TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 5, pp. 1777-1797, October 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

APPROXIMATE CONTROLLABILITY OF NONLINEAR DETERMINISTIC AND STOCHASTIC SYSTEMS WITH UNBOUNDED DELAY

R. Sakthivel, Juan J. Nieto and N. I. Mahmudov

Abstract. In this paper, we consider approximate controllability for nonlinear deterministic and stochastic systems with resolvent operators and unbounded delay. We study the problem of approximate controllability of deterministic nonlinear differential equations with impulsive terms, resolvent operators and unbounded delay. Next, approximate controllability results are being established for a class of nonlinear stochastic differential equations with resolvent operators in a real separable Hilbert spaces. By using the resolvent operators and fixed point technique, sufficient conditions have been formulated and proved. In this paper, we prove the approximate controllability of nonlinear deterministic and stochastic control systems under the assumption that the corresponding linear system is approximately controllable. Examples are presented to illustrate the utility and applicability of the proposed method.

1. Introduction

Controllability of the deterministic systems in infinite dimensional spaces has been extensively studied. Several authors [3, 4, 5, 6, 8, 9, 14, 15] have studied the concept of exact controllability for systems represented by nonlinear evolutions equations, in which the authors have effectively used fixed point technique. From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. In infinite-dimensional spaces the concept of exact controllability is usually too strong and, indeed has limited applicability (see [29] and references therein). Approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications (see

Received January 5, 2009, accepted February 17, 2009.
Communicated by Boris S. Mordukhovich.
2000 *Mathematics Subject Classification*: 93B05, 93E03, 93E20.
Key words and phrases: Approximate controllability, Resolvent operators, Impulsive differential equations, Neutral equations, Stochastic differential equations.

[10, 29] and references therein). Therefore, it is important, in fact, necessary to study the weaker concept of controllability, namely approximate controllability for nonlinear integrodifferential systems.

The theory of impulsive differential equations is emerging as an important area of investigation since it is richer than the theory of classical differential equations. In recent years, existence of solutions of impulsive differential equations have been investigated in several works [24, 25, 28, 41]. The applications of the impulsive differential equations emerge in epidemiology [17, 18, 40], pharmacokinetics, fed-batch culture in fermentative production [16], population dynamics [37] etc. Impulsive control problems can also arise in investment decisions in economics and the injection of a medical drug into a patient in mathematical models in pharmacology. Yang has derived impulsive control for a class of nonlinear systems and interesting applications of impulsive control in chaotic systems and chaotic spread spectrum communications can be found in [39]. Impulsive control problems for systems governed by ordinary differential equations have been studied in [32]. Wang et al.[38] have studied the dynamics complexity of a preypredator system with Beddington-type functional response and impulsive control strategy by using theories and methods of ecology and ordinary differential equations. Recently, Chang and Chalishajar [7] have established sufficient conditions for the controllability of semilinear mixed Volterra-Fredholm-type integrodifferential inclusions using Bohnenblust-Karlin's fixed point theorem. Exact controllability of various types of nonlinear impulsive differential systems has been studied by several authors [2, 5]. However, the approximate control theory of impulsive differential equations is not yet sufficiently elaborated, compared to that of ordinary differential equations.

Recently many works report approximate controllability results for first order nonlinear systems [11, 29, 31, 36]. Approximate controllability of first order functional differential equations with finite delay was considered in [11] with the aid of Schauder's fixed point theorem. Approximate controllability for semilinear deterministic and stochastic control systems can be found in Mahmudov [29]. Mahmudov [31] have studied the approximate controllability for the abstract evolution equations with nonlocal conditions. More recently Sakthivel et al [36] have established approximate controllability for the nonlinear differential and neutral functional differential equations with impulses but the result obtained in [36] is only in connection with finite delay. Since many systems arising from realistic models can be described as functional differential systems with unbounded delay (see [24] and references therein), it is natural to discuss approximate controllability of this kind of problems. On the other hand, the resolvent operator is similar to the evolution operator for autonomous differential equations but a number of results comes directly from the definition of the resolvent operator [19, 20, 21]. However, up to now approximate controllability problems for nonlinear integrodifferential systems with resolvent operators and unbounded delay have not been considered in the literature. In order to fill this gap, this paper studies the approximate controllability of the following nonlinear integrodifferential equation of the form

$$
\begin{aligned}
x'(t) &= Ax(t) + \int_0^t G(t-s)x(s)ds + Bu(t) + f(t, x_t), t \in J = [0, b] \\
x_0 &= \phi \in \mathbf{P}, \\
\triangle x(t_k) &= I_k(x_{t_k}), \quad k = 1, \cdots, m
\end{aligned}
\tag{1}
$$

where $A : D(A) \subset X \to X$ and $G : D(G(t)) \subset X \to X, t \geq 0$, are closed linear operators. X is a Hilbert space; B is a bounded linear operator from a Hilbert space U into X; the control $u(\cdot) \in L_2(J, U)$, a Hilbert space of admissible control functions. The functions $x_t : (-\infty, 0] \to X, x_t(\theta) = x(t+\theta)$, belong to some abstract phase space $\mathbf{P}$ defined axiomatically. Here $0 < t_1 <, \cdots < t_k < b$ are prefixed numbers, $f : J \times \mathbf{P} \to X, I_k : \mathbf{P} \to X$ are appropriate functions and the symbol $\triangle \xi(t)$ represent the jump of the function ξ at t, which is defined by $\triangle \xi(t) = \xi(t^+) - \xi(t^-)$.

On the other hand, stochastic differential equations are well known to model problems from many areas of science and engineering. The qualitative properties of solutions of stochastic differential equations in infinite dimensions have been investigated by many authors because of its importance in applications. In particular, recently approximate controllability results for various types of first order stochastic nonlinear equations have been established [10, 12, 30]. There are only few works on approximate controllability of stochastic nonlinear systems. The work on existence of solutions of abstract stochastic differential equations with the help of resolvent operators was initiated by Kech and McKibben [27]. The stochastic systems with resolvent operator arises in various applications such as viscoelasticity, heat equations and many other physical phenomena (see [27] and references therein). The study of approximate controllability result for stochastic nonlinear equations with resolvent operators is an untreated topic and it is also the motivation of this paper. Henry [26] has discussed the approximate controllability for a nonlinear parabolic equation. He pointed out that if the range BU of the operator B in (1) is dense in $L_2(J; X)$ then under some hypotheses on the nonlinear function $f(.)$, the nonlinear parabolic system (1) is approximately controllable. Later Zhou [42] has obtained a set sufficient conditions for the approximate controllability of nonlinear parabolic control system which improves the results in [26] with some more restrictions. Based on the work in [42], in this paper, we prove the approximate controllability of nonlinear deterministic and stochastic control systems under the assumption that the corresponding linear system is approximately controllable.

2. Preliminaries

To consider the impulsive condition, it is convenient to introduce some additional concepts and notations. We say that a function $x : [\mu, \eta] \to X$ is a normalized piecewise continuous function on $[\mu, \eta]$ if x is piecewise continuous, and left continuous on $(\mu, \eta]$. Let $PC([\mu, \eta]; X)$ be the space formed by the normalized piecewise continuous functions from $[\mu, \eta]$ to X. The notation PC stands for the space formed by all functions $x : [0, b] \to X$ such that $x(\cdot)$ is continuous at $t \neq t_k, x(t_k^-) = x(t_k)$ and $x(t_k^+)$ exists for all $k = 1, ..., m.$ In this section $(PC, \|\cdot\|_{PC})$ is a Banach space endowed with the norm $\|x\|_{PC} = \sup_{s\in J} \|x(s)\|$.

In this work, we employ an axiomatic definition for the phase space $\mathbf{P}$ introduced in [22], specifically $\mathbf{P}$ will be a linear space of functions mapping $(-\infty, 0]$ to X endowed with a seminorm $\|\cdot\|_{\mathbf{P}}$ and verifying the following axioms:

(A_1) If $x : (-\infty, \mu + \sigma] \to X, \sigma > 0$ is such that $x_\mu \in \mathbf{P}$ and $x\mid_{[\mu,\mu+\sigma]} \in PC([\mu, \mu + \sigma] : X)$ then for every $t \in [\mu, \mu + \sigma]$ the following conditions hold:

- (i) x_t is in $\mathbf{P}$,
- (ii) $\|x(t)\| \leq \tilde{H}\|x_t\|_{\mathbf{P}}$,
- (iii) $\|x_t\|_{\mathbf{P}} \leq \tilde{K}(t-\mu)\sup\{\|x(s)\| : \mu \leq s \leq t\} + \tilde{M}(t-\mu)\|x_\mu\|_{\mathbf{P}}$, where $\tilde{H} > 0$ is a constant; $\tilde{K}, \tilde{M} : [0, \infty) \to [1, \infty), \tilde{K}$ is continuous, $\tilde{M}$ is locally bounded and $\tilde{H}, \tilde{K}, \tilde{M}$ are independent of $x(\cdot)$.

(A_2) For the function $x(\cdot)$ in (A_1), the function $t \to x_t$ is continuous from $[\mu, \mu+\sigma]$ into $\mathbf{P}$.

(A_3) The space $\mathbf{P}$ is complete.

Let A and $G(t), t \geq 0$, be closed linear operators defined on a common domain D which is dense in X. To obtain our results, we assume that the integro-differential abstract Cauchy problem

$$
\begin{aligned}
x'(t) &= Ax(t) + \int_0^t G(t-s)x(s)ds \\
x(0) &= x_0 \in X,
\end{aligned}
\tag{2}
$$

has an associated resolvent operator of bounded linear operators $(R(t))_{t\geq 0}$ on X.

Definition 2.1. A family of bounded linear operators $(R(t)_{t\geq 0}$ is a resolvent operator family for (2) if the following conditions are verified.

- (i) $R(0) = I$ and $R(\cdot)x \in C([0, \infty) : X)$ for every $x \in X$.
- (ii) For $x \in D(A), AR(\cdot)x \in C([0, \infty) : X)$ and $R(\cdot)x \in C^1([0, \infty) : X)$

(iii) For all $x \in D(A)$ and every $t \geq 0$, the following resolvent equations are verified.

$$R'(t)x = AR(t)x + \int_0^t G(t-s)R(s)xds$$
$$R'(t)x = R(t)Ax + \int_0^t R(t-s)G(s)xds$$

For additions details related to resolvent of operator associated to integro-differential equations, see [19, 20].

Let $x_b(x_0;u)$ be the state value of (1) at terminal time b corresponding to the control u and the initial value $x_0 = \phi \in \mathbf{P}$. Introduce the set

$$\Re(b, x_0) = \{x_b(x_0;u)(0) : u(\cdot) \in L_2(J, U)\},$$

which is called the reachable set of system (1) at terminal time b, its closure in X is denoted by $\overline{\Re(b, x_0)}$.

Definition 2.2. The system (1) is said to be approximately controllable on the interval J if $\overline{\Re(b, x_0)} = X$.

It is convenient at this point to define operators

$$\Gamma_0^b = \int_0^b R(b-s)BB^*R^*(b-s)ds$$

$$\tilde{R}(\alpha, \Gamma_0^b) = (\alpha I + \Gamma_0^b)^{-1}.$$

(S_1) $\alpha\tilde{R}(\alpha, \Gamma_0^b) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.

The assumption (S_1) holds iff the linear integro-differential Cauchy problem corresponding to (1) is approximately controllable on J.

3. Approximate Controllability of Deterministic Systems

In this section, we prove the result on approximate controllability of nonlinear deterministic systems. To do this, we first prove the existence of solutions using Schauder's fixed point theorem. Second, we show that under certain assumptions, the approximate controllability of (1) is implied by the approximate controllability of the corresponding linear system.

To establish our results, we introduce the following assumptions:

(I) $R(t), t > 0$ is compact.

(II) The function $f : J \times \mathbf{P} \to X$ satisfies the following conditions:

(i) For every $x : (-\infty, b] \to X$ be such that $x_0 = \phi$ and $x|_J \in PC$, the function $t \to f(t, x_t)$ is strongly measurable.

(ii) For each $t \in J$, the function $f(t, .) : \mathbf{P} \to X$ is continuous.

(iii) For each $q > 0$, there exists a function $\lambda_q \in L^1(J, R^+)$ such that

$$\sup_{\|\xi\| \le q} \|f(t, \xi)\| \le \lambda_q(t), \qquad \text{for a.e. } t \in J,$$

and

$$\lim_{q \to \infty} \inf \int_0^b \frac{\lambda_q(t)}{q} dt = \delta < \infty.$$

(III) The maps I_k are continuous and there exists a positive constant d_k such that $\|I_k(\psi)\| \le d_k$ for every $\psi \in \mathbf{P}$.

(IV) The function $f : J \times \mathbf{P} \to X$ is continuous and uniformly bounded and there exists $N > 0$ such that $\|f(t, \phi)\| \le N$ for all $(t, \phi) \in J \times \mathbf{P}$

Let $Z = \{x : (-\infty, b] \to X : x_0 = 0, x|_J \in PC\}$ endowed with the norm of the uniform convergence topology.

In this section, it will be shown that the system (1) is approximately controllable if for all $\alpha > 0$ there exists a continuous function $x(\cdot) \in Z$ such that

(3) $u(t) = B^* R^*(b - t)\tilde{R}(\alpha, \Gamma_0^b)p(x(\cdot))$

(4) $x(t) = R(t)\phi(0) + \int_0^t R(t-s)[Bu(s) + f(s, x_s)]ds + \sum_{0<t_k<t} R(t-t_k)I_k(x_{t_k})$

where

$$p(x(\cdot)) = x_b - R(b)\phi(0) - \int_0^b R(b-s)f(s, x_s)ds - \sum_{k=1}^m R(b-t_k)I_k(x_{t_k})$$

The following notations are introduced for convenience

$M = \max\{\|R(t)\| : 0 \le t \le b\}$.

$M_B = \|B\|$.

$K_1 = \|x_b\| + M\tilde{H}\|\phi\|_{\mathbf{P}} + M \sum_{k=1}^m d_k$.

$K_2 = M\tilde{H}\|\phi\|_{\mathbf{P}} + \frac{1}{\alpha} bM^2 M_B^2 K_1 + M \sum_{k=1}^m d_k$.

Theorem 3.1. *Assume that conditions (I)-(III) are satisfied. Further, suppose that for all* $\alpha > 0$

$$(1 + \frac{1}{\alpha} bM^2 M_B) M\tilde{K}_b \delta < 1,$$

then the system (1) has a solution on J.

Proof. Along this theorem, let $y : (-\infty, b] \to X$ be the function defined by $y_0 = \phi$ and $y(t) = R(t)\phi(0)$ for $t \in [0, b]$.

The main aim in this section is to find conditions for solvability of system (3) and (4) for $\alpha > 0$. On the Banach space Z consider a set

$$Q = \{x(\cdot) \in Z, \|x\| \le r\}.$$

where r is a positive constant.

For $\alpha > 0$, define the operator $F_\alpha : Z \to Z$ by

$$\begin{aligned} F_\alpha x(t) &= 0, \ t \in (-\infty, 0] \\ &= z(t), \quad t \in J \end{aligned}$$

where

$$\begin{aligned} z(t) &= R(t)\phi(0) + \int_0^t R(t-s)[Bv(s) + f(s, x_s + y_s)]ds \\ &\quad + \sum_{0<t_k<t} R(t-t_k) I_k(x_{t_k} + y_{t_k}) \\ v(t) &= B^* R^*(b-t) \tilde{R}(\alpha, \Gamma_0^b) p(x(\cdot)), \\ p(x(\cdot)) &= x_b - R(b)\phi(0) - \int_0^b R(b-s) f(s, x_s + y_s) ds \\ &\quad - \sum_{k=1}^m R(b-t_k) I_k(x_{t_k} + y_{t_k}). \end{aligned}$$

It will be shown that for all $\alpha > 0$ the operator F_α from Z into itself has a fixed point.

Step 1. For $\alpha > 0$, there exists $r > 0$ such that $F_\alpha(Q) \subset Q$. In fact, if we assume that the assertion is false, then there exists $\alpha > 0$ such that for every $r > 0$, there exists $x' \in Q$ and $t' \in J$ such that $r < \|F_\alpha x'(t')\|$.

For such $\alpha > 0$, we find that

$$\begin{aligned} r &< \|F_\alpha x'(t')\| \\ &\le M\tilde{H}\|\phi\|_{\mathbf{P}} + MM_B \int_0^t \|v(s)\|ds + M\int_0^t \|f(s, x_s + y_s)\|ds + M\sum_{k=1}^m d_k. \end{aligned}$$

We observe that for every $x \in Q$ and $t \in J$

$$\|x_t + y_t\| \le (\tilde{M}_b + \tilde{K}_b M\tilde{H})\|\phi\|_{\mathbf{P}} + \tilde{K}_b r = r^*,$$

where $\tilde{M}_b$ and $\tilde{K}_b$ are constants defined by $\tilde{M}_b = \sup_{t\in J} \tilde{M}(t), \tilde{K}_b = \sup_{t\in J} \tilde{K}(t)$.

We denote by r^* the right hand side of the above expression, we have

$$\|x_t + y_t\| \leq (\tilde{M}_b + \tilde{K}_b M \tilde{H})\|\phi\|_{\mathbf{P}} + \tilde{K}_b r = r^*,$$

where r^* is a positive constant. Hence we obtain

$$\begin{aligned} r &\leq M\tilde{H}\|\phi\|_{\mathbf{P}} + bMM_B[\frac{1}{\alpha}MM_B(K_1 + M\int_0^b \lambda_{r^*}(s)ds)] \\ &\quad + M\int_0^b \lambda_{r^*}(s)ds + M\sum_{k=1}^m d_k \\ &\leq (1 + \frac{1}{\alpha}bM^2M_B)M\int_0^b \lambda_{r^*}(s)ds + K_2. \end{aligned}$$

We note that K_2 is independent of r and $r^* \to \infty$ as $r \to \infty$. Now

$$\lim_{r\to\infty} \inf \int_0^b \frac{\lambda_{r^*}(s)}{r} ds = \lim_{r\to\infty} \inf \int_0^b \frac{\lambda_{r^*}(s)}{r^*} \cdot \frac{r^*}{r}) ds = \delta \tilde{K}_b.$$

Hence we have for $\alpha > 0$,

$$(1 + \frac{1}{\alpha}bM^2M_B)M\tilde{K}_b\delta \geq 1,$$

which is a contradiction to our assumption. Thus $\alpha > 0$, there exists $r > 0$ such that F_α maps Q into itself.

Step 2. For each $\alpha > 0$, the operator F_α maps Q into a relatively compact subset of Q.

First we prove that (i) for arbitrary $t \in J$ the set $V(t) = \{(F_\alpha x)(t) : x(\cdot) \in Q\}$ is relatively compact.

The case $t = 0$ is obvious, since $V(0) = \{\phi(0)\}$. For $0 < \tau < t \leq b$, define

$$(F_\alpha^\tau x)(t) = R(\tau)z(t - \tau).$$

since $R(t)$ is compact and $z(t - \tau)$ is bounded on Q, the set

$$V_\tau(t) = \{(F_\alpha^\tau x)(t) : x(\cdot) \in Q\}$$

is relatively compact set in X. That is, a finite set $\{\tilde{y}_i, 1 \leq i \leq n\}$ in X exists such that

$$V_\tau(t) \subset \bigcup_{i=1}^m \hat{N}(\tilde{y}_i, \epsilon/2),$$

where $\hat{N}(\tilde{y}_i, \epsilon/2)$ is an open ball in X with center at $\tilde{y}_i$ and radius $\epsilon/2$. On the other hand,

$$\begin{aligned}\|(F_\alpha x)(t) - (F_\alpha^\tau x)(t)\| &= \| \int_{t-\tau}^{t} R(t-s)[Bv(s) + f(s, x_s + y_s)]ds\| \\ &\leq \frac{1}{\alpha} M M_B \left(\|x_b\| + M\tilde{H}\|\phi\|_{\mathbf{P}} + M \int_0^b \lambda_{r^*}(s)ds \right. \\ &\quad \left. + M \sum_{k=1}^{m} d_k \right) \tau + M \int_{t-\tau}^{t} \lambda_{r^*}(s)ds \leq \frac{\epsilon}{2}.\end{aligned}$$

Consequently

$$V(t) \subset \bigcup_{i=1}^{n} \hat{N}(\tilde{y}_i, \epsilon)$$

Hence for each $t \in J$, $V(t)$ is relatively compact in X.

Second to prove (ii) for arbitrary $\epsilon > 0$, there exists $\hat{\delta} > 0$ such that $\|(F_\alpha x)(t_1) - (F_\alpha x)(t_2)\| < \epsilon$ if $\|x\| \leq r, \quad |t_1 - t_2| \leq \hat{\delta}, \quad t_1, t_2 \in J$.

To prove (ii), we have to show that $V = \{(F_\alpha x)(.) \mid x(\cdot) \in Q\}$ is equicontinuous on $[0, b]$. For $0 < t_1 < t_2 \leq b$, we have

$$\begin{aligned}&\|z(t_1) - z(t_2)\| \\ \leq\ & \|R(t_1) - R(t_2)\|\tilde{H}\|\phi\|_{\mathbf{P}} + M M_B \int_{t_1}^{t_2} \|v(s)\|ds \\ & + M_B \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\|\|v(s)\|ds \\ & + \sum_{0<t_k<t_1} \|R(t_2 - t_k) - R(t_1 - t_k)\|\|I_k(x_{t_k} + y_{t_k})\| \\ & + \sum_{t_1 \leq t_k < t_2} \|R(t_2 - t_k)\|\|I_k(x_{t_k} + y_{t_k})\| + M \int_{t_1}^{t_2} \lambda_{r^*}(s)ds \\ & + \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\|\lambda_{r^*}(s)ds \\ \leq\ & \|R(t_1) - R(t_2)\|\tilde{H}\|\phi\|_{\mathbf{P}} \\ & + \frac{1}{\alpha} M^2 {M_B}^2 \int_{t_1}^{t_2} (\|x_b\| + M\tilde{H}\|\phi\|_{\mathbf{P}} + M \sum_{k=1}^{m} d_k + M \int_0^b \lambda_{r^*}(s)ds)d\eta \\ & + \sum_{0<t_k<t_1} \|R(t_2 - t_k) - R(t_1 - t_k)\|d_k + \sum_{t_1 \leq t_k < t_2} \|R(t_2 - t_k)\|d_k \\ & + \frac{1}{\alpha} M {M_B}^2 \int_0^{t_1} \|R(t_1 - s) - R(t_2 - s)\|(\|x_b\| + M\tilde{H}\|\phi\|_{\mathbf{P}}\end{aligned} \tag{5}$$

$$+ M \sum_{k=1}^{m} d_k + M \int_0^b \lambda_{r^*}(s)ds)d\eta + M \int_{t_1}^{t_2} \lambda_{r^*}(s)ds$$
$$+ \sum_{i=1}^{q} \int_0^{t_1} \|R(t_1 - s) - R(t_2 - s)\| \lambda_{r^*}(s)ds$$

Thus the right hand side of (5) does not depend on particular choices of $x(\cdot)$ and tends to zero as $t_1 - t_2 \to 0$, since the compactness of $R(t)$ for $t > 0$ implies the continuity in the uniform operator topology. So we obtain the equicontinuity of V. We have considered here only the case $0 < t_1 < t_2$, since other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple. Thus $F_\alpha(Q)$ is equicontinuous and also bounded. By the Ascoli-Arzela theorem $F_\alpha(Q)$ is relatively compact in Z. It is easy to show that for all $\alpha > 0$, F_α is continuous on Z. Hence from the Schauder's fixed point theorem F_α has a fixed point. Thus the problem (1) has a solution on J.

Theorem 3.2. *Assume assumptions* $(I), (S_1)$ *and* (IV) *are satisfied. Then the system (1) is approximately controllable on* J.

Proof. Let $\hat{x}_\alpha(\cdot)$ be a fixed point of F_α in Q. Any fixed point of F_α is a mild solution of (1) under the control

$$\hat{u}_\alpha(t) = B^* R^*(b - t) \tilde{R}(\alpha, \Gamma_0^b) p(\hat{x}_\alpha)$$

and satisfies the inequality

$$\hat{x}_\alpha(b) = x_b + \alpha \tilde{R}(\alpha, \Gamma_0^b) p(\hat{x}_\alpha)$$

By the condition (IV)

$$\int_0^b \|f(s, \hat{x}_\alpha(s))\|^2 ds \le N^2 b$$

and consequently the sequence $\{f(s, \hat{x}_\alpha(s))\}$ is bounded in $L_2(J, X)$. Then there is a subsequence denoted by $\{f(s, \hat{x}_\alpha(s))\}$, that weakly converges to say $f(s)$ in $L_2(J, X)$. Now, thanks to the compactness of an operator $l(\cdot) \to \int_0^{\cdot} R(\cdot - s) l(s) ds :$ $L_2(J, X) \to C(J, X)$ we obtain

$$\begin{aligned} \|p(\hat{x}_\alpha) - \hat{w}\| &= \| \int_0^b R(b - s)[\|f(s, \hat{x}_\alpha(s)) - f(s)] ds\| \\ &\le \sup_{0 \le t \le b} \| \int_0^t R(b - s)[f(s, \hat{x}_\alpha(s)) - f(s)] ds\| \to 0 \end{aligned}$$

as $\alpha \to 0^+$, where

$$\hat{w} = R(b)\phi(0) + \int_0^b R(b - s) f(s) ds - x_b$$

Then from

$$\begin{aligned}\|\hat{x}_\alpha(b)-x_b\| &\le \|\alpha\tilde{R}(\alpha,\Gamma_0^b)(\hat{w})\|+\|\alpha\tilde{R}(\alpha,\Gamma_0^b)\|\|p(\hat{x}_\alpha)-\hat{w}\| \\ &\le \|\alpha\tilde{R}(\alpha,\Gamma_0^b)(\hat{w})\|+\|p(\hat{x}_\alpha)-\hat{w}\|\to 0\end{aligned}$$

as $\alpha\to 0^+$. This proves the approximate controllability of (1).

Remark 3.3. Neutral integro-differential equations with unbounded delay arises in the description of heat conduction in materials with fading memory [34]. Now, we consider the approximate controllability of neutral differential equations with impulses and unbounded delay of the form

$$\begin{aligned}\frac{d\tilde{D}(t,x_t)}{dt} &= A\tilde{D}(t,x_t)+\int_0^t G(t-s)\tilde{D}(s,x_s)ds \\ &\quad +Bu(t)+f(t,x_t), t\in J=[0,b] \\ x_0 &= \phi\in\mathbf{P}, \\ \triangle x(t_k) &= I_k(x_{t_k}),\quad k=1,\cdots,m\end{aligned}\tag{6}$$

where $\tilde{D}(t,\phi)=\phi(0)+g(t,\phi)$ and $g:J\times P\to X$ is an appropriate function. Recently few works reporting exact controllability results for impulsive neutral differential systems (see [36]) and references therein. However, in these works the authors impose compactness assumptions on the operator generated by A, which imply that the underlying space X has finite dimension. With the proceeding reason, in this remark, we discuss the approximate controllability of integrodifferential equations of the form (6). A function $x:(-\infty,b]\to X$ is a mild solution of (6) on J if $x\in C([0,b],X); x_0=\phi$ and satisfies

$$\begin{aligned}x(t) &= R(t)(\phi(0)+g(0,\phi))-g(t,x_t)+\int_0^t R(t-s)[Bu(s)+f(s,x_s)]ds \\ &\quad +\sum_{0<t_k<t}T(t-t_k)I_k(x_{t_k})\end{aligned}$$

By suitably applying the above Theorems, one can easily prove that the system (6) is approximately controllable.

Remark 3.4. Differential inclusions play an important role in characterizing many social, physical, biological and engineering problems. In particular, the problems in physics, especially in solid mechanics, where non-monotone and multivalued constitutive laws lead to differential inclusions. The above result can be extended to study the controllability of nonlinear impulsive differential inclusions by suitably introducing the multivalued map defined in [14, 15].

Example 3.5. Consider the following control partial differential equations with impulses

$$
\begin{aligned}
\frac{\partial D(t,z_t)(y)}{\partial t} &= \frac{\partial^2 D(t,z_t)(y)}{\partial y^2} + \bar{\mu}(t,y) \\
&\quad + \int_0^t (t-s)^{\beta-1} e^{-p(t-s)} \frac{\partial^2 D(s,z_s)(y)}{\partial y^2} ds + \int_\infty^t c(t-s) z(s,y) ds \\
z(\tau,y) &= \phi(\tau,y), \tau \le 0, y \in [0,\pi] \\
z(t,0) &= z(t,\pi) = 0,\ t \in [0,b] \\
\Delta z(t_k)(y) &= \int_{-\infty}^{t_k} \nu_k(t_k-s) z(s,y) ds, \quad k = 1,\cdots,m,
\end{aligned} \tag{7}
$$

where $0 < t_1 < ... < t_m < b$ are pre-fixed numbers, $\beta \in (0,1), p > 0$ and $\phi \in \mathbf{P}$. Here $D(t,z_t)(y) = z(t,y) - \int_{-\infty}^0 c_0(t)c_1(s)z(t+s,y)ds$ and $c_i(\cdot)$ are continuous functions.

Let $X = L^2[0,\pi]$ and $\mathbf{P} = PC_0 \times L^2(\tilde{g}, X)$ ($\tilde{g} : (-\infty, -r] \to R$ be a positive function) be the phase space introduced in [24, 25]. We define the operator A by $Az = z''$ with domain $D(A) = \{w \in X : w$ and w' are absolutely continuous, $w'' \in X, w(0) = w(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge 0}$ on X and from Grimmer [19] we assert that the integrodifferential system

$$
\begin{aligned}
x'(t) &= Ax(t) + \int_0^t G(t-s)Ax(s)ds \\
x(0) &= x_0 \in X,
\end{aligned}
$$

where $G(t) = t^{\beta-1}e^{-pt}$ has an associated resolvent of operator $(R(t))_{t\ge 0}$ on X.

To study the above system, by assuming the function $c_0(\cdot)$ is bounded and that

$$
\begin{aligned}
L_1 &= \left(\int_{-\infty}^0 \frac{(c_1(s)^2)}{\tilde{g}(s)} ds \right)^{1/2}, \\
L_h &= \left(\int_{-\infty}^0 \frac{(c(-s))^2}{\tilde{g}(s)} ds \right)^{1/2}
\end{aligned}
$$

and

$$
L_k = \left(\int_{-\infty}^0 \frac{(\nu_k(-s))^2}{\tilde{g}(s)} ds \right)^{1/2}
$$

are finite.

Choosing suitably the functions $D, f, g, I_k : \mathbf{P} \to X$ by

$$\begin{aligned} D(t,\psi)(y) &= \psi(0,y) - \int_{-\infty}^{0} c_0(t)c_1(s)\psi(s,y)ds, \\ g(t,\psi)(y) &= \int_{-\infty}^{0} c_0(t)c_1(s)\psi(s)(y)ds, \\ f(t,\psi)(y) &= \int_{-\infty}^{0} c(-s)\psi(s)(y)ds, \\ I_k(\psi)(y) &= \int_{-\infty}^{0} \nu_k(-s)\psi(s,y)ds, \quad k = 1, \cdots, m \end{aligned}$$

and

$$Bu(t)(y) = \bar{\mu}(t,y),$$

the partial differential equation (7) can be written in the abstract form (6). Further the maps f, g, I_k are bounded linear operators such that $\|f\| \le L_1, \|g\| \le L_h$ and $\|I_k\| \le L_k$. Because of the compactness of $R(t)$, the associated linear system of (7) is not exactly controllable but it is approximately controllable. Hence by Theorem 3.1 and 3.2, system (7) is approximately controllable on $[0, b]$.

Remark 3.6. Now we briefly comment on the non-autonomous versions of systems (1) and (6), where the operator A is replaced by $\{A(t) : 0 \le t \le b\}$. In order to proceed to prove the approximate controllability results in a similar manner employed in the above theorem, a resolvent family $\{R(t,s), 0 \le t \le s < \infty\}$ is guaranteed to exist. Conditions guaranteeing existence of $R(t,s)$ can be found in [20] and hence the above theorems can be extended to the time-dependent case by making suitable modifications involving the use of properties of the time-dependent resolvent family in the above arguments.

4. Approximate Controllability of Stochastic Systems

In this section, we study the approximate controllability of stochastic nonlinear differential equation using resolvent opertators, together with the natural assumption that associated linear system is approximately controllable.

Let (Ω, Γ, P) be a complete probability space equipped with a normal filtration $(\Gamma_t), t \in J = [0, b]$. Let X, U and E are the separable Hilbert spaces and w is a Q-Weiner process on (Ω, Γ, P) with the linear bounded covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}$ in E, a bounded sequence of nonnegative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda_n e_n$, n=1,2,..., and a sequence $\{\beta_n\}$ of independent Brownian motions such that

$$< w(t), e >= \sum_{k=1}^{\infty} \sqrt{\lambda_n} < e_n, e > \beta_n(t), \quad e \in E, t \in J$$

and $\Gamma_t = \Gamma_t^w$, where Γ_t^w is the σ- algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}E$ to X with the inner product $< \psi, \pi >_{L_2^0} = tr[\psi Q \pi^*]$. Let $L_2(\Gamma_b, X)$ be a Hilbert space of all Γ_b-measurable square integrable random variables with values in the Hilbert space X, $L_2^\Gamma([0,b], X)$ is the Hilber space of all square integrable Γ_t- adapted process with values in X.

Let $C([0,b]; L_2(\Gamma, X))$ be the Banach space of continuous maps from $[0,b]$ into $L_2(\Gamma, X)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$. Let $H_2([0,b]; X)$ is the closed subspace of $C([0,b]; L_2(\Gamma, X))$ consisting of measurable and Γ_t- adapted process $x(t)$. Then H_2 is a Banach space with norm topology given by $\|x\|_{H_2} = (\sup_{t\in[0,b]} E\|x(t)\|^2)^{1/2}$.

The focus of this section is to investigate the approximate controllability problem for the class of nonlinear stochastic differential equations of the form

$$\begin{aligned} dx(t) &= \left[A\left\{x(t) + \int_0^t a(t-s)x(s)ds\right\}\right. \\ &\qquad \left. + Bu(t) + f(t, x(t))\right]dt + g(t, x(t))dw(t) \\ x(0) &= x_0 \end{aligned} \tag{8}$$

in a Hilbert space X, where $A : D(A) \subset X \to X$ is a linear, closed, densely-defined (possible unbounded) operator; B is a bounded linear operator from the Hilber space U into X; the control $u \in L_2^\Gamma([0,b], U)$; w is a E-valued Wiener process; $a : [0,b] \times \Omega \to R$ is a stochastic kernel; $f : J \times X \to X$, $g : J \times X \to L_2^0$ are appropriate functions.

The following are the main assumptions in this section:

(i) $R(t), t > 0$ is compact.

(ii) The functions $f : J \times X \to X$ and $g : J \times X \to L_2^0$ are continuous and there exists a constant $\tilde{N} > 0$ such that

$$\begin{aligned} \|f(t,x) - f(t,y)\| + \|g(t,x) - g(t,y)\| &\leq \tilde{N}\|x - y\| \\ \|f(t,x)\| + \|g(t,x)\| &\leq \tilde{N} \end{aligned}$$

(iii) For each $0 \leq t < b$, the operator $\alpha(\alpha I + \Psi_t^b)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0$, where

$$\Psi_t^b = \int_t^b R(b-s)BB^*R^*(b-s)ds$$

i.e., the linear deterministic system corresponds to (8) is approximately controllable on every $[t, b]$.

Definition 4.1. A stochastic process $x \in H_2([0,b],X)$ is a mild solution of (8) if for each $u \in L_2^{\Gamma}([0,b],U)$, it satisfies the following integral equation

$$x(t)=R(t)x_0+\int_0^t R(t-s)[Bu(s)+f(s,x(s))]ds+\int_0^t R(t-s)g(s,x(s))dw(s),$$

where $R(t)$ is a resolvent family for stochastic systems which is defined in [27].

For details related to resolvent of operator associated to stochastic integro-differential equations and additional background, we refer the reader to [27] and the references therein.

Definition 4.2. System (8) is approximately controllable on $[0,b]$ if $\overline{\Re(b)} = L_2(\Gamma_b, H)$, where

$$\Re(b) = \{x(b;u) : u \in L_2^{\Gamma}([0,b],U)\}.$$

Theorem 4.3. *Assume hypotheses* (i)-(iii) *are satisfied. Then the system* (8) *is approximately controllable.*

Proof. For any $\alpha > 0$ define the operator $\tilde{F_\alpha}$ on $H_2([0,b],X)$ by $\tilde{F_\alpha}(x) = z$ where

$$\begin{aligned} z(t) &= R(t)x_0+\int_0^t R(t-s)[Bu(s)+f(s,x(s))]ds+\int_0^t R(t-s)g(s,x(s))dw(s),\\ u(t) &= B^*R^*(b-t)\left[(\alpha I+\Psi_0^b)^{-1}(\mathbf{E}\tilde{x}_b-R(b)x_0)+\int_0^t(\alpha I+\Psi_s^b)^{-1}\tilde{\phi}(s)dw(s)\right]\\ &\quad -B^*R^*(b-t)\int_0^t(\alpha I+\Psi_s^b)^{-1}R(b-s)f(s,x(s))ds\\ &\quad -B^*R^*(b-t)\int_0^t(\alpha I+\Psi_s^b)^{-1}R(b-s)g(s,x(s))dw(s) \end{aligned}$$

and $\tilde{\phi}(\cdot) \in L_2^{\Gamma}(J, L_2^0)$ from the representation $\tilde{x}_b = \mathbf{E}\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$ of $\tilde{x}_b \in L_2(\Gamma_b, X)$, see [29]. It will be shown that the system (8) is approximately controllable, if for all $\alpha > 0$ there exists a fixed point of the operator $\tilde{F_\alpha}$. One can easily prove that for all $\alpha > 0$, $\tilde{F_\alpha}$ has a fixed point in H_2, by employing the contraction mapping principle, see [29],[30].

Let x_α be a fixed point of $\tilde{F_\alpha}$ in H_2. By using the stochastic Fubini theorem it is easy to see that

$$\begin{aligned} x_\alpha(b) &= \tilde{x}_b - \alpha(\alpha I+\Psi_0^b)^{-1}(\ \mathbf{E}\tilde{x}_b - R(b)x_0)\\ &\quad +\alpha\int_0^b(\alpha I+\Psi_s^b)^{-1}R(b-s)f(s,x_\alpha(s))ds\\ &\quad +\alpha\int_0^b(\alpha I+\Psi_s^b)^{-1}[R(b-s)g(s,x_\alpha(s))-\tilde{\phi}(s)]dw(s) \end{aligned}$$

By the condition (ii)

$$\|f(s, x_\alpha(s))\|^2 + \|g(s, x_\alpha(s))\|^2 \leq \tilde{N}$$

in $[0, b] \times \Omega$. Then there is a subsequence denoted by $\{f(s, x_\alpha(s)), g(s, x_\alpha(s))\}$ weakly converging to say $(f(s, w), g(s, w))$ in $X \times L_2^0$. Now, the compactness of $R(t)$ implies that $R(t-s)f(s, x_\alpha(s)) \to R(t-s)f(s), R(t-s)g(s, x_\alpha(s)) \to R(t-s)g(s)$ in $J \times \Omega$. On the other hand by assumption (iii) , for all $0 \leq s < b$ the operator

$$\alpha(\alpha I + \Psi_s^b)^{-1} \to 0 \text{ strongly as } \quad \alpha \to 0^+$$

and moreover

$$\|\alpha(\alpha I + \Psi_s^b)^{-1}\| \leq 1$$

Thus by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
E\|x_\alpha(b) - \tilde{x}_b\|^2 \leq\ & \|\alpha(\alpha I + \Psi_0^b)^{-1}(\mathbf{E}\tilde{x}_b - R(b)x_0)\|^2 \\
& + \mathbf{E}\left(\int_0^b \|\alpha(\alpha I + \Psi_s^b)^{-1}\|\|\tilde{\phi}(s)\|^2 ds\right)^{1/2} \\
& + \mathbf{E}\left(\int_0^b \|\alpha(\alpha I + \Psi_s^b)^{-1}\|\|R(b-s)[f(s, x_\alpha(s)) - f(s)]\| ds\right)^{1/2} \\
& + \mathbf{E}\left(\int_0^b \|\alpha(\alpha I + \Psi_s^b)^{-1}\|\|R(b-s)[g(s, x_\alpha(s)) - g(s)]\| ds\right)^{1/2} \\
& \to 0 \text{ as } \alpha \to 0^+.
\end{aligned}$$

This gives the approximate controllability.

Example 4.4. Consider the stochastic version of the heat equation with control

$$\begin{aligned}
\partial z(t, y) &= \left(-\frac{\partial}{\partial y} q(t, y) + f(t, z(t, y)) + \hat{\mu}(t, y)\right)\partial t + g(t, z(t, y))\partial w(t), \\
q(t, y) &= -\left(z_y(t, y) + \int_0^t \tilde{a}(t-s) z_y(s, y) ds\right), t \in [0, b], 0 \leq y \leq \pi \qquad (9) \\
z(0, y) &= z_0(y)
\end{aligned}$$

where $z : [0, b] \to R, \tilde{a} : [0, b] \to R$, $\tilde{a} \in L^1((0, b); R)$ is Γ_t -adapted, positive, nonincreasing and convex kernel. $w(t)$ is standard one dimensional Weiner process. Let $X = E = L_2[0, \pi]$ and define $A : D(A) \subset X \to X$ by $A(z(t, \cdot) = -\frac{\partial}{\partial^2 y} z(t, \cdot)$. It is well known that A is a positive definite, self adjoint operator in X. Moreover using the properties of A and a, it follows from [33], the resolvent operator $\|R(t)\| \leq 1$. The functions $f : [0, b] \times X \to X, g : [0, b] \times X \to L_2(E, X)$ satisfy the conditions of Theorem 4.3. The equation (9) can be written in the abstract form of (8). One can invoke Theorem 4.3, to prove that system (9) is approximately controllable.

Remark 4.5. The technique used here can be extended to investigate the approximate controllability of nonlinear stochastic functional differential equations with unbounded delay by suitably introducing the phase space defined in [1].

Remark 4.6. Consider the following perturbed infinite dimensional stochastic equation

$$
\begin{aligned}
dx(t) &= \left[A\left\{x(t)+\int_0^t a(t-s)x(s)ds\right\}+Bu(t)\right.\\
&\quad \left.+\ f(t,x(t))+F(t,x(t))\right]dt+g(t,x(t))dw(t)+G(t,x(t))dw(t) \qquad (10)\\
x(0) &= x_0
\end{aligned}
$$

where $f,F:J\times X\to X,\ g,G:J\times X\to L_2^0$ are measurable, locally bounded mappings.

It is important to note that mathematically the above equation (10) is more general than (8). On the other hand, from a practical applications point of view, (10) allows some long-range dependence of the noise in the models under consideration.

Mild solution of the above equation is

$$
\begin{aligned}
x(t) &= R(t)x_0+\int_0^t R(t-s)[Bu(s)+f(s,x(s)+F(s,x(s))]ds\\
&\quad +\int_0^t R(t-s)g(s,x(s))dw(s)+\int_0^t R(t-s)G(s,x(s))dw(s)
\end{aligned}
$$

One can easily prove that by adopting and employing the method used in the previous Theorem 4.3, the system (10) is approximately controllable.

Remark 4.7. We can consider the nonlinear stochastic control systems (8) and (10) with $w^{\tilde H}$, where $w^{\tilde H}$ is the fractional Brownian motion with $0<\tilde H<1$. Fractional Brownian motion denotes a family of Gaussian processes with continuous sample paths that are indexed by the Hurst parameter $\tilde H\in(0,1)$ and that have properties that appear empirically in a wide variety of physical phenomena, such as hydrology, economic data, telecommunications, and medicine [13, 23]. Since some physical phenomena are naturally modeled by stochastic partial differential equations and the randomness can be described by a fractional Gaussian noise. More recently, complete controllability of stochastic differential equations in finite dimensions driven by fractional Brownian motion have been considered in [35]. Existence and uniqueness of mild solutions for stochastic semilinear differential equations in Hilbert spaces are obtained with fractional Brownian motion in [13]. Instead of using the Ito isometry, using the fractional Ito isometry discussed in [13, 35], One can establish the approximate controllability of systems (8) and (10) with the fractional Brownian motion.

ACKNOWLEDGMENTS

The research of J. J. Nieto was partially supported by Ministerio de Educacion y Ciencia and FEDER, project MTM2004-06652-C03-01, and by Xunta de Galicia and FEDER, project PGIDIT05PXIC20702PN.

REFERENCES

1. P. Balasubramaniam and S. K. Ntouyas, Controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space, *Journal of Mathematical Analysis and Applications*, **324** (2006), 161-176.

2. M. Benchohra, L. Gorniewicz, S. K. Ntouyas and A. Ouahab, Controllability results for impulsive functional differential inclusions, *Reports on Mathematical Physics*, **54** (2004), 211-228.

3. M. Benchohra and A. Ouahab, Controllability results for functional semilinear differential inclusions in Frechet spaces, *Nonlinear Analysis*, **61** (2005), 405-423.

4. M. Benchohra, L. Gorniewicz, S. K. Ntouyas and A. Ouahab, Controllability results for nondensely defined semilinear functional differential equations, *Zeitschrift für Analysis und ihre Anwendungen*, **25** (2006), 311-325.

5. Y. K. Chang, Controllability of impulsive functional differential systems with infinite delay in Banach spaces, *Chaos, Solitons & Fractals*, **33** (2007), 1601-1609.

6. Y. K. Chang, W. T. Li and J. J. Nieto, Controllability of evolution differential inclusions in Banach spaces, *Nonlinear Analysis*, **67** (2007), 623-632.

7. Y. K. Chang and D. N. Chalishajar, Controllability of mixed Volterra-Fredholm-type integro-differential inclusions in Banach spaces, *Journal of the Franklin Institute*, **345** (2008), 480-486.

8. D. N. Chalishajar, Controllability of mixed Volterra Fredholm-type integro-differential systems in Banach space, *Journal of the Franklin Institute*, **344** (2007), 12-21.

9. D. N. Chalishajar, Controllability of nonlinear integro-differential third order dispersion system, *Journal of Mathematical Analysis and Applications*, **348** (2008), 480-486.

10. J. P. Dauer, N. I. Mahmudov and M. M. Matar, Approximate controllability of backward stochastic evolution equations in Hilbert spaces, *Journal of Mathematical Analysis and Applications*, **323** (2006), 42-56.

11. J. P. Dauer and N. I. Mahmudov, Approximate controllability of semilinear functional equations in Hilbert spaces, *Journal of Mathematical Analysis and Applications*, **273** (2002), 310-327.

12. J. P. Dauer and N. I. Mahmudov, Controllability of stochastic semilinear functional differential equations in Hilbert spaces, *Journal of Mathematical Analysis and Applications*, **290** (2004), 373-394.

13. T. E. Duncan, Y. Hu and B. Pasik-Duncan, Stochastic calculus for fractional Brownian motion I. Theory, *SIAM Journal on Control and Optimization*, **38** (2000), 582-612.

14. L. Gorniewicz, S. K. Ntouyas and D. O'Regan, Controllability of semilinear differential equations and inclusions via semigroup theory in Banach spaces, *Reports on Mathematical Physics* **56** (2005), 437-470.

15. L. Gorniewicz, S. K. Ntouyas and D. O. Regan, Existence and Controllability Results for First-and Second-Order Functional Semilinear Differential Inclusions with Nonlocal Conditions, *Numerical Functional Analysis and Optimization*, **28** (2007), 53-82.

16. C. Gao, K. Li, E. Feng and Z. Xiu, Nonlinear impulsive system of fed-batch culture in fermentative production and its properties, *Chaos Solitons* & *Fractals*, **28** (2006), 271-277.

17. S. Gao, L. Chen, J. J. Nieto and A. Torres, Analysis of a delayed epidemic model with pulse vaccination and saturation incidence, *Vaccine*, **24** (2006), 6037-6045.

18. S. Gao, Z. Teng, J. J. Nieto and A. Torres, Analysis of an SIR Epidemic Model with Pulse Vaccination and Distributed Time Delay, *Journal of Biomedicine and Biotechnology* (to appear).

19. R. Grimmer and F. Kappel, Series expansions for resolvents of Volterra integro-differential equations in Banach space, *SIAM Journal on Mathematical Analysis*, **15** (1984), 595-604.

20. R. Grimmer and J. Pruss, On linear Volterra equations in Banach spaces. Hyperbolic partial differential equations, II. *Computers and Mathematics with Applications*, **11** (1985), 189-205.

21. R. Grimmer, Resolvent operators for integral equations in a Banach space, *Transactions of the American Mathematical Society*, **273** (1982), 333-349.

22. Y. Hino, S. Murakami and T. Naito, *Functional differential equations with infinite delay*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Vol. 1473, 1991.

23. Y. Hu and B. Oksendal, Fractional white noise calculus and applications to finance, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, **6** (2003), 1-32.

24. E. M. Hernandez, D. Santos and P. C. Jose, Existence results for partial neutral integro-differential equation with unbounded delay, *Applicable Analysis*, **86** (2007), 223-237.

25. E. M. Hernandez, M. Rabello and H. R. Hernandez, Existence of solutions for impulsive partial neutral functional differential equations, *Journal of Mathematical Analysis and Applications*, **331** (2007), 1135-1158.

26. J. Henry, Etude de la controlabilité de certains équations paraboliques non-linéaires, Thése détat, Paris, June 1978.

27. D. N. Keck and M. A. McKibben, Abstract semilinear stochastic Ito-Volterra integrodifferential equations, *Journal of Applied Mathematics and Stochastic Analysis*, (2006), 1-22.

28. J. Li, J. J. Nieto and J. Shen, Impulsive periodic boundary value problems of first-order differential equations, *Journal of Mathematical Analysis and Applications*, **325** (2007), 226-236.

29. N. I. Mahmudov, Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces, *SIAM Journal on Control and Optimization*, **42** (2003), 1604-1622.

30. N. I. Mahmudov, Controllability of semilinear stochastic systems in Hilbert spaces, *Journal of Mathematical Analysis and Applications*, **288** (2003), 197-211.

31. N. I. Mahmudov, Approximate controllability of evolution systems with nonlocal conditions, *Nonlinear Analysis*, **68** (2008), 536-546.

32. B. M. Miller and E. YA. Rubinovich, Impulsive Control in Continuous and Discrete-Continuous systems, Kluwer, New York, 2003.

33. Yu. S. Mishura, Abstract Volterra equations with stochastic kernels, *Theory of Probability and Mathematical Statistics*, **64** (2002), 139-151.

34. J. W. Nunziato, On heat conduction in materials with memory, *Quarterly Journal of Applied Mathematics*, **29** (1971), 187-204.

35. R. Sakthivel, J. H. Kim and N. I. Mahmudov, On controllability of nonlinear stochastic systems, *Reports on Mathematical Physics*, **58** (2006), 437-447.

36. R. Sakthivel, N. I. Mahmudov and J. H. Kim, Approximate Controllability of nonlinear impulsive differential systems, *Reports on Mathematical Physics*, **60** (2007), 85-96.

37. S. Tang and L. Chen, Density-dependent birth rate, birth pulses and their population dynamic consequences, *Journal of Mathematical Biology*, **44** (2002), 185-199.

38. W. Wang, H. Wang and Z. Li, The dynamic complexity of a three-species Beddington-type food chain with impulsive control strategy, *Chaos, Solitons & Fractals*, **32** (2007), 1772-1785.

39. T. Yang, Impulsive Systems and Control: Theory and applications, Berlin, Germany, Springer-Verlag, 2001.

40. H. Zhang, L. Chen and J. Nieto, A delayed epidemic model with stage-structure and pulses for pest management strategy, *Nonlinear Anal., Real World Problems*, (to appear).

41. S. T. Zavalishchin and A. N. Sesekin, Dynamic impulse systems, Theory and applications, Kluwer Academic Publishers Group, Dordrecht, 1997.

42. H. X. Zhou, Approximate controllability for a class of semilinear abstract equations, *SIAM Journal on Control and Optimization*, **21**, (1983), 551-565.

R. Sakthivel
Department of Mathematics,
Sungkyunkwan University,
Suwon 440-746,
South Korea

Juan J. Nieto
Departamento de Analisis Matematico,
Universidad de Santiago de Compostela,
15782 Spain

N. I. Mahmudov
Department of Mathematics,
Eastern Mediterranean University,
Gazimagusa, TRNC,
via Mersin 10,
Turkey