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PERSISTENCE OF UNIFORMLY HYPERBOLIC LOWER DIMENSIONAL INVARIANT TORI OF HAMILTONIAN SYSTEMS

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Abstract. In this paper, we prove that the normally uniform-hyperbolic lower dimensional invariant tori of the un-perturbed system will persist under small perturbations. The proof is based on the theory of exponentially dichotomous linear systems and an improved KAM machinery adapted for the perturbations of angle dependent unperturbed parts.

1. INTRODUCTION

In recent decades, persistence of invariant tori has been extensively studied by many authors (see, e.g., [5, 6, 9, 8, 13, 15, 20, 21, 25, 27, 28]). The first persistence result of the hyperbolic lower dimensional tori, given by Moser in [18], was for the following Hamiltonian system:

$$H = e + \langle \omega, y \rangle + \frac{1}{2}\langle y, Ay \rangle + \frac{1}{2}\langle z, Mz \rangle + P(x, y, z)$$

where $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2m}$, $\omega \in \mathbb{R}^n$ is a fixed Diophantine toral frequency, A, M are $n \times n, 2m \times 2m$ non-singular constant matrices respectively, JM (J is the standard symplectic matrix in the normal phase space) is hyperbolic with all eigenvalues being real and distinct, and P is a small perturbation. In [10], Graff generalized Moser's result under the hyperbolic condition that the eigenvalues of JM have nonzero real part. For the Lindstedt series approach to the persistence of hyperbolic tori in Hamiltonian systems, we refer the reader to [7, 12]. In the papers mentioned above, M is often taken to be a constant matrix. In [29], under the positive condition

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$$\mathrm{Re}\langle \Omega(x)\xi, \xi\rangle \geq \mu|\xi|^2, \quad \xi \in \mathbb{C}^m, \mu > 0,$$

Zehnder proved the persistence of lower dimensional invariant tori of the Hamiltonian system

$$h(x, y, z_+, z_-) = e + \langle \omega, y\rangle + \langle \Omega(x)z_+, z_-\rangle + O_3(|y| + |z_+| + |z_-|),$$

by implicit function theorem, where $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{C}^{2m}$, $z = (z_+, z_-)$. In [3] ([4]), under the similar positive condition as that in [29], the persistence of the lower dimensional hyperbolic invariant tori of the Hamiltonian system (with degeneracy)

$$H(x, y, z) = h(y) + \langle z_-, \Omega(x, y)z_+\rangle + R(x, y, z)$$

is achieved by KAM theory. Here the coefficients matrix Ω could depend on the action variable y, but it's not essential. In the above three cases, the coefficients matrix M in the normal direction $z = (z_+, z_-)$ may be far from x-independent matrices, but M has the special form

$$M = \begin{pmatrix} 0 & \Omega \\ \Omega^T & 0 \end{pmatrix}.$$

In 2005, Li and Yi [16] further generalized the Graff-Zehnder result to the following more general Hamiltonian systems

$$H = e(\lambda) + \langle \omega(\lambda), y\rangle + \frac{1}{2}\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \mathcal{M}(x, \lambda)\begin{pmatrix} y \\ z \end{pmatrix}\right\rangle + h(x, y, z, \lambda) + P(x, y, z, \lambda)$$

where $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2m}$, λ is a parameter, the matrix

$$\mathcal{M}(x, \lambda) = \begin{pmatrix} A(x, \lambda) & B(x, \lambda) \\ B(x, \lambda)^T & M(x, \lambda) \end{pmatrix}$$

is symmetric and the matrix B and M are close to some constant matrix (i.e., close to matrices independent on x), $h(x, y, z, \lambda) = O(|(y, z)|^3)$ and P is a small perturbation. The lower dimensional tori considered by Li and Yi in [16] is hyperbolic. The main innovation of their paper is to define the hyperbolicity by the average of $\mathcal{M}(x, \lambda)$ instead of $M(x, \lambda)$ which applies to more general situations.

In this paper, we give a persistence result of lower dimensional invariant tori of Hamiltonian systems with more general form of M by assuming that the unperturbed tori are uniformly hyperbolic (or exponentially dichotomous in the other terminology). For simplicity, we consider the real analytic Hamiltonian systems of the form

$$H = e(\lambda) + \langle \omega(\lambda), y\rangle + \frac{1}{2}\langle z, M(x,\lambda)z\rangle + P(x,y,z,\lambda), \tag{1.1}$$

where $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2m}$, λ is a parameter in a bounded closed connected domain $\mathcal{O} \subset \mathbb{R}^k$. The functions e, ω, M and P are real analytic on $\mathcal{O}$. The matrix function M is symmetric, real analytic in $x \in D(s) = \{x \in \mathbb{C}^n/(2\pi\mathbb{Z}^n) : |\mathrm{Im}\, x| \le s\}$ and the perturbation P is real analytic in a complex neighborhood $D(s, r) = \{(x, y, z) : |\mathrm{Im}\, x| \le s, |y| \le r^2, \|z\| \le r\}$ of $T^n \times \{0\} \times \{0\}$. we remark that (1.1) can not be reduced to the special case considered by Zehnder in [29], and the proof in this paper is valid for a more general case as that considered by Li and Yi.

In the present paper, we shall use the symbol $\|\cdot\|$ to denote the Eulidean norm of vectors and the operator norm of matrices, the symbol $|\cdot|$ to denote the standard l_1-norm in the lattice $\mathbb{Z}^n$ and the Lebesgue measure of some set in $\mathbb{R}^k$ and $[\cdot]$ to denote the average of a function on the torus. For any two complex vectors ξ, ζ of the same dimension, $\langle \xi, \zeta\rangle$ is the standard inner product. Expand P as

$$P = \sum_{k,p,q} P_{k,p,q}(\lambda) e^{\sqrt{-1}\langle k,x\rangle} y^p z^q.$$

Define:

$$|P|^l_{D(s,r)} = \sup_{|y|\le r^2, \|z\|\le r} \left| \sum_{k,p,q} |P_{k,p,q}(\lambda)|^l e^{s|k|} y^p z^q \right|,$$

where $|\cdot|^l$ denotes C^l norm. The Hamiltonian vector field of P is $X_P = (P_y, -P_x, JP_z)$, where J is the $2m \times 2m$ standard symplectic matrix. Define $|P_y|^l_{D(s,r)} = \max_{1\le i\le n} |P_{y_i}|^l_{D(s,r)}$ and $\|P_z\|^l_{D(s,r)} = (\sum_{j=1}^{2m} (|P_{z_j}|^l_{D(s,r)})^2)^{1/2}$. $|P_x|^l_{D(s,r)}$ is similarly defined. A weight norm of X_P is defined by:

$$\|X_P\|^l_{r;D(s,r)} = |P_y|^l_{D(s,r)} + \frac{1}{r^2}|P_x|^l_{D(s,r)} + \frac{1}{r}\|P_z\|^l_{D(s,r)}.$$

The equations of motion associated to (1.1) read

$$\begin{cases} \dot{x} = \omega(\lambda) + \partial_y P \\ \dot{y} = -\dfrac{1}{2}\partial_x \langle z, M(x,\lambda)z\rangle - \partial_x P \\ \dot{z} = JMz + J\partial_z P \end{cases} \tag{1.2}$$

Thus, the unperturbed system associated to (1.1) admits an invariant torus $\mathbb{T}^n \times \{0\} \times \{0\}$ with toral frequencies $\omega(\lambda)$ for each $\lambda \in \mathcal{O}$. The normal behavior of the invariant torus is determined by the linear skew product systems

(1.3) $$\frac{dz}{dt} = JM(x,\lambda)z, \quad \frac{dx}{dt} = \omega(\lambda),$$

To consider the perturbation of this torus, we assume that

$(H1)$ (Hyperbolicity): *The invariant tori $\mathbb{T}^n \times \{0\} \times \{0\}$ of the unperturbed system is uniformly hyperbolic, i.e., (1.3) is uniformly hyperbolic for all λ with uniform constants K and β independent of λ.*

Precise definition will be given in the next section.

$(H2)$ (Non-degeneracy): *The frequency map $\omega(\lambda)$ satisfies the Rüssmann's condition*

$$\max_{\lambda\in\mathcal{O}} \operatorname{rank}\{\partial^\alpha \omega(\lambda) : |\alpha| \le n-1\} = n.$$

The Rüssmann condition is known to be the weakest non-degenerate condition for the persistence of maximal dimensional invariant tori in nearly integrable analytic Hamiltonian systems [1, 22, 23, 26].

The main result of this paper is the following

Theorem 1. Consider (1.1). *Assume that the conditions (H1), (H2) hold, $l_0 \ge \max\{m, 2\}$ and there is a constant $\mu = \mu(s, r, l_0, M, \omega)$ sufficiently small such that*

(1.4) $$\|X_P\|^l_{D(s,r)} < \gamma^{3l_0+4}\mu, \qquad |l| \le l_0.$$

Then there is a Cantor-like set $\mathcal{O}_\gamma \subset \mathcal{O}$, with $|\mathcal{O} \setminus \mathcal{O}_\gamma| = O(\gamma^{l_0-1})$ for which the following holds. There is a C^{l_0-1} Whitney smooth family of real analytic, symplectic transforms

$$\Phi = \Phi_\lambda : D(\frac{s}{2}, \frac{r}{2}) \to D(s, r),\ \lambda \in \mathcal{O},$$

which are C^{l_0} uniformly closed to the identity such that

$$H \circ \Phi = e_* + \langle \omega_*, y\rangle + \frac{1}{2}\langle z, M_*(x,\lambda)z\rangle + P_*(x, y, z, \lambda),$$

where

$$|e_* - e|^l_{\mathcal{O}_\gamma} = O(\gamma^{n+1}\mu r^2),$$

$$|\omega_* - \omega|^l_{\mathcal{O}_\gamma} = O(\gamma^{n+1}\mu^{\frac{5}{8}} r^2),$$

$$\|M_* - M\|^l_{D(\frac{s}{2})\times\mathcal{O}_\gamma} = O(\gamma^{n+1}\mu^{\frac{1}{4}} r^2)$$

for all $|l| \le l_0$. Moreover,

$$\|\partial_y^p \partial_z^q P_*\|_{(y,z)=(0,0)} \equiv 0, \qquad |2p| + |q| \leq 2.$$

Thus all unperturbed tori T_λ ($\{y=0,\ z=0\}$ at given λ) with $\lambda \in \mathcal{O}_\gamma$ will persist and give rise to a C^{l_0-1} Whitney smooth family of slightly deformed analytic, quasi-periodic, exponentially dichotomous invariant n-tori of the perturbed system with Diophantine toral frequency $\omega_*(\lambda)$.

Theorem 1 implies the following persistence result of uniformly hyperbolic lower dimensional tori of the analytic Hamiltonian system

$$H = H_0(y) + \frac{1}{2}\langle z, M(x)z\rangle + P(x,y,z), \tag{1.5}$$

where $(x,y,z) \in \mathbb{T}^n \times \Sigma \times \mathbb{R}^{2m} \subset \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2m}$.

Theorem 2. *Assume that H_0 is Rüssmann degenerate, i.e.,*

$$\mathit{max}_{y\in D}\ \mathit{rank}\{\partial^\alpha \nabla H_0(y) : |\alpha| \leq n-1\} = n,$$

and the invariant cylinder $z=0$ is uniformly hyperbolic when $P=0$. Then most of invariant tori of the unperturbed system persist under small perturbations P.

2. The Homological Equation

The result will be proved by the KAM iteration. A key ingredient in each KAM iteration step is to solve the homological equation

$$\partial_\omega F^{01} + MJF^{01} = P^{01}, \tag{2.1}$$

where $\partial_\omega = \langle \omega, \partial_x\rangle$ and P^{01} is the coefficient of z of the perturbation P. As M may depend on the angular variable x and may not be close to a constant, it is almost impossible to solve (2.1) by the Fourier series expansion method. We will show that the equation (2.1) has a real analytic solution with some estimations under the assumption (H1) in this sections.

Firstly, let's state precisely the hypothesis (H1). Let $A(x,\lambda) = M(x,\lambda)J$ in the following for simplicity. Consider a family of quasi-periodic linear systems (parameterized by λ)

$$\frac{dz}{dt} = A(x + \omega(\lambda)t, \lambda)z, \tag{2.2}$$

associated to (1.3). Let $\Psi_s^t(x,\lambda)$ be the fundamental matrix of (2.2), i.e.,

$$\frac{\partial \Psi_s^t(x,\lambda)}{\partial t} = A(x+\omega(\lambda),\lambda)\Psi_s^t(x,\lambda),\ \Psi_s^s(x,\lambda) \equiv I_{2m}.$$

Definition 2.1. (1.3) is uniformly hyperbolic if there are projections $C(x,\lambda):\mathbb{C}^{2m}\to\mathbb{C}^{2m}$ dependent continuously on $x\in D(s)$, $\lambda\in\mathcal{O}$ and positive constants K and β independent of $x\in D(s)$, $\lambda\in\mathcal{O}$, such that

$$\|\Psi_0^t(x)C(x)\Psi_\tau^0(x)\| \le K\,e^{-\beta(t-\tau)}, \qquad \tau\le t,$$
$$\|\Psi_0^t(x)(I_{2m}-C(x))\Psi_\tau^0(x)\| \le K\,e^{-\beta(t-\tau)}, \qquad \tau> t,$$

where I_{2m} is the $2m\times 2m$ identity matrix.

Define

$$G_0(\tau,x,\lambda)=\begin{cases}\Psi_\tau^0(x,\lambda)C(x+\omega\tau,\lambda), & \tau\le 0,\\ -\Psi_\tau^0(x,\lambda)(I_{2m}-C(x+\omega\tau,\lambda)), & \tau>0.\end{cases}\tag{2.3}$$

Then,

$$G_t(\tau,x,\lambda)=\Psi_0^t(x,\lambda)G_0(\tau,x,\lambda),\tag{2.4}$$

and

$$\|G_t(\tau,x,\lambda)\|\le K\,e^{-\beta|t-\tau|},\tag{2.5}$$

uniformly for $x\in D(s)$, $\lambda\in\mathcal{O}$; $t,\,\tau\in\mathbb{R}$. Hence, $G_0(\tau,x,\lambda)$ can be referred as the Green's function of the system (2.2). Moreover,

$$\int_{-\infty}^{\infty}\|G_0(\tau,x,\lambda)\|\,d\tau\le\frac{2K}{\beta}=K_1<\infty.\tag{2.6}$$

From [17], we know that the system (2.2) has an unique Green's function $G_0(\tau,x)$ under the assumption (H1) and the matrix $C(x)=\lim_{t\to 0-}G_t(\tau,x)=G_{0-}(\tau,x)$ satisfies $C^2(x)=C(x)$ for any $x\in D(s)$, which is real for the real system (2.2). As the fundamental matrix $\Psi_s^t(x,s,\lambda)$ is also real for the real system (2.2), from (2.3), the Green's function $G_0(\tau,x)$ and the projecting matrix $C(x)$ are real for real $x\in\mathbb{T}^n$.

Through out this section, (H1) is always assumed.

Lemma 2.1. *Suppose that A and f are continuous in $D(s)\times\mathcal{O}$. Then the quasi-pereiodic non-homogeneous linear equation*

$$\frac{dz}{dt}=A(x+\omega t,\lambda)z+f(x+\omega t,\lambda),\tag{2.7}$$

has an unique solution continuously depending on x, λ, and bounded by $K_1\|f\|$.

Denote by $C^0_\omega(D(s))$ the space of all those $F \in C^0(D(s), \mathbb{C}^{2m})$, which admits a continuous directional derivative in the direction ω, and set $\|F\|_{0,\omega} = \|F\| + \|D_\omega F\|$. Although the function u is in general not differentiable, it does admit a directional derivative in the direction ω, which we denote by $D_\omega u$. Consider the following partial differential equations of the first order

$$D_\omega u - Au - f = 0, \tag{2.8}$$

where A and f are continuous functions on $T^n \times D(s) \times \mathcal{O}$.

Lemma 2.1 implies that (2.8) has a solution in $C^0_\omega(D(s))$. Similar to Lemma 2.3 and Corollary 2.4 in [19], we have the following regularity result:

Lemma 2.2. *Suppose that A and f are analytic in $D(s)\times\mathcal{O}$. Then the equation* (2.8) *has an unique solution*

$$u(x) = \int_{-\infty}^{\infty} G_0(\tau, x) f(x + \omega\tau)\, d\tau$$

which is real analytic in $D(s)$, and $\|u\|_{D(s)} \le K_1 \|f\|_{D(s)}$.

Proof. Denote the coordinate of the $2m \times 2m$ dimensional matrix N by

$$N = (n_{11}, \cdots, n_{1,2m}, \cdots, n_{2m,1}, \cdots, n_{2m,2m}).$$

Consider the analytic map Ψ from $C^0(D(s), \mathbb{C}^{2m\times 2m}) \times C^0(D(s), \mathbb{C}^{2m}) \times C^0_\omega(D(s), \mathbb{C}^{2m})$ into $C^0(D(s), \mathbb{C}^{2m})$ given by $(N, g, v) \to D_\omega v - Nv - g$, which vanishes at (A, f). For any $h \in C^0(D(s), \mathbb{C}^{2m})$, the linear map taking $F \in C^0_\omega$ into

$$D_\omega F - AF = h \tag{2.9}$$

has a bounded inverse, since under the assumption (H1) the equation (2.9) has an unique solution $F \in C^0_\omega$ which can be written in the form

$$F(x) = G_A h(x) = \int_{-\infty}^{\infty} G_0(\tau, x) h(x + \omega\tau)\, d\tau.$$

By the implicit function theorem, there exists a neighborhood $U(\epsilon_0)$ of (A, f) and an unique analytic map

$$\Phi : U(\epsilon_0) \to C^0_\omega(D(s), \mathbb{C}^{2m}), \qquad \Phi(A, f) = u,$$

such that for all $(N, h) \in U(\epsilon_0)$, $v = \Phi(N, h)$ satisfies the equation $D_\omega v - Nv - h = 0$. Write $x = a + \sqrt{-1}b$, where $|b| < s$ and $a, b \in \mathbb{R}^n$. Then $u(x) = u(a + \sqrt{-1}b) = \Phi(A(\cdot + \sqrt{-1}b),\ f(\cdot + \sqrt{-1}b))(a)$. Denote by $x =$

$(x_1, \cdots, x_n)^T$, $u(x) = (u_1(x), \cdots, u_{2m}(x))^T$, $f(x) = (f_1(x), \cdots, f_{2m})^T$, $A(x) = (a_{11}, \cdots, a_{1,2m}, \cdots, a_{2m,1}, \cdots, a_{2m,2m})$, and $\Phi = (\Phi_1, \cdots, \Phi_{2m})^T$. Let $u_i(x) = \Phi_i(A, f) = \phi_i(a_{11}(x), \cdots, a_{1,2m}(x), \cdots, a_{2m,1}(x), \cdots, a_{2m,2m}(x), f_1(x), \cdots, f_{2m}(x))$, $i = 1, \cdots, 2m$. If A and f are real analytic in $D(s)$, so are a_{kl} and f_i, where $k, l = 1, \cdots, 2m$, $i = 1, \cdots, 2m$. Then u is continuously differentiable in a_j and b_j, and

$$\frac{\partial u_i(x)}{\partial \overline{x}_j} = \sum_{k,l} \frac{\partial \phi_i}{\partial a_{kl}} \frac{\partial a_{kl}(x)}{\partial \overline{x}_j} + \sum_k \frac{\partial \phi_i}{\partial f_k} \frac{\partial f_k}{\partial \overline{x}_j} = 0,$$

where $j = 1, \cdots, n$; $i = 1, \cdots, 2m$. Hence u satisfies the Cauchy- Riemann equations and is analytic in $D(s)$. As the Green function $G_0(\tau, x)$ is real for real x, $u(x)$ is real analytic in $D(s)$ and

$$\begin{aligned} \|u(x)\|_{D(s)} &= \| \int_{-\infty}^{\infty} G_0(\tau, x) f(x + \omega\tau)\, d\tau \| \\ &\leq \int_{-\infty}^{\infty} \|G_0(\tau, x)\|\, d\tau \cdot \|f\|_{D(s)} \\ &\leq K_1 \|f\|_{D(s)}. \end{aligned}$$

This completes the proof of the lemma.

Note that the hyperbolicity of (1.3) implies the hyperbolicity of

$$\frac{dz}{dt} = -JM(x, \lambda) z, \qquad \frac{dx}{dt} = \omega(\lambda). \tag{2.10}$$

For simplicity, we denote by $G_0(\tau, x, \lambda)$ the Green function of the system (2.10) in the following.

Corollary 1. *For all $\lambda \in \mathcal{O}$ the homological equation (2.1) has an unique real analytic solution*

$$F^{01}(x, \lambda) = -\int_{-\infty}^{\infty} JG_0(\tau, x, \lambda) JP^{01}\, d\tau \tag{2.11}$$

in $x \in D(s)$.

Proof. Let's first consider the linear equation

$$\frac{dz}{dt} = -JM(x + \omega(\lambda)t, \lambda) z - JP^{01},$$

where $x \in D(s)$. From the above analysis, it has an unique real analytic solution

$$f^{01}(x) = -\int_{-\infty}^{\infty} G_0(\tau, x, \lambda) JP^{01}\, d\tau$$

in $x \in D(s)$. Then, it is easy to verify that the real analytic function $F^{01} = Jf^{01}$ solves the homological equation (2.1). This completes the proof.

As the homological equations in each KAM step is a small perturbation of the first step, we will point out that the hypotheses (H1) is kept if the initial perturbation is sufficiently small and the related positive constants K, β of the Green function $G_0(\tau, x, \lambda)$ in the form of the inequality (2.5) at each step can be controlled. The following two lemmas are deduced from [17].

Lemma 2.3. *Assume* $(H1)$ *and the matrix* $\widetilde{M}$ *is analytic in* $D(s)$. *Then the system*

$$\frac{dz}{dt} = J\widetilde{M}(x + \widetilde{\omega}t)z \tag{2.12}$$

is also exponentially dichotomous on $\mathbb{R}$, *if* $|\widetilde{\omega} - \omega|$, $\widetilde{M} - M\|_{D(s)} \le \epsilon_1 = \epsilon_1(M, \omega)$ *for some positive* ϵ_1 *small enough.*

Lemma 2.4. *The assumption* $(H1)$ *is equivalent to the following: there exists a non-degenerate symmetric matrix* $S(x) \in C^1(D(s))$, *for which the matrix* $\widehat{S}(x) = \partial_\omega S(x) + S(x)(JM(x)) + (JM(x))^* S(x)$ *is negative definite for all* $x \in D(s)$, *where* $(JM)^*$ *is the conjugate transpose of* JM. *Moreover, if*

$$\langle \widehat{S}(x)z, z\rangle \le -b\|z\|^2, \tag{2.13}$$

where b *is a constant* ≥ 0, *then the positive constants* K *and* β *in the estimate (2.5) can be represented by the inequality:*

$$K = (2+\sqrt{2})\left(\frac{\|JM\|_{D(s)}\|S\|_{D(s)}}{b}\right)^{\frac{3}{2}}, \quad \beta = \frac{b}{2\|S\|_{D(s)}}. \tag{2.14}$$

With the above two lemmas, we prove that the hyperbolic is preserved under small perturbations. Let $\widehat{\widehat{S}}(x) = \partial_\omega S(x) + S(x)(JM(x)) + (JM(x))^* S(x)$, where $\widetilde{\omega} = \omega + \widehat{\omega}$, $\widetilde{M} = M + \widehat{M}$ and $\|\omega\|$, $\|M\|_{D(s)} \le \epsilon_1$. Then

$$\begin{aligned}
&\langle \widehat{S}(x)z, z\rangle \\
&= \langle \widehat{S}(x)z, z\rangle + \langle \partial_{\widehat{\omega}} S(x)z, z\rangle + \langle S(x)J\widehat{M}(x)z, z\rangle + \langle (JM)^* S(x)z, z\rangle \\
&\le -b\|z\|^2 + 2\epsilon_1 \|S(x)\|_1 \cdot \|z\|^2 \\
&\le -(b - 2\epsilon_1\|S(x)\|_1)\|z\|^2,
\end{aligned}$$

where $\|S(x)\|_1 - \|S(x)\| + \|\frac{\partial S(x)}{\partial x}\|$. As $S(x) \in C^1(D(s))$, there exists a constant $c_1 \ge 0$ such that $\|S(x)\|_1 \le c_1$ for all $x \in D(s)$. If

$$\epsilon_1 \le \frac{b}{4c_1}, \tag{2.15}$$

then $\langle\widehat{\widetilde{S}}(x)z,z\rangle \leq -\frac{1}{2}\|z\|^2$. So the constants K, β corresponding to the Green function $\widetilde{G}_0(t,x)$ of the system (2.12) have the following estimates:

$$\widetilde{K}=(2+\sqrt{2})\left(\frac{\|J\widetilde{M}\|_{D(s)}\|S\|_{D(s)}}{\frac{b}{2}}\right)^{\frac{3}{2}} \leq 2\sqrt{2}\left(1+\frac{\epsilon_1}{\|M\|_{D(s)}}\right)^{\frac{3}{2}} K,\ \widetilde{\beta}=\frac{1}{2}\beta.$$

If

$$\epsilon_1 \leq (\sqrt[3]{2}-1)\|M\|_{D(s)}, \tag{2.16}$$

then we have

$$\widetilde{K} \leq 4K, \qquad \widetilde{\beta}=\frac{1}{2}\beta. \tag{2.17}$$

At the end of this section we consider the continuity and differentiability of the solution $F^{01}(x,\lambda)$ in the parameter $\lambda \in \mathcal{O}$. From the expression of F^{01} in (2.11), the continuity and differentiability of $F^{01}(x,\lambda)$ in λ depends on that's of the perturbation P and the Green function $G_0(\tau,x,\lambda)$. And for the Green function $G_0(\tau,x,\lambda)$ we have the following result [17].

Lemma 2.5. *Assume* (*H1*) *and that the matrix function* $M(x,\lambda)$ *is* C^{l_0} *differentiable in the parameter* $\lambda \in \mathcal{O}$. *Then the Green function* $G_0(\tau,x,\lambda)$ *of the equation (2.1) is continuously differentiable in the parameter* λ *up to order* l_0. *Moreover, for any constant* $0<\alpha<\frac{\beta}{4l_0}$ *such that* $\frac{\beta}{2}-|l|\alpha>\frac{\beta}{4}$, *the estimate*

$$\|G_0(\tau,x,\lambda)\|^l_{D(s)\times\mathcal{O}} \leq c(K,l_0)e^{-(\beta-l\alpha)|\tau|} \tag{2.18}$$

is valid, where $1\leq l\leq l_0$, $c(K,l_0)$ *is a constant independent of* τ, x *and* $\lambda\in\mathcal{O}$.

By (2.11) and (2.18), we have

$$\begin{aligned}
&\|F^{01}(x,\lambda)\|^l_{D(s)\times\mathcal{O}}\\
&\leq \|\int_{-\infty}^{\infty}\sum_{k=0}^{l}\partial_\lambda^k G_0(\tau,x,\lambda)\,\partial_\lambda^{l-k}P^{01}(x+\omega\tau)\,d\tau\|_{D(s)\times\mathcal{O}}\\
&\leq \int_{-\infty}^{\infty}\sum_{k=0}^{l}\|\partial_\lambda^k G_0(\tau,x,\lambda)\|_{D(s)\times\mathcal{O}}\cdot\|\partial_\lambda^{l-k}P^{01}(x+\omega\tau,\lambda)\|_{D(s)\times\mathcal{O}}\,d\tau\\
&\leq \int_{-\infty}^{\infty}c(m,k,l_0)e^{-(\beta-l\alpha)|\tau|}\,d\tau\,\|P^{01}\|^l_{D(s)\times\mathcal{O}}\\
&\leq c(m,K,\beta,l_0)\|P^{01}\|^l_{D(s)\times\mathcal{O}},
\end{aligned} \tag{2.19}$$

for any $1<l\leq l_0$, where $c(m,K,\beta,l_0)$ are positive constants independent of $x\in D(s)$ and $\lambda\in\mathcal{O}$.

3. Proof of Theorem 1

With the above preparation, the main results of this paper can be proved by standard KAM iteration. Fix $\tau \geq \max\{n(n-1)-1, 0\}$. For simplicity, we set $l_0 = n$.

3.1. Outline of KAM steps

Below we give the ideas of one KAM step. In the following, all the quantities represent the quantities in the νth KAM step. The quantities with subscript $+$ represent the quantities in the $(\nu+1)$th KAM step.
At each KAM step, we will consider a Hamiltonian of the form:

$$H = N + P,$$

where

$$N = e(\lambda) + \langle \omega(\lambda), y\rangle + \frac{1}{2}\langle z, Mz\rangle,$$

P is a small perturbation.

Moreover, we assume that

$$\|X_P\|^l_{D(s)} \leq \gamma^{n+1}\mu, \qquad |l| \leq n. \tag{3.1}$$

Truncate P as $P = R + \widetilde{P}$, where

$$R = \sum_{k, 2|p|+|q|\leq 2} P_{k,p,q} e^{\sqrt{-1}\langle k, x\rangle} y^p z^q, \;\; \widetilde{P} = P - R.$$

It follows that $\|X_R\|^l_{D(s)} \leq \gamma^{n+1} r^2 \mu$. We further write R as

$$R = P^0 + \langle P^{10}, y\rangle + \langle P^{01}, z\rangle + \frac{1}{2}\langle z, Mz\rangle, \tag{3.2}$$

where P^{10} is n-dimensional vector, P^{01} is $2m$-dimensional vector, P^{02} is $2m \times 2m$ matrix.

At each KAM step, we will construct a symplectic map Φ such that $H_+ = H \circ \Phi = N_+ + P_+$ with P_+ being much smaller.

3.2. The symplectic change of variables

As usual, we construct the desired symplectic map Φ by the time 1-map of the flow X^t_F of a Hamiltonian vector field X_F.

Let

$$F = F^0(x) + \langle F^{10}(x), y\rangle + \langle F^{01}(x), z\rangle$$

$$= \sum_{0\neq k\in\mathbb{Z}^n} F_k^0 e^{\sqrt{-1}\langle k,x\rangle} + \sum_{0\neq k\in\mathbb{Z}^n} \langle F_k^{10}, y\rangle e^{\sqrt{-1}\langle k,x\rangle} + \langle F^{01}(x), z\rangle, \tag{3.3}$$

where F_k^{10} is a n-dimensional vector, F_k^{01} is a $2m$-dimensional vector.

It follows that

$$\begin{aligned} H\circ\Phi &= N + R + \{N,F\} + \int_0^1 (1-t)\{\{N,F\}+R,F\}\circ X_F^t dt + \tilde{P}\circ X_F^1 \\ &= N_+ + \{N,F\} + \tilde{R} + P_+ \end{aligned}$$

where

$$N_+ = N + [P^0] + \langle [P^{10}], y\rangle + \frac{1}{2}\langle z, P^{02}z\rangle, \tag{3.4}$$

$$\tilde{R} = P^0 - [P^0] + \langle P^{10} - [P^{10}], y\rangle + \langle P^{01}, z\rangle + \frac{1}{2}\langle\langle z, \partial_x M z\rangle, F^{10}\rangle, \tag{3.5}$$

$$\begin{aligned} P_+ &= \int_0^1 \{(1-t)\{N,F\}+R,F\}\circ X_F^t\,dt + \tilde{P}\circ X_F^1 + \frac{1}{2}\langle\langle z, \partial_x M z\rangle, F^{10}\rangle \\ &= \int_0^1 \{R_t, F\}\circ X_F^t\,dt + \tilde{P}\circ X_F^1 + \frac{1}{2}\langle\langle z, \partial_x M z\rangle, F^{10}\rangle, \end{aligned} \tag{3.6}$$

where

$$R_t = R + (1-t)\{N,F\}.$$

We shall prove that

$$\{N,F\} + \tilde{R} = 0 \tag{3.7}$$

is solvable and P_+ is much smaller. Taking F as the solution of the above equation, the time 1-map of the flow X_F^t is the desired map.

$$\begin{aligned} &-\{N,F\} \\ &= \partial_\omega F^0 \end{aligned} \tag{3.8}$$

$$+\langle \partial_\omega F^{10}, y\rangle \tag{3.9}$$

(3.10) $$+\langle \partial\omega F^{01} + MJF^{01}, z\rangle$$

(3.11) $$-\frac{1}{2}\langle\langle z, \partial_x Mz\rangle, F^{10}\rangle$$

By (3.5) (3.7) and (3.8), we have:

(3.12) $$\partial_\omega F^0 = P^0 - [P^0],$$

i.e.,

$$F_k^0 = \frac{1}{\sqrt{-1}\langle\omega, k\rangle} P_k^0,\ k \neq 0$$

If the small divisor conditions

(3.13) $$|\langle\omega, k\rangle| \geq \frac{\gamma}{|k|^\tau},\ k \neq 0$$

hold, by Lemma A.2. in [25], we have

$$|F_k^0|^l \leq \frac{|k|^{(l+1)\tau+l}}{\gamma^{l+1}} |P_k^0|^l,\ k \neq 0$$

Thus we have

(3.14) $$\frac{1}{r^2}|F^0|^l_{D(s-\rho)} \leq \frac{c}{r^2\gamma^{l+1}\rho^v}|P^0|^l_{D(s)} \leq \frac{c\mu}{\rho^v},$$

with $v \geq (l+1)\tau + n + l$.

Now we consider the terms of degree one with respect to z. By (3.5) (3.7) and (3.10), we have:

$$\partial_\omega F^{01} + MJF^{01} = P^{01}.$$

By Corollary 1, the above equation has an unique real analytic solution defined by (2.11)

$$F^{01}(x) = -\int_{-\infty}^{\infty} JG_0(\tau, x)JP^{01}\,d\tau,$$

in $D(s)$. Moreover, by (2.19) there exists a constant $c = c(m, K, \beta, l_0)$ such that

(3.15) $$\frac{1}{r}\|F^{01}\|^l_{D(s-\rho)} \leq c\|P^{01}\|^l_{D(s)} \leq c\gamma^{n+1}\mu,$$

where the constants K and β is the same constants as in Definition 2.1.

By (3.5), (3.9), we have

$$\partial_\omega F^{10} = P^{10} - [P^{10}],$$

i.e.,

$$F_k^{10} = \frac{1}{\sqrt{-1}\langle \omega, k\rangle} P_k^{10},\ k \neq 0.$$

If (3.13) holds, we have:

$$\frac{1}{r^2}\|F^{10}\|^l_{D(s-\rho)} \leq \frac{c}{r^2\gamma^{l+1}\rho^v}\|P^{10}\|^l \leq \frac{c\mu}{\rho^v}. \tag{3.16}$$

Combining (3.14)–(3.16), we find a function F such that

$$\{N, F\} + \tilde{R} = 0.$$

And

$$\begin{aligned} \|X_F\|^l_{r,D(s-2\rho)} &= \frac{1}{r^2}|F_x|^l_{D(s-2\rho)} + |F_y|^l_{D(s-2\rho)} + \frac{1}{r}\|F_z\|^l_{D(s-2\rho)} \\ &\leq c(\frac{1}{\rho r^2}|F^0|^l_{D(s-\rho)} + |F^{10}|^l_{D(s-\rho)} + \frac{1}{r}\|F^{01}\|^l_{D(s-\rho)}) \leq \frac{c\mu}{\rho^v}, \end{aligned} \tag{3.17}$$

if $\lambda \in \mathcal{O}_+$ satisfies (3.13).

Moreover,

$$|e_+ - e|^l = |[P^0]|^l \leq c\gamma^{n+1}r^2\mu, \tag{3.18}$$

$$|\omega_+ - \omega|^l = |[P^{10}]|^l \leq c\gamma^{n+1}r\mu, \tag{3.19}$$

$$\|M_+ - M\|^l_{D(s)} = \|P^{02}\|^l_{D(s)} \leq c\gamma^{n+1}\mu. \tag{3.20}$$

Remark 3.1. By the Whitney's extension theorem in [24], a function defined on $\mathcal{O}_\gamma$ can be extended to $\mathcal{O}$ such that all the estimates still hold on $\mathcal{O}$. So we always regard all functions of λ in the KAM steps to be defined on $\mathcal{O}$ and ignore the domain in the estimates, but it makes sense only for $\lambda \in \mathcal{O}_\gamma$.

3.3. Estimates for the new perturbation

To complete the KAM step, we have to estimate the new perturbation P_+.

For small constant $\delta > 0$,

$$\|X_{\tilde{P}}\|^l_{r^{1+\delta},D(s,2r^{1+\delta})} \leq c\gamma^{n+1}\mu r^\delta. \tag{3.21}$$

If the inequality

$$\|M(x)\|^l_{D(s)} \leq 2\|M_0(x)\|^l_{D(s)} \tag{3.22}$$

is satisfied, we have

$$\begin{aligned}
&|\langle\langle z,\partial_x M(x)z\rangle, F^{01}\rangle|^l_{D(s-2\rho,2r^{1+\delta})}\\
&\le cr^{2(1+\delta)}\|\partial_x M(x)\|^l_{D(s-\rho)}\cdot\|F^{10}\|^l_{D(s-\rho)}\\
&\le \frac{cr^{2(1+\delta)}}{\rho}\|M_0\|^l_{D(s)}\cdot\frac{c\mu r}{\rho^v}\\
&\le \frac{c\mu r^{3+2\delta}}{\rho^{v+1}}.
\end{aligned}$$

So

$$\begin{aligned}
&\|X_{\langle\langle z,\partial_x M(x)z\rangle,F^{01}\rangle|}\|^l_{r^{1+\delta};D(s-3\rho,r^{1+\delta})}\\
&= |\partial_y\langle\langle z,\partial_x M(x)z\rangle, F^{01}\rangle|^l_{D(s-3\rho,r^{1+\delta})}\\
&\quad+\frac{1}{r^{2(1+\delta)}}|\partial_x\langle\langle z,\partial_x M(x)z\rangle, F^{01}\rangle|^l_{D(s-3\rho,r^{1+\delta})}\\
&\quad+\frac{1}{r^{1+\delta}}\|\partial_z\langle\langle z,\partial_x M(x)z\rangle, F^{01}\rangle\|^l_{D(s-3\rho,r^{1+\delta})}\\
&\le c\frac{|\langle\langle z,\partial_x M(x)z\rangle, F^{01}\rangle|^l_{D(s-2\rho,2r^{1+\delta})}}{\rho r^{2+2\delta}}\le\frac{c\mu r}{\rho^{v+2}}.
\end{aligned}\tag{3.23}$$

To estimate P_+ in (3.6), we first estimate the symplectic map X_F^t.

Lemma 3.6. *If X_F satisfies (3.17) and*

$$2E=\frac{2c\mu r}{\rho^{v+2}}\le 1,\tag{3.24}$$

then we have

$$\frac{1}{\rho}\|X_F^t-id\|^l_{D(s-2\rho,\frac{r}{2})},\ \|DX_F^t-Id\|^l_{r;D(s-3\rho,\frac{r}{4})}\le cE\le\frac{1}{2},$$

$$\|D^2X_F^t-Id\|_{r;r;D(s-4\rho,\frac{r}{8})}\le cE,$$

for $|t|\le 1$, where D is the differential operator with respect to (x,y,z). c is independent of KAM steps.

The preceding estimates imply that for each $\lambda\in\mathcal{O}_+$,

$$X_F^t(\cdot,\lambda): D(s-5\rho,r^{1+\delta})\to D(s-4\rho,2r^{1+\delta}),\qquad \forall|t|\le 1.$$

By (3.5) and (3.23), we have

$$\begin{aligned}\|X_{\tilde{R}}\|^l_{r^{1+\delta};D(s-3\rho,r^{1+\delta})} &\le c\left(\|X_P\|^l_{r,D(s,r)} + \|X_{\langle\langle z,\partial_x M(x)z\rangle,F^{01}\rangle|}\|^l_{r^{1+\delta};D(s-3\rho,r^{1+\delta})}\right)\\ &\le c(\gamma^{n+1}\mu + \frac{\mu r}{\rho^{v+2}}).\end{aligned}$$

So by (3.2) (3.7) and (3.5) we have

$$\|X_{R_t}\|^l_{r^{1+\delta};D(s-3\rho,r^{1+\delta})} \le c(\gamma^{n+1}\mu + \frac{\mu r}{\rho^{v+2}}). \tag{3.25}$$

Similar to Lemma A.4 in [25], we have the following lemma:

Lemma 3.7. *If a Hamiltonian vector field $W(\cdot,\lambda)$ is analytic on $V = D(s-2\rho, 3r^{1+\delta})$ depending on the parameter λ with $\|W\|^l_{r;V} < +\infty$, and $\Phi = X^t_F : U \to \overline{V}$, where $U = D(s-4\rho, r^{1+\delta})$, $\overline{V} = D(s-3\rho, 2r^{1+\delta})$, then $\Phi^* W = (D\Phi)^{-1} W \circ \Phi$. Moreover, if*

$$\frac{1}{\rho}\|\Phi - id\|^l_{r^{1+\delta};U},\ \|D\Phi - Id\|^l_{r^{1+\delta};r^{1+\delta};U} \le cE \le \frac{1}{2},$$

we have $\|\Phi^* W\|^l_{r^{1+\delta},U} \le 4\|W\|^l_{r^{1+\delta},V}$.

So if r_0 is sufficiently small, then

$$\begin{aligned}&\|X_{P_+}\|^l_{r^{1+\delta};D(s-5\rho,r^{1+\delta})}\\ \le\ & 4\|X_P\|^l_{r^{1+\delta};D(s-4\rho,2r^{1+\delta})} + 4\int_0^1 \|[X_{R_t}, X_F]\|^l_{r^{1+\delta};D(s-4\rho,2r^{1+\delta})}dt\\ &+\|X_{\langle\langle z,\partial_x M(x)z\rangle,F^{01}\rangle|}\|^l_{r^{1+\delta};D(s-3\rho,r^{1+\delta})}\end{aligned}$$

By Cauchy's inequality and Lemma 3.6,

$$\begin{aligned}\|[X_{R_t}, X_F]\|^l_{r^{1+\delta};D(s-4\rho,2r^{1+\delta})} &\le \frac{1}{r^{2\delta}}\|DX_{R_t}X_F - DX_F X_{R_t}\|^l_{r;D(s-4\rho,2r^{1+\delta})}\\ &\le \frac{c}{r^{2\delta}\rho}\|X_{R_t}\|^l_{r;D(s,r)}\|X_F\|^l_{r;D(s-\rho,r)}\\ &\le \frac{c\mu^2 r^{1-2\delta}}{\rho^{v+3}}(\gamma^{n+1} + \frac{r}{\rho^{v+2}}).\end{aligned}$$

So

$$\|X_{P_+}\|^l_{r^{1+\delta};D(s-5\rho,r^{1+\delta})} \le c\gamma^{n+1}\mu r^{1+\delta} + \frac{c\mu^2 r^{1-2\delta}}{\rho^{v+3}}(\gamma^{n+1} + \frac{r}{\rho^{v+2}}) + \frac{c\mu r}{\rho^{v+2}}. \tag{3.26}$$

The KAM step is now complete.

3.4. Iteration Lemma and convergence

For given γ, μ, s, r in the introduction, we set $e_1 = e$, $\omega_1 = \omega$, $M_1 = M$, $N_1 = N$, $P_1 = P$, $E_1 = \frac{\mu_1 r_1}{\rho_1^{v+2}}$, $\mathcal{O}_1 = \mathcal{O}$, $\gamma_1 = \gamma$, $s_1 = s$, $r_1 = \mu^{\frac{3}{8}}$, $\mu_1 = \mu^{\frac{1}{4}} r^2$, $\rho_1 = \frac{s_1}{20}$ initially.

Define some sequences inductively as follows:

$$r_{\nu+1} = r^{\frac{11}{9}},\ s_{\nu+1} = s_\nu - 5\rho_\nu,\ \rho_{\nu+1} = \frac{1}{20}\rho_\nu$$

$$\mu_{\nu+1} = \mu_\nu^{\frac{10}{9}},\ E_{\nu+1} = \frac{\mu_{\nu+1} r_{\nu+1}}{\rho_{\nu+1}^{v+2}},\ \gamma_{\nu+1} = \frac{\gamma}{2}(1 + 2^{-\nu-1}).$$

Then by (3.26)

$$\begin{aligned} \|X_{P_+}\|^l_{D(s_+,r_+)} \le \gamma_+^{n+1}\mu_+ c \Bigg(\left(\frac{\gamma}{\gamma_+}\right)^{n+1} \frac{r^{\frac{11}{9}}}{\mu^{\frac{1}{9}}} \\ + \left(\frac{\gamma}{\gamma_+}\right)^{n+1} \frac{\mu^{\frac{8}{9}} r^{\frac{5}{9}}}{\rho^{v+3}} + \frac{r}{\rho^{v+2}\gamma_+^{n+2}} \Bigg). \end{aligned} \tag{3.27}$$

If

$$\begin{aligned} \mu \le \epsilon_2 = \min\Bigg\{ \left[\left(\frac{1}{2}\right)^{n+1} (3c)^{-1} r^{\frac{2}{9}}\right]^{\frac{72}{31}}, \left[(3c)^{-1}\left(\frac{s}{20}\right)^{v+2}\left(\frac{1}{2}\gamma\right)^{n+2}\right]^{\frac{8}{3}}, \\ \left[(3c)^{-1}\left(\frac{1}{2}\right)^{n+1}\left(\frac{s}{20}\right)^{v+3} r^{-\frac{16}{9}}\right]^{\frac{72}{31}} \Bigg\}, \end{aligned} \tag{3.28}$$

then by (3.27) we will have

$$\|X_{P_+}\|^l_{D(s_+,r_+)} \le \gamma_+^{n+1}\mu_+.$$

Let

$$D_\nu = D(s_\nu, r_\nu).$$

The proceeding analysis may be summarized as the following iteration lemma.

Lemma 3.8. *If*

$$\mu \le \epsilon_3 = min\left\{ \left(\frac{s^{v+2}}{20^{v+2} 2cr^2}\right)^{\frac{8}{5}},\ \left(\frac{1}{20\rho^{\frac{1}{9}}}\right)^{\frac{8(v+2)}{3}},\ \frac{s^{v+2}}{20^{v+2}2c},\ \epsilon_2 \right\}, \tag{3.29}$$

the following holds for all $\nu \geq 1$*: Suppose* $H_\nu = H \circ \Phi^\nu = N_\nu + P_\nu$, *where*

$$N_\nu = e_\nu + \langle \omega_\nu, y \rangle + \frac{1}{2}\langle z, M_\nu(x) z \rangle,$$

defined on $D_\nu \times \mathcal{O}_\nu$ *with*

(3.30) $$|e_{\nu+1} - e_\nu|^l \leq c\gamma_\nu^{n+1} r^2 \mu_\nu,$$

(3.31) $$|\omega_{\nu+1} - \omega_\nu|^l \leq c\gamma_\nu^{n+1} r \mu_\nu,$$

(3.32) $$\|M_{\nu+1} - M_\nu\|^l \leq c\gamma_\nu^{n+1} \mu_\nu.$$

$\mathcal{O}_\nu$ *is the set such that for* $\lambda \in \mathcal{O}_\nu$, *the small divisor conditions*

$$|\langle \omega_\nu, k \rangle| \geq \frac{\gamma_\nu}{|k|^\tau}, \ \forall 0 \neq k \in \mathbb{Z}^n$$

hold at the ν*th KAM iteration step.*

Finally, we have that

$$\|X_{P_\nu}\|_{D_\nu, \mathcal{O}_\nu} \leq \gamma^{n+1} \mu_\nu.$$

Then there is a subset $\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu$,

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus \cup_{|k| \geq 2^\nu} \mathcal{R}_k^{\nu+1}(\gamma_\nu),$$

where $\mathcal{R}_k^{\nu+1}(\gamma_{\nu+1}) = \{\lambda \in \mathcal{O}_\nu | \ |\langle k, \omega_{\nu+1} \rangle^{-1}| > \frac{|k|^\tau}{\gamma_\nu}\}$, with $\omega_{\nu+1} = \omega_\nu + [P_\nu^{10}]$, and a symplectic change of variables

(3.33) $$\Phi_\nu : D_{\nu+1} \times \mathcal{O}_{\nu+1} \to D_\nu,$$

such that $H_{\nu+1} = H_\nu \circ \Phi_\nu$, defined on $D_{\nu+1} \times \mathcal{O}_{\nu+1}$, satisfies the same assumptions with $\nu + 1$ in place of ν.

If μ in Theorem 1 satisfies the condition (3.29), then μ satisfies the conditions in Lemma 3.6. So for $\forall \lambda \in \mathcal{O}_{\nu+1}$ we have the map $\Phi_\nu : \ D_{\nu+1} \to D_\nu$ satisfying

(3.34) $$\frac{1}{\rho_\nu}\|\Phi_\nu - id\|^l_{r_\nu; D_{\nu+1}}, \ \|D\Phi_\nu - Id\|^l_{r_\nu; r_\nu; D_{\nu+1}} \leq cE_\nu.$$

Let $\Phi^\nu = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_\nu$, thus $H_\nu = H \circ \Phi^\nu = N_\nu + P_\nu$, where

$$N_\nu = e +_\nu + \langle \omega_\nu, y \rangle + \frac{1}{2}\langle z, Mz \rangle.$$

Let $\mathcal{O}_\gamma = \cap_{\nu\geq 1}\mathcal{O}_\nu$. By the inequalities (3.30), (3.19) and (3.20) in Lemma 3.8 we have:

$$|e_{\nu+1} - e_\nu|^l \leq c\gamma_1^{n+1} r_1^{2(\frac{11}{9})^\nu} \mu_1^{(\frac{10}{9})^\nu},$$

$$|\omega_{\nu+1} - \omega_\nu|^l \leq c\gamma_1^{n+1} r_1^{(\frac{11}{9})^\nu} \mu_1^{(\frac{10}{9})^\nu},$$

$$\|M_{\nu+1} - M_\nu\|_{D_\nu}^l \leq c\gamma_1^{n+1}\mu_1^{(\frac{10}{9})^\nu}$$

for $\lambda \in \mathcal{O}_\gamma$. If $\mu \leq \left(\frac{c}{\gamma}\right)^{\frac{162}{55}} \left(\frac{1}{r}\right)^8$, then it follows that $c\mu_1^{(\frac{10}{9})^{\nu+1}} \leq \left(c\mu_1^{(\frac{10}{9})^\nu}\right)^{\frac{1}{2}}$ for all $\nu \geq 1$. It follows that

$$|e_{\nu+1} - e|^l \leq \sum_{\nu\geq 1} c\gamma_1^{n+1} r_1^{2(\frac{11}{9})^\nu} \mu_1^{(\frac{10}{9})^\nu} \leq 2c\gamma^{n+1} r^2 \mu, \tag{3.35}$$

$$|\omega_{\nu+1} - \omega|^l \leq \sum_{\nu\geq 1} c\gamma_1^{n+1} r_1^{(\frac{11}{9})^\nu} \mu_1^{(\frac{10}{9})^\nu} \leq 2c\gamma^{n+1} r^2 \mu^{\frac{5}{8}}, \tag{3.36}$$

$$\|M_{\nu+1} - M\|_{D_\nu}^l \leq \sum_{\nu\geq 1} c\gamma^{n+1}\mu_\nu \leq 2c\gamma^{n+1} r^2 \mu^{\frac{1}{4}}. \tag{3.37}$$

Since $E_{\nu+1} \leq E^{\frac{10}{9}}$, we have $E_{\nu+1} \leq (\frac{1}{2})^{(10/9)^\nu}$ under the condition (3.24). It follows that

$$\sum_{\nu\geq 1} cE_\nu \leq 2cE_1.$$

Now we prove $\{\Phi^\nu\}$ is convergent on $D_* \times \mathcal{O}_\gamma = \bigcap_{\nu\geq 1} D_\nu \times \mathcal{O}_\nu$ with $D_* = D(\frac{1}{2}s) \times \{0\} \times \{0\}$. From the proceeding analysis, Φ_ν maps $D_{\nu+1}$ into $D(s_\nu - 4\rho_\nu, 2r_\nu^{\frac{11}{9}}) \subset D(s_\nu - 2\rho_\nu, \frac{1}{2}r_\nu)$. Since the distance $\|\cdot\|$ from $D(s_\nu - 5\rho_\nu, 2r_\nu^{\frac{10}{9}})$ to the boundary of $D(s_\nu - 4\rho_\nu, \frac{1}{2}r_\nu)$ is more than ρ_ν, if E_1 is sufficiently small, we have

$$\|\Phi_{\nu-1} \circ \Phi_\nu - id\|_{D_{\nu+1}}^l \leq \|\partial_\lambda^l(\Phi_{\nu-1} - id)\|_{D_\nu}.$$

Inductively it follows that for any $\nu \geq 1$ and $\nu' \geq 1$,

$$\|\Phi_\nu \circ \Phi_{\nu+1} \circ \cdots \circ \Phi_{\nu+\nu'} - id\|_{D_{\nu+\nu'+1}}^l \leq \|\Phi_\nu - id\|_{D_{\nu+1}}^l.$$

Since $\Phi^{\nu+1} = \Phi^\nu \circ \Phi_{\nu+1}$, we have

$$\|\Phi^{\nu+1} - \Phi^\nu\|_{D_{\nu+2}}^l \leq \|D\Phi^\nu\|_{D_{\nu+1}}^l \|\Phi_{\nu+1} - id\|_{D_{\nu+2}}^l.$$

By the inequality of the operator norm $\|\cdot\|$, we have

$$\|D\Phi^\nu\|^l_{D_{\nu+1}} \leq \|D\Phi_1\|_{D_2}\|D\Phi_2\|^l_{D_3}\cdots\|D\Phi\|^l_{D_{\nu+1}} \\ \leq \prod_{\nu'=1}^{\nu}(1+cE_{\nu'}) < +\infty \tag{3.38}$$

So

$$\|\Phi^{\nu+1}-\Phi^\nu\|^l_{D_{\nu+2}} \leq c\|\Phi_{\nu+1}-id\|^l_{D_{\nu+2}} \leq cE_\nu,$$

thus $\{\Phi^\nu\}$ is convergent on $D_*\times\mathcal{O}_\gamma$, say, to Φ. Now we give the ideas of the proof of the convergency of Φ^ν on $D(\frac{1}{2}s,\frac{1}{2}r)\times\mathcal{O}_\gamma$. We can use the estimates about $D\Phi_\nu$ to prove that $\{D\Phi^\nu\}$ are convergent on $D_*\times\mathcal{O}_\gamma$ as in [21]. It is clear that Φ_ν is affine in y and z, and so are their composition mappings Φ^ν. Thus the fact that $\{\Phi^\nu\}$ and $\{D\Phi^\nu\}$ are convergent on $D_*\times\mathcal{O}_\gamma$ implies that $\{\Phi^\nu\}$ is actually convergent on $D(\frac{1}{2}s,\frac{1}{2}r)\times\mathcal{O}_\gamma$. Since $\|X_{P_\nu}\|^l_{D_\nu}\leq\gamma_\nu^{n+1}r^2\mu$ and $\lim_{\nu\to\infty}\|X_{P_\nu}-X_{P_*}\|^l_{D_\nu}=0$, it follows that $P_*=0$ on $D_*\times\mathcal{O}_\gamma$ and $\frac{\partial^{p+q}P_*}{\partial y^p\partial z^q}|_{D_*}=0$ for $2|p|+|q|\leq 2$. So $P_*=\sum_{k\in\mathbb{Z}^n,2|p|+|q|\geq 3}P_{*kpq}y^lz^qe^{\sqrt{-1}\langle k,x\rangle}$. Let $\lim_{\nu\to+\infty}\Phi^\nu=\Phi$. Then $H\circ\Phi=N_*+P_*$ on $D(\frac{1}{2}s,\frac{1}{2}r)\times\mathcal{O}_\gamma$, where

$$N_*=\lim_{\nu\to\infty}N_\nu=e_*+\langle\omega_*,y\rangle+\frac{1}{2}\langle z,M_*z\rangle$$

and $e_*=\lim_{\nu\to\infty}e_\nu$, $\omega_*=\lim_{\nu\to\infty}\omega_\nu$, $M_*=\lim_{\nu\to\infty}M_\nu$. From (3.35)-(3.37), it follows that

$$|e_*-e|^l_{\mathcal{O}_\gamma}=O(\gamma^{n+1}r^2\mu),$$

$$|\omega_*-\omega|^l_{\mathcal{O}_\gamma}=O(\gamma^{n+1}r^2\mu^{\frac{5}{8}}),$$

$$\|M_*-M\|^l_{D(\frac{s}{2})\times\mathcal{O}_\gamma}=O(\gamma^{n+1}r^2\mu^{\frac{1}{4}})$$

for any $|l|\leq l_0$. From the above iteration, it is easy to see that the map Φ is close to the identity map with $\|\Phi-Id\|^l=O(\mu^{\frac{5}{8}}r^2s^{-(v+2)})$. The measure estimates are standard, see for example [26] (also [2, 14]). This completes the proof of Theorem 1.

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