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COMMON FIXED POINTS OF NONCOMMUTING DISCONTINUOUS WEAKLY CONTRACTIVE MAPPINGS IN CONE METRIC SPACES

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Abstract. In this paper we prove the existence of coincidence points and common fixed points for a large class of noncommuting discontinuous contractive type mappings in cone metric spaces. These results generalize, extend and unify several well-known recent related results in literature.

1. Introduction

In a very recent paper [14], the authors established some fixed point theorems in cone metric spaces, an ambient space which is obtained by replacing the real axis in the definition of the distance, by an ordered real Banach space whose order is induced by a (positive) cone.

Thereafter, Abbas and Jungck [1] used this setting as ambient space in order to formulate and prove several common fixed point theorems that extend well-known fixed point theorems for contractive type mappings from the case of usual metric spaces. In direct relation to these results, in [19] the authors pointed out that all the fixed point theorems, established in [14] for the case of a cone metric space ordered by a a normal cone P with normal constant K, could fe formulated and proved in the more general case of a cone metric space. Moreover, they presented several interesting and useful facts about normal and regular cones, illustrated with appropriate examples.

On the other hand, the author [11] extended all the coincidence and common fixed point theorems in [1] to a more general class of discontinuous noncommuting mappings, but by restricting the ambient space to the case of usual metric spaces.

As the cone metric spaces clearly form a bigger category than the one of usual metric spaces, it is the main aim of the present paper to extend and unify all the

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results in [14, 1, 11] and [19], in view of the important considerations from [19]. To this end, we present in the following some definitions and basic results that will be needed to state our main results.

The classical Banach's contraction principle is one of the most useful results in fixed point theory. In a metric space setting its statement is given by the next theorem.

Theorem B. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a map satisfying*

$$d(Tx, Ty) \le a\, d(x, y)\,, \quad \text{for all } x, y \in X\,, \tag{1.1}$$

where $0 \le a < 1$ *is constant. Then:*

$(p1)$ T *has a unique fixed point* x^* *in* X*;*

$(p2)$ *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

converges to x^**, for any* $x_0 \in X$*.*

$(p3)$ *The following estimate holds:*

$$d(x_{n+i-1}, x^*) \le \frac{a^i}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots; \; i = 1, 2, \ldots \tag{1.3}$$

$(p4)$ *The rate of convergence of Picard iteration is given by*

$$d(x_n, x^*) \le a\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots \tag{1.4}$$

Remarks. A map satisfying $(p1)$ and $(p2)$ in Theorem B is said to be a *Picard operator*, see [23], [24], while a mapping satisfying (1.1) is usually called *strict contraction* or a*-contraction* or *Banach contraction*. Theorem B shows therefore that any strict contraction is a Picard operator.

Theorem B has many applications in solving nonlinear equations. Its merit is not only to state the existence and uniqueness of the fixed point of the strict contraction T but also to show that the fixed point can be approximated by means of Picard iteration (1.2). Moreover, for this iterative method both *a priori*

$$d(x_n, x^*) \le \frac{a^n}{1-a}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots$$

and *a posteriori*

$$d(x_n, x^*) \le \frac{a}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots; \; i = 1, 2, \ldots$$

error estimates are available, which are both contained in (1.3).

On the other hand, the inequality (1.4) shows that the rate of convergence of Picard iteration is linear in the class of strict contractions.

Despite these important features, Theorem B suffers from one drawback - the contractive condition (1.1) forces T be continuous on X.

It was then natural to ask if there exist or not contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [17] in 1968, who proved a fixed point theorem which extends Theorem B to mappings that need not be continuous on X (but are continuous at their fixed point), see [22], by considering instead of (1.1) the next condition: there exists $b \in \left[0, \frac{1}{2}\right)$ such that

(1.5) $$d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X .$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see for example, [23], [10] and the references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [12], is based on a condition similar to (1.5): there exists $c \in \left[0, \frac{1}{2}\right)$ such that

(1.6) $$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X.$$

For a presentation and comparison of such kind of fixed point theorems, see [20, 21, 18] and [10].

These fixed point results were then complemented by important results regarding the existence of common fixed points of such contractive type mappings. So, Jungck [15] proved in 1976 a common fixed point theorem for commuting maps, thus generalizing Theorem B. In the same spirit, very recently M. Abbas and G. Jungck [1], obtained coincidence and common fixed point theorems for the class of Banach contractions, Kannan contractions and Chatterjea contractions in cone metric spaces, without making use of the commutative property, but based on the so called concept of weakly compatible mappings, introduced by Jungck [16].

On the other hand, in 1972, Zamfirescu [28] obtained a very interesting fixed point theorem, by combining the contractive conditions (1.1) of Banach, (1.5) of Kannan and (1.6) of Chatterjea.

Note that, as shown by Rhoades [20], the contractive conditions (1.1) and (1.5), as well as (1.1) and (1.6), respectively, are independent.

We give here the complete statement of Zamfirescu's fixed point theorem, including also the error and rate of convergence estimates, similar to that given in the very recent paper [11], in view of its extension to coincidence and common fixed points. A complete proof of Theorem 1 can be found in [9] and [10].

Theorem 1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping for which there exist $a \in [0, 1)$, $b, c \in [0, \frac{1}{2})$ such that for all $x, y \in X$, at least one of the following conditions is true:*

(z_1) $d(Tx, Ty) \leq a\, d(x, y)$;

(z_2) $d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big]$;

(z_3) $d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big]$.

Then the Picard iteration $\{x_n\}$ defined by (1.2) and starting from $x_0 \in X$ converges to the unique fixed point x^* of T with the following error estimate

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots$$

where $\delta = \max\left\{a, \dfrac{b}{1-b}, \dfrac{c}{1-c}\right\}$.

Moreover, the convergence rate of the Picard iteration is given by

$$d(x_n, x^*) \leq \delta \cdot d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots \tag{1.7}$$

It is therefore the main aim of this paper to extend and unify all the results in [14, 1, 11] and [19], in view of the important considerations from [19].

2. Preliminaries

Definition 1. Let E be a real Banach space and P a subset of E. P is called a *cone* if

(i) P *is closed, non-empty and* $P \neq \{0\}$;

(ii) $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b;

(iii) $P \cap (-P) = \{0\}$.

Note that the relations $intP + intP \subseteq intP$ and $\lambda intP \subseteq intP (\lambda > 0)$ hold.

For a given cone $P \subseteq E$, we can define on E a partial ordering $\leq$ with respect to P by putting $x \leq y$ if and only if $y - x \in P$. Further, $x < y$ stands for $x \leq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in int\, P$, where $int\, P$ denotes as usually the interior of P.

Definition 2. Let X be a non-empty set. A mapping $d : X \times X \to E$ satisfying

$(d1)$ $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(d3)$ $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. is called a cone metric on X, and (X,d) is called a cone metric space [1].

Note that in the papers [2, 23] these notions were termed ”generalized metric” and ”generalized metric space”, while in [3, 26, 27, 13] and [25] they are called ”K-metric” and ”K-metric spaces” or ”E-metric” and ”E-metric spaces”, respectively. The obtained results there are concerned with fixed point theorems only, directly related to Theorem B.

Definition 3. Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then

- (i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $\varepsilon \in E$ with $0 \ll \varepsilon$, there is a natural number N such that $d(x_n, x) \ll \varepsilon$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
- (ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $\varepsilon \in E$ with $0 \ll \varepsilon$ there is a natural number N such that $d(x_{n+p}, x_n) \ll \varepsilon$ for all $n, p \geq N$;
- (iii) (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 4. ([1]). Let S and T be selfmaps of a nonempty set X. If there exists $x \in X$ such that $Sx = Tx$ then x is called a coincidence point of S and T, while $y = Sx = Tx$ is called a value of coincidence of S and T. If $Sx = Tx = x$, then x is a common fixed point of S and T.

Definition 5. ([16]). Let S and T be selfmaps of a nonempty set X. The pair of mappings S and T is said to be weakly compatible if they commute at their coincidence points.

The next Proposition, which is given in [1] as Proposition 1.4, will be needed to prove our main result.

Proposition 1. *Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

3. Main Result

In [1], the authors obtained three coincidence and common fixed point theorems, corresponding to Banach contraction condition (Theorem 2.1), Kannan's contractive condition (Theorem 2.3) and Chatterjea's contractive conditon (Theorem 2.4), respectively, in cone metric spaces.

Then, the author [11] extended all the coincidence and common fixed point theorems in [1] to a more general class of discontinuous noncommuting mappings, by restricting the ambient space to the case of usual metric spaces. We now establish the corresponding results from [1, 14, 11] and and [19] in a cone metric space setting. Note that our technique of proof, adapted from [11], is significantly different from the one used in [1].

Theorem 2. *Let (X,d) be a cone metric space and let T, $S:X\to X$ be two mappings for which there exist $a\in[0,1)$, $b,c\in[0,\frac{1}{2})$ such that for all $x,y\in X$, at least one of the following conditions is true:*

(z_1) $d(Tx,Ty)\le a\,d(Sx,Sy)$;

(z_2) $d(Tx,Ty)\le b\big[d(Sx,Tx)+d(Sy,Ty)\big]$;

(z_3) $d(Tx,Ty)\le c\big[d(Sx,Ty)+d(Sy,Tx)\big]$.

If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, the iteration $\{Sx_n\}$ defined by (3.5) converges to the unique (coincidence) common fixed point x^* of S and T, for any $x_0\in X$.

Proof. We first fix $x,y\in X$. At least one of (z_1), (z_2) or (z_3) is true. If (z_2) holds, then

$$\begin{aligned} d(Tx,Ty) &\le b[d(Sx,Tx)+d(Sy,Ty)] \\ &\le b\{d(Sx,Tx)+[d(Sy,Sx)+d(Sx,Tx)+d(Tx,Ty)]\}. \end{aligned}$$

So

$$(1-b)\ d(Tx,Ty)\le 2b\,d(Sx,Tx)+b\,d(Sx,Sy),$$

which yields

$$d(Tx,Ty)\le \frac{2b}{1-b}\ d(Sx,Tx)+\frac{b}{1-b}\ d(Sx,Sy). \tag{3.1}$$

If (z_3) holds, then similarly we get

$$d(Tx,Ty)\le \frac{2c}{1-c}\ d(Sx,Tx)+\frac{c}{1-c}\ d(Sx,Sy). \tag{3.2}$$

Therefore, by denoting

$$\delta=\max\left\{a,\frac{b}{1-b},\frac{c}{1-c}\right\},$$

we have $0 \leq \delta < 1$ and then, by (z_1), (3.1) and (3.2), we find that, for all $x, y \in X$, the following inequality

$$d(Tx, Ty) \leq 2\delta \cdot d(Sx, Tx) + \delta \cdot d(Sx, Sy) \tag{3.3}$$

holds. In a similar manner we obtain

$$d(Tx, Ty) \leq 2\delta \cdot d(Sx, Ty) + \delta \cdot d(Sx, Sy), \tag{3.4}$$

valid for all $x, y \in X$.

Let now x_0 be an arbitrary point in X. Since $T(X) \subset T(X)$, we can choose a point x_1 in X such that $Tx_0 = Sx_1$ Continuing in this way, for a x_n in X, we can find $x_{n+1} \in X$ such that

$$Sx_{n+1} = Tx_n,\ n = 0, 1, \ldots \tag{3.5}$$

If $x := x_n,\ y := x_{n-1}$ are two successive approximations defined by (3.5), then by (3.4) we have

$$d(Sx_{n+1}, Sx_n) = d(Tx_n, Tx_{n-1}) \leq 2\delta \cdot d(Sx_n, Tx_{n-1}) + \delta \cdot d(Sx_n, Sx_{n-1}),$$

which in view of (3.5) yields

$$d(Sx_{n+1}, Sx_n) \leq \delta \cdot d(Sx_n, Sx_{n-1}),\ n = 0, 1, 2 \ldots . \tag{3.6}$$

Now by induction, from (3.6) we obtain

$$d(Sx_{n+k}, Sx_{n+k-1}) \leq \delta^k \cdot d(Sx_n, Sx_{n-1}),\ n = 0, 1, 2 \ldots;\ k = 1, 2, \ldots, \tag{3.7}$$

and then, for $p > i$, we get after straightforward calculations

$$d(Sx_{n+p}, Sx_{n+i-1}) \leq \frac{\delta^i(1 - \delta^{p-i+1})}{1 - \delta} \cdot d(Sx_n, Sx_{n-1}),\ n \geq 0;\ i \geq 1. \tag{3.8}$$

For $i = 1$ and then by an inductive process, (3.8) yields

$$d(Sx_{n+p}, Sx_n) \leq \frac{\delta}{1 - \delta} \cdot d(Sx_n, Sx_{n-1}) \leq \frac{\delta^n}{1 - \delta} \cdot d(Sx_1, Sx_0),\ n = 0, 1, 2 \ldots,$$

Let $0 \ll \varepsilon$ be given. Choose $\delta > 0$ such that $\varepsilon + N_\delta(0) \subset P$, where $N_\delta(0) = \{y \in E : \|y\| \leq \delta\}$.

Also choose a natural number N_1 such that

$$\frac{\delta^n}{1 - \delta} \cdot d(Sx_1, Sx_0) \in N_\delta(0), \text{ for all } n \geq N_1.$$

Then

$$\frac{\delta^n}{1-\delta} \cdot d(Sx_1, Sx_0) \ll \epsilon, \text{ for all } n \geq N_1$$

and hence

$$d(Sx_{n+p}, Sx_n) \leq \frac{\delta}{1-\delta} \cdot d(Sx_n, Sx_{n-1}) \ll \varepsilon, \text{ for all } n, p \geq N_1$$

which shows that $\{Sx_n\}$ is a Cauchy sequence.

Since $S(X)$ is complete, there exists x^* in $S(X)$ such that

$$\lim_{n\to\infty} Sx_{n+1} = x^*. \tag{3.9}$$

We can find $p \in X$ such that $Sp = x^*$. By (3.5) and (3.6) we further have

$$d(Sx_n, Tp) = d(Tx_{n-1}, Tp) \leq \delta d(Sx_{n-1}, Sp) \leq \delta^{n-1} d(Sx_1, Sp),$$

which shows that we also have

$$\lim_{n\to\infty} Sx_n = Tp. \tag{3.10}$$

By (3.9) and (3.10) it results now that $Tp = Sp$, that is, p is a coincidence point of T and S (or x^* is a point of coincidence of T and S).

Now let us show that T and S have a unique point of coincidence. Assume there exists $q \in X$ such that $Tq = Sq$. Then, by (3.3) we get

$$d(Sq, Sp) = d(Tq, Tp) \leq 2\delta d(Sq, Tq) + \delta d(Sq, Tp) = \delta d(Sq, Sp)$$

which yields

$$(1-\delta) d(Sq, Sp) \leq 0.$$

As by definition, $0 \leq d(Sq, Sp)$, that is, $d(Sq, Sp) \in P$, by the previous inequality we obtain $-d(Sq, Sp) \in P$, since $\frac{1}{1-\delta} > 0$. This means that $d(Sq, Sp) = 0$, which shows that $Sq = Sp = x^*$, that is T and S have a unique point of coincidence, x^*.

Now if T and S have are weakly compatible, by Proposition 1 it follows that x^* is their unique common fixed point. ■

Particular cases

(1) If in Theorem 2, condition (z_1) holds for all $x, y \in X$, then by Theorem 2 we obtain Theorem 2.1 in [1].

(2) If in Theorem 2, condition (z_2) holds for all $x, y \in X$, then by Theorem 2 we obtain Theorem 2.3 in [1], that is, Corollary 1 in this paper.

Corollary 1. *Let (X, d) be a cone metric space and P a cone with normal constant K. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \leq k\,[d(fx, gx) + d(fy, gy)]\,,\ \forall x, y \in X, \tag{3.11}$$

where $k \in [0, \frac{1}{2})$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then then f and g have a unique common fixed point in X.

(3) If in Theorem 2, condition (z_3) holds for all $x, y \in X$, then by Theorem 2 we obtain Theorem 2.4 in [1].

Corollary 2. *Let (X, d) be a metric space and let $T, S : X \to X$ be two mappings for which there exist $c \in [0, \frac{1}{2})$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq c\big[d(Sx, Ty) + d(Sy, Tx)\big].$$

If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, the iteration $\{Sx_n\}$ defined by (3.5) converges to the unique (coincidence) common fixed point x^* of S and T, for any $x_0 \in X$.

(4) If in Theorem 2, the cone $P = \mathbb{R}_+$, the nonnegative real semi-axis, then by Theorem 2 we obtain the main result (Theorem 3) in [11].

4. Conclusions

By Theorem 2 we can also obtain a common fixed point result for mappings that satisfy a single contractive condition, as in the next theorem.

Theorem 3. *Let (X, d) be a cone metric space and let $T, S : X \to X$ be two mappings for which there exist $h \in [0, 1)$ such that*

$$\begin{aligned} d(Tx, Ty) \leq h \cdot \max\big\{&d(Sx, Sy), [d(Sx, Tx) + d(Sy, Ty)]/2, \\ &[d(Sx, Ty) + d(Sy, Tx)]/2\big\}, \text{for all } x, y \in X\,. \end{aligned} \tag{4.1}$$

If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, the iteration $\{Sx_n\}$ defined by (3.5) converges to the unique (coincidence) common fixed point x^* of S and T, for any $x_0 \in X$.

Proof. The contractive condition (4.1) is equivalent to Zamfirescu's conditions, see [20], so the theorem follows by Theorem 2. ■

Remark 1. Note that all results in this paper remain valid if we replace the assumption "$S(X)$ is a complete subspace of X" by "(X,d) is a complete (cone) metric space".

The next example shows that the generalizations given by our results in this paper are effective.

Example 1. Let $E=\mathbb{R}^2$ be the Euclidean plane, and

$$P=\{(x,y)\in\mathbb{R}^2 : x,y\geq 0\}$$

be the positive cone of E.

Let $X=\{(x,0)\in\mathbb{R}^2 : 0\leq x\leq 1\}$ and define $d: X\times X\to P$ by

$$d((x,0),(y,0))=\left(|x-y|,\frac{1}{2}|x-y|\right),\ \forall (x,0),(y,0)\in X. \tag{4.2}$$

It is a simple task to show that (X,d) is a complete cone metric space. Let $T,S: X\to X$ be defined by

$$T(x,0)=\begin{cases}\left(\dfrac{x}{4},0\right), & 0\leq x<1\\ \left(\dfrac{1}{3},0\right), & x=1\end{cases}$$

and

$$S(x,0)=\begin{cases}(x,0), & 0\leq x<1\\ \left(\dfrac{2}{3},0\right), & x=1,\end{cases}$$

respectively.

We have $T(X)=\{(x,0): 0\leq x<1/4\}\cup\{(1/3,0)\}\subset\{(x,0): 0\leq x<1\}=S(X)$. Moreover, $(0,0)$ is the unique coincidence point of S and T and, since, obviously, T and S commute at $(0,0)$, S and T are weakly compatible. In order to show that S and T do satisfy the contractive conditions in Theorem 2 (and also in Theorem 3), let us denote

$$M_1=[0,1)\times[0,1);\ M_2=[0,1)\times\{1\}\cup\{1\}\times[0,1);\ M_3=\{1\}\times\{1\}.$$

Clearly, $[0,1]\times[0,1]=M_1\cup M_2\cup M_3$. For $(x,y)\in M_1$, S and T satisfy condition (z_1), with constant $a=1/4$. Indeed, by (z_1) we get

$$\left(\left|\frac{x}{4}-\frac{y}{4}\right|,\frac{1}{2}\left|\frac{x}{4}-\frac{y}{4}\right|\right)\leq a\cdot\left(|x-y|,\frac{1}{2}|x-y|\right)$$

and both components reduce to the obvious relation

$$\left|\frac{x}{4}-\frac{y}{4}\right|\leq a\,|x-y|\,.$$

For $(x,y)\in M_3$, $T(x,0)=T(y,0)$ and so (z_2) is obviously satisfied. Consider now $(x,y)\in M_2$. Due to the symmetry of the contractive condition (z_3), it suffices to show that (z_3) is satisfied for all $x\in[0,1)$ and $y=1$. As $T(x,0)=\left(\frac{x}{4},0\right)$, $T(1,0)=\left(\frac{1}{3},0\right)$, $S(x,0)=(x,0)$ and $S(1,0)=\left(\frac{2}{3},0\right)$, in view of (4.2), condition (z_3) reduces to show that there exists a constant c, $0\leq c<\frac{1}{2}$, such that

$$\left|\frac{x}{4}-\frac{1}{3}\right|\leq c\left(\left|x-\frac{1}{3}\right|+\left|\frac{2}{3}-\frac{x}{4}\right|\right),\ \forall x\in[0,1). \tag{4.3}$$

We shall prove that (4.3) holds with $c=\frac{3}{7}<\frac{1}{2}$, that is,

$$\left|\frac{x}{4}-\frac{1}{3}\right|\leq \frac{3}{7}\left(\left|x-\frac{1}{3}\right|+\left|\frac{2}{3}-\frac{x}{4}\right|\right),\ \forall x\in[0,1). \tag{4.4}$$

As, for $x\in[0,1)$, $\frac{x}{4}<\frac{1}{4}<\frac{1}{3}$, we have $\left|\frac{x}{4}-\frac{1}{3}\right|=\frac{1}{3}-\frac{x}{4}$. Similarly, since $\frac{2}{3}>\frac{1}{4}>\frac{x}{4}$, we get $\left|\frac{2}{3}-\frac{x}{4}\right|=\frac{2}{3}-\frac{x}{4}$ and therefore, (4.4) becomes

$$\frac{1}{3}-\frac{x}{4}\leq \frac{3}{7}\left(\left|x-\frac{1}{3}\right|+\frac{2}{3}-\frac{x}{4}\right),\ \forall x\in[0,1). \tag{4.5}$$

If $0\leq x\leq\frac{1}{3}$, then $\left|x-\frac{1}{3}\right|=\frac{1}{3}-x$ and (4.5) reduces to $x\leq\frac{1}{3}$, which is obviously true.

If $\frac{1}{3}\leq x<1$, then $\left|x-\frac{1}{3}\right|=x-\frac{1}{3}$ and (4.5) reduces to $x\geq\frac{1}{3}$, which is also true. This proves that (4.3) is satisfied and therefore, S and T do satisfy condition (z_3) on M_3.

Note that T and S do not satisfy condition (z_1) on the whole space X. Indeed, this would imply that there exists a constant a, $0 \le a < 1$, such that

$$d\left(T(x,0), T(y,0)\right) \le a \cdot d\left(S(x,0), T(y,0)\right),\ \forall x, y \in [0,1]. \tag{4.6}$$

Take $0 \le x < 1$ and $y = 1$. In view of (4.2), condition (4.6) reduces to

$$\frac{1}{3} - \frac{x}{4} \le a(1-x),$$

which by letting $x \to 1$ yields the contradiction $\dfrac{1}{12} \le 0$.

So, Theorem 2.1 in [1] do not apply. Moreover, S and T do not satisfy (z_2) on the whole X. Indeed, for $0 \le x < 1$ and $y = 1$, in view of (4.2), condition (z_2) reduces to

$$\left|\frac{x}{4} - \frac{1}{3}\right| \le b\left(\left|x - \frac{x}{4}\right| + \left|1 - \frac{2}{3}\right|\right),\ \forall x \in [0,1), \tag{4.7}$$

where $0 \le b < \dfrac{1}{2}$. If we take $x = 0$ in (4.7), we get

$$\frac{1}{3} \le b \cdot \frac{1}{3} < \frac{1}{6},$$

a contradiction. This shows that S and T do not satisfy (z_2) on the whole X and hence, Theorem 2.3 in [1] cannot be applied here. In view of Remark 1, both Theorem 2 and Theorem 3 in our paper do apply and $(0,0)$ is the unique common fixed point of S and T.

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