DIMENSION FORMULAS FOR MODULES FINITE OVER LOCAL HOMOMORPHISMS

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Abstract. We extend some major theorems in commutative algebra to the class of modules that are not necessarily finitely generated. The novelty of our extension is that the hypothesis of finite generation over R is replaced by one over S, where R and S are commutative Noetherian local rings and there is a local homomorphism $\varphi : R \to S$. Among the results that we extend are: Intersection Theorem and Intersection dimension formula.

1. INTRODUCTION

I. S. Cohen Structure Theorem is one of the most fundamental results of commutative algebra which shows that a complete local ring may be presented as a homomorphic image of a ring of formal power series over a complete discrete valuation domain; thus some questions on complete local rings are reduced to questions on ideals in specific rings.

On the other hand, a classical chapter of commutative algebra started by Grothendieck, deals with the correlation between properties of commutative Noetherian local rings R and S where there exists a local homomorphism $\varphi : R \to S$. The idea behind it is that if $\varphi : R \to S$ is a faithfully flat homomorphism, then the properties of R and of the fibers of φ determine and are determined by the properties of S; this is in accordance with geometric intuition, which perceives such homomorphisms as algebraic substitutes for fiber bundles.

By considering these two stages, Avramov, Foxby and B. Herzog, developed a powerful technique, the so-called 'Technology of Cohen-factorizations'. They proved that for each local homomorphism $R \to S$, the induced map from R to the completion $\hat{S}$ of S has a Cohen-factorization $R \to R' \to \hat{S}$ into a local flat extension

with regular closed fiber, followed by a surjective homomorphism [1]. Using this, the corresponding problem is first treated in the minimal ideal-adic completion of a ring, and then the flatness of completion is used in order to descend the information to the initial ring.

In view of this structure, several homological invariants related to a finitely generated (henceforth finite) R-module, has extended to a much wider class of modules: those that are finite over some local ring S, for which there exists a local homomorphism $\varphi : R \to S$ (see [2] and [7]).

The purpose of this paper is to pursue this point of view to extend some major results in commutative algebra, to the class of R-modules that are not necessarily finite over R. In geometric language, we prove the theorems for direct images of coherent sheaves under a local morphism. Note that if $f : X \to Y$ is a morphism of (even noetherian) schemes, it is not true in general that f_* of a coherent sheaf is coherent [5, II.5.8.1]. So the extended category is larger than the original one. In commutative algebra there are plenty of examples known for finite S-modules that are not finite over R. For instance, the completion of a noetherian local ring R is not finite over R, in general.

The paper is structured as follows: Section 2 is devoted to extend a known classical result about the behavior of Cohen-Macaulayness of modules under flat extension of rings.

In Section 3, the notion of projective dimension over local homomorphisms is introduced. Using this, we provide an extension of the Intersection Theorem. The version we aim to extend is due to Peskin and Szpiro that states $\dim_R N \le \mathrm{pd}_R M$, whenever R contains a field and M and N are non-zero finite R-modules such that $M \otimes_R N$ is of finite length. To this end, we use the technology of Cohen factorization.

The last section of the paper deals with the Intersection Dimension formula. Let $(R, \mathfrak{m})$ be a regular local ring. Serre proved that for all finite R-modules M, N such that $M \otimes_R N$ is of finite length,

$$\dim_R M + \dim_R N \le \dim R.$$

There are some results related to this formula. In [10, Proposition 1.3], Simon proved that the formula holds when M is an R-module of finite injective dimension over the equicharacteristic local ring $(R, \mathfrak{m})$, provided the natural morphism $\mathrm{Ext}_R^s(k, M) \to H_{\mathfrak{m}}^s(M)$ is non-zero, where $s = \dim_R M \ge 1$.

In Section 4, by introducing the notion of injective dimension over local homomorphisms, we prove an extension of Simon's result to the class of R-modules that are finite over some local ring S, where there is a local homomorphism $\varphi : R \to S$.

1.1. Notations and conventions

Throughout this paper $(R, \mathfrak{m}, k)$ denotes a commutative Noetherian local ring

with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We fix a local homomorphism

$$\varphi : (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l),$$

that is a homomorphism of Noetherian local rings such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. M denotes an arbitrary R-module and N denotes a finite S-module. We write $\hat{N}$ for $N \otimes_S \hat{S}$, where $\hat{S}$ denotes the completion of S with respect to $\mathfrak{n}$.

We use notation $(R', \mathfrak{m}', k')$ to denote a (commutative Noetherian) local ring which is flat over R, $f : R \rightarrow R'$ always denote the (local) flat homomorphism of rings.

Support

By the support of M over R, denoted $\mathrm{Supp}_R M$, we shall mean the set of all prime ideals $\mathfrak{p}$ in $\mathrm{Spec}(R)$ such that $M_{\mathfrak{p}}$, the localization of M at $\mathfrak{p}$, is non-zero. Now, as above, let $\varphi : R \longrightarrow S$ be a local homomorphism and N be a finite S-module. Then one can see that $\mathrm{Supp}_R N = \mathrm{Var}(\mathrm{Ann}_R N)$, see proof of the Theorem 2.5 below.

Dimension

As usual the (Krull) dimension of M over R, denoted $\dim_R M$ or $\dim M$, is the supremum of lengths of chains of prime ideals in the support of M if this supremum exists, and ∞ otherwise. By convention, the dimension of the zero R-module is -1.

Depth

By [6, Theorem 6.1], the depth of M over R, denoted $\mathrm{depth}_R M$, is defined by

$$\mathrm{depth}_R M = \inf\{\ i \mid \mathrm{Ext}_R^i(k, M) \neq 0\}.$$

Note that $\mathrm{depth}_R M$ is either a non-negative integer or ∞. It is easy to see that when $\varphi : R \longrightarrow S$ is a local homomorphism and N is a finite S-module, $\mathrm{Ass}_R N = {}^a\varphi(\mathrm{Ass}_S N)$, where ${}^a\varphi : \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ is the associated map of spectra, see e.g. [8, pp. 129-132]. This, in particular, implies that $\mathrm{Ass}_R N$ is finite and $\mathrm{depth}_R N \leq \mathrm{depth}_S N$. So $\mathrm{depth}_R N$ is always finite. Moreover, it can be seen easily that in these situations, $\mathrm{depth}_R N$ is equal to the length of a maximal M-sequences contained in $\mathfrak{m}$.

1.2. Embedding dimension

The embedding dimension, $\mathrm{edim}\, R$, of R is the minimal number of generators of its maximal ideal $\mathfrak{m}$. We shall refer to the embedding dimension of the local ring $S/\mathfrak{m}S$ as the embedding dimension of φ, where $\varphi : R \longrightarrow S$ is a local homomorphism. It will be denoted by $\mathrm{edim}\, \varphi$.

2. Depth Formula over Local Homomorphisms

There is a classical known result on the behavior of Cohen-Macaulay modules under flat extensions, that is if $R \to R'$ is a flat morphism of local rings and M is a finite R-module, then $M \otimes_R R'$ is Cohen-Macaulay R'-module if and only if M is Cohen-Macaulay R-module and $R'/\mathfrak{m}R'$ is Cohen-Macaulay R'-module. This result can be generalized to the class of modules that are finite over some local ring S with a local homomorphism $R \to S$. Our main theorem in this section states as follows.

Theorem 2.1. *Let* $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a homomorphism of Noetherian local rings and* N *be a finite* S*-module. Suppose* $f : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ *is a flat homomorphism of local rings. Then the following are equivalent.*

(a) $\operatorname{depth}_{R'} N \otimes_R R' = \dim_{R'} N \otimes_R R'$.

(b) $\operatorname{depth}_R N = \dim_R N$ *and* $R'/\mathfrak{m}R'$ *is Cohen-Macaulay.*

We should mention that some of our results in this section, which are required for the proof of the above theorem, are close to available results in the literature and have precursors in various contexts; see e.g. [6] and [4].

Lemma 2.2. *Let* $\underline{x} = x_1, \ldots, x_n$ *be an* N*-sequence in* R*. Then* $f(\underline{x}) = f(x_1), \ldots, f(x_n)$ *is an* $N \otimes_R R'$*-sequence in* R'.

Proof. We argue by induction on n. First assume that $n = 1$ and let $x_1 \in \mathfrak{m}$ be N-regular. So $x_1 \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \mathfrak{p}$. By [8, Theorem 23.2], $\operatorname{Ass}_{R'} N \otimes_R R' = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \operatorname{Ass}_{R'}(R'/\mathfrak{p}R')$. Hence it follows that $f(x_1) \notin \operatorname{Ass}_{R'} N \otimes_R R'$ and so $f(x_1)$ is $N \otimes_R R'$-regular. Now let $n > 1$ and x_1 be an N-regular element. The R'-isomorphism

$$N/x_1 N \otimes_R R' \cong \frac{N \otimes_R R'}{x_1(N \otimes_R R')}$$

in conjunction with the induction assumption, completes the proof. ∎

Lemma 2.3. *Let* $\underline{y} = y_1, \cdots, y_m$ *be an* $R'/\mathfrak{m}R'$*-sequence in* R'*. Then it is* $X \otimes_R R'$*-sequence, where* X *is an* S*-module.*

Proof. We prove the lemma by induction on m, the length of $R'/\mathfrak{m}R'$- regular sequence. Let $m = 1$ and y_1 be an $R'/\mathfrak{m}R'$-regular element. Let $z \in X \otimes_R R'$ be such that $y_1 z = 0$. Consider the expression of z as $\sum_{i=1}^{s} n_i \otimes r_i'$, where $n_i \in X$ and $r_i' \in R'$. Set $L = \sum_{i=1}^{s} R n_i$. So, considering L as an R-submodule of X, we get $L \otimes_R R'$ as an R'-submodule of $X \otimes_R R'$ and $z \in L \otimes_R R'$. Since $L \otimes_R R'$ is finite R'-module, by Krull's intersection theorem, one has $\bigcap_{i \in \mathbb{N}} \mathfrak{m}^i(L \otimes_R R') = 0$.

Now suppose that $z \neq 0$. So there exists $i \in \mathbb{N}$ such that $0 \neq \bar{z} \in \mathfrak{m}^i(L \otimes_R R')/\mathfrak{m}^{i-1}(L \otimes_R R')$. The latter module is naturally isomorphic to $k^t \otimes_R R' \cong (R'/\mathfrak{m}R')^t$, for some positive integer t. But by our choice of y_1, it is $R'/\mathfrak{m}R'$-regular element. Hence z has to be zero. Now, in order to complete the induction step, one just should note that for an arbitrary ideal J of R', there is a natural isomorphism of R'-modules

$$\frac{R' \otimes_R N}{J(R' \otimes_R N)} \cong \frac{R'}{J} \otimes_R N$$

and use the fact that $R'/\underline{y}R'$ is flat over R.   ∎

As the referee mentioned, one could use [6, Corollary 2.6] to obtain the following theorem. For the sake of completeness, we present here an elementary proof.

Theorem 2.4. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of Noetherian local rings and N be a finite S-module. Suppose $f : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ is a flat homomorphism of local rings. Then*

$$\operatorname{depth}_{R'} N \otimes_R R' = \operatorname{depth}_R N + \operatorname{depth}_{R'} R'/\mathfrak{m}R'.$$

Proof.   Let $\underline{x} = x_1, \ldots, x_n$ (respectively $\underline{y} = y_1, \ldots, y_m$) be maximal N-(respectively $R'/\mathfrak{m}R'$-) sequence in R (respectively R'). We show that

$$f(\underline{x}), \underline{y} = f(x_1), \ldots, f(x_n), y_1, \ldots, y_m$$

is a maximal $N \otimes_R R'$-sequence in R'. First of all, note that by Lemma 2.2, $f(\underline{x}) = f(x_1), \ldots, f(x_n)$ is an $N \otimes_R R'$-sequence. Moreover by Lemma 2.3, $\underline{y}$ is an $\bar{N} \otimes_R R'$-sequence, where $\bar{N} = N/\underline{x}N$. Now it follows from the isomorphism

$$\bar{N} \otimes_R R' \cong \frac{N \otimes_R R'}{f(\underline{x})(N \otimes_R R')},$$

that $f(\underline{x}), \underline{y}$ is an $N \otimes_R R'$-sequence. To complete the proof, it is enough to show that $\mathfrak{m}' \in \bar{\mathrm{A}}\mathrm{ss}_{R'}(N \otimes_R R'/(f(\underline{x}), \underline{y})N \otimes_R R')$. To this end, consider the sequence of isomorphisms

$$\frac{N \otimes_R R'}{(f(\underline{x}), \underline{y})N \otimes_R R'} \cong \frac{N \otimes_R R'/\underline{y}R'}{\underline{x}(N \otimes_R R/\underline{y}R')} \cong N/\underline{x}N \otimes_R R'/\underline{y}R'.$$

By [3, Lemma 1.2.17], $R'/\underline{y}R'$ is a flat R-algebra and so by [8, Theorem 23.2], we have $\mathrm{Ass}_{R'} N/\underline{x}N \otimes_R R'/\underline{y}R' = \{\mathfrak{m}'\}$. The result hence is complete.   ∎

Theorem 2.5. *Let the situation be as in Theorem 2.4. Then*

$$\dim_{R'} N \otimes_R R' = \dim_R N + \dim R'/\mathfrak{m}R'.$$

Proof. First we claim that $\operatorname{Supp}_R N = \operatorname{Var}(\operatorname{Ann}_R N)$. Let $\mathfrak{p} \supseteq \operatorname{Ann}_R N$. Let $N = \sum_{i=1}^n Sx_i$. Suppose contrary that $N_{\mathfrak{p}} = 0$. So there exists $r \in R \setminus \mathfrak{p}$ such that for all $i = 1, \ldots, n$, $rx_i = 0$. Hence $rN = 0$ which is a contradiction. So $\operatorname{Var}(\operatorname{Ann}_R N) \subseteq \operatorname{Supp}_R N$. The other side is clear. So $\dim_R N = \dim R/\operatorname{Ann}_R N$. Set $I = \operatorname{Ann}_R N$ and $\bar{R} = R/I$. For every $\bar{R}$-module X, there is an isomorphism $X \otimes_R R' \cong X \otimes_{\bar{R}} R'/IR'$. So we may (and do) replace R by $\bar{R}$ and R' by R'/IR'. Hence we may assume that $\operatorname{Supp}_R N = \operatorname{Spec}(R)$. Now conditions are as same as condition that exists in part (a) of [3, Theorem A.5]. Hence $\dim_{R'} N \otimes_R R' \leq \dim_R N + \dim R'/\mathfrak{m}R'$. For the converse, modify the proof of [3, Theorem A.11]. ∎
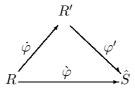
Proof of Theorem 2.1.. It is now a direct consequence of Theorems 2.4 and 2.5.

3. INTERSECTION THEOREM

Our goal in this section is to provide an extension of the Intersection Theorem. The version we aim to extend is due to Peskin and Szpiro that states $\dim_R N \leq \operatorname{pd}_R M$, whenever R contains a field and M and N are non-zero finite R-modules such that $M \otimes_R N$ is of finite length. To this end, we use the technology of Cohen factorization due to Avramov, Foxby and Herzog. Let us first recall the notion.

3.1. Factorization

A factorization of a local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a commutative triangle of local homomorphisms

with $\dot{\varphi}$ flat and φ' surjective. A factorization is said to be regular, if the local ring $R'/\mathfrak{m}R'$ is regular.

A *Cohen factorization* is a regular factorization with a complete local ring R'. It is shown in [1], that for each local homomorphism φ, the composition $\dot{\varphi} : R \to \hat{S}$ of φ and the $\mathfrak{n}$-adic completion map $S \to \hat{S}$ has a Cohen factorization.

3.2. Minimal Cohen Factorization.

Let $\varphi'\grave{\varphi}$ be a Cohen factorization of $\grave{\varphi} : R \to \hat{S}$. It is always the case [1, §1] that

$$\mathrm{edim}\,\varphi \leq \mathrm{edim}\,\grave{\varphi} = \dim(R'/\mathfrak{m}R').$$

The factorization is said to be minimal, when equality holds. It is proved in [1, Proposition 1.5] that the homomorphism $\grave{\varphi}$ always has a minimal Cohen factorization.

Throughout this section, we fix a minimal Cohen factorization of $\grave{\varphi}$. So we always have $\mathrm{edim}\,\varphi = \mathrm{edim}\,\grave{\varphi}$.

Definition 3.1. Let $\varphi : R \to S$ be a local homomorphism and N be a finite S-module. The projective dimension of N over φ, denoted by $\mathrm{pd}_\varphi N$ is defined as $\mathrm{pd}_\varphi N = \mathrm{pd}_{R'}\hat{N}$, were $R \to R' \to \hat{S}$ is a minimal Cohen factorization of $\grave{\varphi}$.

Since we are working just with minimal Cohen factorizations, it follows from [7, Proposition 4.1] that our definition of projective dimension is independent of the choice of minimal Cohen factorizations.

Remark 3.2. In [7, Definition 4.2], there is another notion of projective dimension over a local homomorphism φ. For a finite S-module N, they defined the quantity $\mathrm{pd}_{R'}\hat{N} - \mathrm{edim}\,\grave{\varphi}$ as the projective dimension of N over φ, where as usual $\grave{\varphi} = \varphi'\grave{\varphi}$ is a Cohen factorization of φ. Actually our definition is quiet different, but as it is clear that the finiteness of both dimensions are equivalent. Moreover unlike to the Definition 4.2 of [7], in our definition, projective dimension of a non-zero module is always non-negative. So it seems that our definition is more appropriate to this end.

Lemma 3.3. *With the notation as in 1.1,*

$$\mathrm{Supp}_{R'} N \otimes_R R' = \mathrm{Var}(\mathrm{Ann}_{R'} N \otimes_R R').$$

Proof. Let $x_1, \ldots, x_n$ be a generating set for N over S. Let $\mathfrak{q} \in \mathrm{Var}(\mathrm{Ann}_{R'}(N \otimes_R R'))$. Suppose contrary that $(N \otimes_R R')_\mathfrak{q} = 0$. So $\mathfrak{q}^c \notin \mathrm{Supp}_R N$. Hence there exists $r \in R \setminus \mathfrak{q}^c$ such that $rx_i = 0$ for $i = 1, \ldots, n$. So $r(N \otimes_R R') = 0$ which is a contradiction, because $\mathfrak{q} \supseteq \mathrm{Ann}_{R'}(N \otimes_R R')$. So $\mathfrak{q} \in \mathrm{Supp}_{R'}(N \otimes_R R')$. The converse is elementary. ∎

Lemma 3.4. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism and N be a finitely generated S-module. Then $\mathrm{Supp}_R N = \mathrm{Supp}_R \hat{N}$. In particular, $\dim_R N = \dim_R \hat{N}$, where $\hat{N}$ denotes the completion of N with respect to the $\mathfrak{n}$-adic completion.*

Proof. Clearly $\mathrm{Supp}_R N \subseteq \mathrm{Supp}_R \hat{N}$. Suppose conversely that $\mathfrak{p} \in \mathrm{Supp}_R \hat{N}$. So $\hat{N} \otimes_R R_\mathfrak{p} = \hat{S} \otimes_S N \otimes_R R_\mathfrak{p} \neq 0$. This, in particular, implies that $N \otimes_R R_\mathfrak{p} \neq 0$. Hence $\mathfrak{p} \in \mathrm{Supp}_R N$. The result now is clear. ∎

Lemma 3.5. *Let the situation be as in 1.1. Assume that M is another finite S-module such that $\mathrm{Supp}_R M \cap \mathrm{Supp}_R N = \{\mathfrak{m}\}$. Then*

$$\mathrm{Supp}_{R'}(\hat{M} \otimes_R R') \cap \mathrm{Supp}_{R'}(\hat{N} \otimes_R R'/\underline{y}R') = \{\mathfrak{m}'\},$$

where $\underline{y}$ is a maximal $R'/\mathfrak{m}R'$-sequence.

Proof. To prove the result, it is enough to show that $\mathrm{Ass}_{R'}((\hat{M} \otimes_R \hat{N}) \otimes_R R'/\underline{y}R') = \{\mathfrak{m}'\}$. By the above lemma, $\mathrm{Supp}_R \hat{M} \cap \mathrm{Supp}_R \hat{N} = \{\mathfrak{m}\}$. Now use the fact that $R'/\underline{y}R'$ is an R-flat algebra in Theorem 23.2 of [8] to deduce the result. ∎

Now we are ready to present the main result of this section.

Theorem 3.6. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of rings and M, N be two finite S-module. Assume that $\dim_R(M \otimes_R N) = 0$. Then*

$$\dim_R N \leq \mathrm{pd}_\varphi M.$$

Proof. Let $\varphi'\grave{\varphi}$ be a minimal Cohen factorization of $\grave{\varphi} : R \to \hat{S}$. Clearly we may assume that $\mathrm{pd}_\varphi M$ is finite. Since $\dim_R(M \otimes_R N) = 0$, $\mathrm{Supp}_R M \cap \mathrm{Supp}_R N = \{\mathfrak{m}\}$. So by previous lemma, $\dim_{R'} \hat{M} \otimes_R (\hat{N} \otimes_R R'/\underline{y}R') = 0$, where $\underline{y}$ is a maximal $R'/\underline{y}R'$-sequence. Moreover by Lemma 3.5, $\mathrm{Supp}_{R'}(\hat{N} \otimes_R R'/\underline{y}R') = \mathrm{Var}(\mathrm{Ann}_{R'}(\hat{N} \otimes_R R'/\underline{y}R'))$. Hence by [3, Theorem 9.4.5 and Remark 9.4.8(a)],

$$\dim_{R'} \frac{R'}{\mathrm{Ann}_{R'}(\hat{N} \otimes_R R'/\underline{y}R')} \leq \mathrm{pd}_{R'} \hat{M}.$$

By Theorem 2.5 and Lemma 3.5, the left hand side is equal to $\dim_R N$, while the right hand side is by definition equal to $\mathrm{pd}_\varphi M$. So the result follows. ∎

Remark 3.7. Note that by Auslander-Buchsbaum depth formula, when $\mathrm{pd}_\varphi M$ is finite, we have

$$\begin{aligned}
\mathrm{pd}_\varphi M &= \mathrm{pd}_{R'} \hat{M} \\
&= \mathrm{depth}\, R' - \mathrm{depth}_{R'} \hat{M} \\
&= \mathrm{depth}\, R - \mathrm{depth}_S M + \mathrm{edim}\, \varphi.
\end{aligned}$$

4. Intersection Dimension Formula

Over a commutative Noetherian regular local ring $(R, \mathfrak{m})$, an intersection dimension formula holds. It states that: for all finite R-modules M, N such that $M \otimes_R N$ is of finite length,

$$\dim_R M + \dim_R N \leq \dim R.$$

This was proved by Serre.

When the ring R is not regular, such a formula is not anymore available, as is shown by easy examples. The dimension conjecture states that this formula is still true if one of the modules involved is of finite projective dimension.

Few cases are known. When the module is perfect, the formula is valid for M coupled with any other module N. The intersection's dimension formula is also true for a module of finite projective dimension and of grade one, this was proved by Foxby with the aid of the MacRae invariant. In [9], the case where M is a surjective Buchsbaum module of finite projective dimension over an equicharacteristic Gorenstein local ring is established.

In [10, Proposition 1.3], it is shown that the formula holds when M is an R-module of finite injective dimension over the equicharacteristic local ring $(R, \mathfrak{m})$, provided the natural morphism $\mathrm{Ext}_R^s(k, M) \to H_{\mathfrak{m}}^s(M)$ is non-zero, where $s = \dim_R M \geq 1$.

In this section we extend this result to the case when M and N are not necessary finite over R, but are finite over $(S, \mathfrak{n})$, where there is a local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$. Let us begin with a definition.

Definition 4.1. Let $\varphi : R \to S$ be a local homomorphism and N be a finite S-module. Let $R \to R' \to \hat{S}$ be a minimal Cohen factorization of $\grave{\varphi}$. The injective dimension of N over φ, denoted by $\mathrm{id}_\varphi N$, is defined as the quantity $\mathrm{id}_{R'} \hat{N}$.

Theorem 4.2. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism and M and N be finite S-modules. Assume that R is equicharacteristic, $\mathrm{id}_\varphi M$ is finite, $\dim_R(M \otimes_R N) = 0$ and $\dim_S M = d \geq 1$. Moreover, assume that the canonical map $\mathrm{Ext}_S^d(l, M) \to H_{\mathfrak{n}}^d(M)$ is non-zero. Then*

$$\dim_R M + \dim_R N \leq \dim R + \mathrm{edim}\, \varphi.$$

Proof. Let $R \to R' \to \hat{S}$ be a minimal Cohen factorization of $\grave{\varphi}$. Let

$$0 \to \hat{M} \to E^0 \to E^1 \to \cdots \to E^i \to \cdots$$

(resp. $0 \to \hat{M} \to I^0 \to I^1 \to \cdots \to I^i \to \cdots$) be minimal injective resolution of $\hat{M}$ as $\hat{S}$-module (resp. R'-module). So we get a chain map $E^\bullet \to I^\bullet$. Apply the

functor $\operatorname{Hom}_{R'}(l, \ )$ on it. Since $R' \to \hat{S}$ is an epimorphism, and l and E^i's are $\hat{S}$-module, we may replace the induced exact sequence in the top row $\operatorname{Hom}_{R'}(l, E^{\bullet})$, by $\operatorname{Hom}_{\hat{S}}(l, E^{\bullet})$. So we get a homomorphism $\operatorname{Ext}^d_{\hat{S}}(l, \hat{M}) \to \operatorname{Ext}^d_{R'}(l, \hat{M})$. Since by assumption the map $\operatorname{Ext}^d_S(l, M) \to H^d_{\mathfrak{n}}(M)$ is non-zero, the map $\operatorname{Ext}^d_{\hat{S}}(l, \hat{M}) \to H^d_{\mathfrak{n}}(M)$ and hence the map $\operatorname{Ext}^d_{R'}(l, \hat{M}) \to H^d_{\mathfrak{m}'}(\hat{M})$ also should be non-zero. Moreover by Lemma 3.6, $0 :_{R'} \hat{N} \otimes_R R'/\underline{y}R' + 0 :_{R'} \hat{M}$ is an $\mathfrak{m}'$-primary ideal. So by [10, Proposition 1.3],

$$\dim_{R'} \hat{N} \otimes_R R'/\underline{y}R' + \dim_{R'} \hat{M} \leq \operatorname{id}_{R'} \hat{M}.$$

The left hand side, is greater than or equal to $\dim_R M + \dim_R N$ and the right hand side is by definition, $\operatorname{id}_{\varphi} M$. But since $\hat{M}$ is a finite R'-module, by Bass Theorem, $\operatorname{id}_{R'} \hat{M} = \dim R'$. The latter is equal to $\dim R, + \operatorname{edim} \varphi$. The result hence follows.   ■

REFERENCES

1. L. L. Avramov, H-B. Foxby and B. Herzog, Structure of local homomorphisms, *J. Algebra*, **164(1)** (1994), 124-145.

2. L. L. Avramov, S. Iyengar and C. Miller, Homology over local homomorphisms, *Amer. J. Math.*, **128(1)** (2006), 23-90.

3. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, 1998.

4. L. W. Christensen and H. Holm, Ascent properties of Auslander categories, *Canad. J. Math.*, to appear.

5. R. Hartshorne, *Algebraic Geometry*, Graduate Text in Mathematics, Vol. 52, Berlin, Heidelberg, New York, Springer, 1977.

6. S. Iyengar, Depth for complexes, and intersection theorems, *Math. Z.*, **230** (1999), 547-567.

7. S. Iyengar and S. Sather-Wagstaff, G-dimension over local homomorphisms. Applications to the Frobenius endomorphism, *Illinois J. Math.*, **48(1)** (2004), 241-272.

8. H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.

9. A.-M, Simon, *Intersection dimension and canonical modules*, Rings, Hopf algebras, and Brauer groups (Antwerp/Brussels, 1996), 287-297, Lecture Notes in Pure and Appl. Math., 197, Dekker, New York, 1998.

10. A.-M, Simon, Finite injective dimension and parameter systems, in: *Actas del Congreso Internacional de Teoría de Anillos*, M. J. Asensio, B. Torrecillas and F. Van Oystaeyen, eds., Almería (1993).

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