}^{\beta} \tag{4.1}$$

holds.

Conversely, let us take any point $v = (v_k)$ in the set $\{\ell_1(F)\}^{\beta}$. Then, the series $\sum u_k v_k$ is convergent for each $(u_k) \in \ell_1(F)$. Define the sequence (f_k) of bounded quasilinear operators on $\ell_1(F)$ by

$$f_n(u) = \sum_{k=0}^{n} u_k v_k$$

for all $n \in \mathbb{N}$. One can observe by using Proposition 2.8 and Proposition 3.21 at this stage that the operator f defined by

$$f(u) = \lim_{n\to\infty} f_n(u) = \sum u_k v_k$$

is a bounded quasilinear operator on the space $\ell_1(F)$. Consider the sequence $\{u_k^{(n)}\}_{k=0}^{\infty} \in \ell_1(F)$ defined by

$$u_k^{(n)} = \begin{cases} \overline{1} & , \quad (n = k) \\ \overline{0} & , \quad (n \neq k) \end{cases} \tag{4.2}$$

for every fixed $n \in \mathbb{N}$. Therefore, by the boundedness of f we see from the fact

$$\left\| \sum_k u_k^{(n)} v_k \right\| \leq \|f\| \left\| u_k^{(n)} \right\|_1$$

that $\|(v_k)\| \leq \|f\|$. This yields that $(v_k) \in \ell_\infty(F)$. That is to say that the inclusion

$$\{\ell_1(F)\}^\beta \subseteq \ell_\infty(F) \tag{4.3}$$

holds. Now, the desired result follows by combining the inclusions (4.1) and (4.3). ■

Theorem 4.2. *The $\alpha-$dual of the space $\ell_1(F)$ is the space $\ell_\infty(F)$.*

Proof. Suppose that $(u_k) \in \ell_1(F)$ and $(v_k) \in w(F)$. Then, $\sum \|u_k\| < \infty$. If $(v_k) \in \ell_\infty(F)$, then since $\|v\|_\infty = \sup_{k \in \mathbb{N}} \|v_k\| < \infty$ one can immediately see that

$$\sum \|u_k v_k\| \leq \sum \|u_k\| \|v_k\| \leq \|v\|_\infty \sum \|u_k\| < \infty.$$

Hence, the inclusion

$$\ell_\infty(F) \subseteq \{\ell_1(F)\}^\alpha \tag{4.4}$$

holds.

Conversely, since the inclusion $\{\ell_1(F)\}^\alpha \subseteq \{\ell_1(F)\}^\beta = \ell_\infty(F)$ holds we derive together with (4.4) that the $\alpha-$dual of the space $\ell_1(F)$ is the space $\ell_\infty(F)$. ■

Now, we may state the definition of the concept of perfectness of a set of sequences of fuzzy numbers and give our easy result concerning the perfectness of the spaces $\ell_\infty(F)$ and $\ell_1(F)$.

Definition 4.3. A set $\mu(F) \subset w(F)$ is said to be *perfect* if $\{\mu(F)\}^{\alpha\alpha} = \mu(F)$.

Proposition 4.4. *The spaces $\ell_\infty(F)$ and $\ell_1(F)$ are perfect.*

Proof. One can immediately see by bearing in mind the consequences $\{\ell_\infty(F)\}^\alpha = \ell_1(F)$ (cf. [15, Theorem 3.2]) and Theorem 4.2 of the present section that

$$\{[\ell_1(F)]^\alpha\}^\alpha = \{\ell_\infty(F)\}^\alpha = \ell_1(F)$$

and

$$\{[\ell_\infty(F)]^\alpha\}^\alpha = \{\ell_1(F)\}^\alpha = \ell_\infty(F),$$

as desired. ■

Theorem 4.5. $A \in (\ell_1(F) : \ell_p(F))$ *if and only if*

(i) $M = \sup_{k\in\mathbb{N}} \sum_n \|a_{nk}\|^p < \infty, \ (1 \le p < \infty)$.

(ii) $\sup_{k,n\in\mathbb{N}} \|a_{nk}\| < \infty, \ (p = \infty)$.

Proof. Since the proofs of cases (i) and (ii) are similar, we give the proof only for case (i).

(i) Suppose that $(u_k) \in \ell_1(F)$. Then, by taking into account the fact that the order of summation can be reversed by the absolute convergence of the series $\sum_n \sum_k a_{nk}u_k$ we obtain from Minkowski's inequality that

$$\begin{aligned}\left(\sum_n \left\|\sum_k a_{nk}u_k\right\|^p\right)^{1/p} &\le \left[\sum_n \left(\sum_k \|a_{nk}u_k\|\right)^p\right]^{1/p} \\ &\le \sum_k \left(\sum_n \|a_{nk}u_k\|^p\right)^{1/p} \\ &\le \sum_k \|u_k\| \left(\sum_n \|a_{nk}\|^p\right)^{1/p} \\ &\le M^{1/p}\|u\|_1 < \infty.\end{aligned}$$

This means that $A \in (\ell_1(F) : \ell_p(F))$.

Conversely, let us suppose that $A \in (\ell_1(F) : \ell_p(F))$. Then, since

$$\sum_i \|(Au)_i\|^p < \infty$$

for all $u \in \ell_1(F)$, the series $\sum_k a_{ik}u_k$ converges for all $u \in \ell_1(F)$ and for each $i \in \mathbb{N}$. Hence, $\sup_{k\in\mathbb{N}} \|a_{ik}\| < \infty$ by Theorem 4.2. Define the operators q_n's on the space $\ell_1(F)$ by

$$q_n(u) = \left[\sum_{i=0}^{n} \| \ (Au)_i\|^p\right]^{1/p}, \ (n \in \mathbb{N}).$$

The bounded quasilinearity of A_i's on the space $\ell_1(F)$ implies the bounded quasilinearity of q_n's on the space $\ell_1(F)$. Therefore, there exists a number $H > 0$ by Proposition 3.21 such that

$$\sup_{n\in\mathbb{N}} q_n(u) = \sum_{i=0}^{\infty} \| \ (Au)_i\|^p \le H\|u\|_1. \tag{4.5}$$

Thus, (4.5) gives for the sequence $\{u_k^{(n)}\}$ defined by (4.2) that $\sum_{i=0}^{\infty} \|a_{ik}\|^p \le H$, as desired.

This step completes the proof. ∎

Theorem 4.6. *Let* $1 < p < \infty$. *If* $A \in (\ell_\infty(F) : \ell_\infty(F)) \cap (\ell_1(F) : \ell_1(F))$, *then* $A \in (\ell_p(F) : \ell_p(F))$.

Proof. Suppose that $A \in (\ell_\infty(F) : \ell_\infty(F)) \cap (\ell_1(F) : \ell_1(F))$ and $1 < p < \infty$. Then, Lemma 3.19 and Theorem 4.5 together yield that

$$M = \sup_{n\in\mathbb{N}} \sum_k \|a_{nk}\| < \infty, \ N = \sup_{k\in\mathbb{N}} \sum_n \|a_{nk}\| < \infty.$$

Therefore, we derive for $(u_k) \in \ell_p(F)$ that

$$\begin{aligned}\left\|\sum_k a_{nk}u_k\right\| &\leq \sum_k \|a_{nk}u_k\| \\ &\leq \sum_k \|a_{nk}\|^{1/p}\|a_{nk}\|^{1/q}\|u_k\| \\ &\leq \left(\sum_k \|a_{nk}\|\|u_k\|^p\right)^{1/p} \left(\sum_k \|a_{nk}\|\right)^{1/q}\end{aligned}$$

which leads us to the consequence that

$$\left\|\sum_k a_{nk}u_k\right\|^p \leq \left(\sum_k \|a_{nk}\|\|u_k\|^p\right) \left(\sum_k \|a_{nk}\|\right)^{p/q}.$$

Thus, since

$$\begin{aligned}\sum_n \|(Au)_n\|^p = \sum_n \left\|\sum_k a_{nk}u_k\right\|^p &\leq \sum_n \sum_k \|a_{nk}\|\|u_k\|^p M^{p/q} \\ &\leq M^{p/q} \sum_k \|u_k\|^p \sum_n \|a_{nk}\| \\ &\leq M^{p/q} N \sum_k \|u_k\|^p < \infty,\end{aligned}$$

$Au \in \ell_p(F)$, as desired. ∎

5. Conclusion

Nanda introduced the classical sets $\ell_\infty(F)$, $c(F)$ and $\ell_p(F)$ of sequences of fuzzy numbers, and proved that $(\ell_\infty(F), D_\infty)$, $(c(F), D_\infty)$ and $(\ell_p(F), D_p)$ are complete metric spaces, in [10]. Quite recently Talo and Başar have dealt essentially with the determination of the $\alpha-$, $\beta-$ and $\gamma-$duals of the classical sets of sequences of fuzzy numbers and the characterization of the classes of infinite matrices of fuzzy numbers transforming one of the classical sets to the another one, (see [15]).

The table on the characterizations of the matrix transformations between certain spaces of sequences with real or complex terms were given in the famous article by Stieglitz and Tietz [12]. To prepare the fuzzy analogues of this table, Talo and Başar have characterized the classes $(\mu(F) : \ell_\infty(F))$, $(c_0(F) : c(F))$, $(c_0(F) : c_0(F))$, $(c(F) : c(F); p)$, $(\ell_p(F) : c(F))$ and $(\ell_p(F) : c_0(F))$ of infinite matrices of fuzzy numbers, as a beginning; where $\mu \in \{\ell_\infty,\ c,\ c_0,\ \ell_p\}$, (see [15]). We have just added the characterization of the class $(\ell_1(F) : \ell_p(F))$ to this list.

Of course, during the completion of the table of matrix transformations from the set $\mu_1(F)$ to the set $\mu_2(F)$, there are several open problems depending on the choice of $\mu_1(F)$ and $\mu_2(F)$. Başar and Altay [3] have determined the $\alpha-$, $\beta-$ and $\gamma-$duals of some new spaces of sequences with real or complex terms by using the characterization of related matrix classes, which was a new development of the matrix transformations. As a new approach, Altay and Başar have derived recently some topological properties of certain spaces of sequences with real or complex terms from the characterization of the related matrix classes, (see [1]). It is natural that both of these techniques can apply to the sets of sequences of fuzzy numbers. Then, the table of characterizations of the matrix classes between certain sets of sequences of fuzzy numbers is needed. Indeed, if we study the new set $\mu(F)_A$ obtained by the domain of an infinite matrix A of fuzzy numbers in a set $\mu(F)$ then it is necessary to know the characterization of the matrix transformations from $\mu(F)$ to $\ell_1(F)$, to $c(F)$ and to $\ell_\infty(F)$ for calculating the $\alpha-$, $\beta-$ and $\gamma-$duals of the set $\mu(F)_A$.

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