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H^1 BOUNDEDNESS FOR RIESZ TRANSFORM RELATED TO SCHRÖDINGER OPERATOR ON NILPOTENT GROUPS

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Abstract. Let $\mathbb{G}$ be a nilpotent Lie groups equipped with a Hörmander system of vector fields $X = (X_1, \ldots, X_m)$ and $\Delta = \sum_{i=1}^{m} X_i^2$ be the sub-Laplacians associated with X. Let $A = -\Delta + W$ be the Schrödinger operator with the potential function W belongs to the reverse Hölder class B_q for some $q \geq D/2$, where D denote the dimension at infinity. In this paper, we prove that the Riesz transform $\nabla A^{-1/2}$ related to Schrödinger operator A is bounded from the Hardy space $H^1(\mathbb{G})$ to itself.

1. Introduction and Main Results

Let $\mathbb{G}$ be a nilpotent Lie group associated to the Lie algebra $\mathcal{G}$. Given $X = \{X_1, \ldots, X_m\}$ a Hörmander system of left invariant vector fields on a nilpotent group $\mathbb{G}$ and ρ be the Carnot-Carathéodory distance with respected to X. For each $x \in \mathbb{G}$ and each $r > 0$, we denote by $B(x, r) = \{y \in \mathbb{G} : \rho(x, y) < r\}$ the ball with center x and radius r. We fix a Haar measure dx on $\mathbb{G}$. For $E \subset \mathbb{G}$ measurable, we use $|E|$ to denote the measure of E. Throughout this paper, $0 \in \mathbb{G}$ denotes the unit element of $\mathbb{G}$. Denote $V(t) = |B(0, t)| = |B(x, t)|$ on each $x \in \mathbb{G}$ and each $t > 0$. Then, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} &C_1^{-1} t^d \leq V(t) \leq C_1 t^d, && \forall\ 0 \leq t \leq 1 \\ &C_1^{-1} t^D \leq V(t) \leq C_1 t^D, && \forall\ 1 \leq t < \infty \end{aligned} \tag{1.1}$$

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where d and D are local dimension and the dimension at infinity of $\mathbb{G}$. From this, we can see that there exists constants $C_2 = C_2(C_1, d, D)$ and $C_3 > 1$ such that

$$C_2^{-1}\left(\frac{R}{r}\right)^d \leq \frac{V(R)}{V(r)} \leq C_2\left(\frac{R}{r}\right)^D \qquad \forall\ 0 < r < R < \infty \tag{1.2}$$

$$V(2r) \leq C_3 V(r) \qquad \forall\ r > 0. \tag{1.3}$$

Let $\Delta = \sum_{i=1}^m X_i^2$ be the sub-Laplacian and $\nabla f = (X_1 f, \ldots, X_k f)$ be the gradient. Consider the Schrӧdinger operator A on $\mathbb{G}$ defined by

$$A = -\Delta + W,$$

where W is a nonnegative potential. We say that W belongs to the reverse Hӧlder class B_q for some $q > 1$ if W satisfies the following inequality:

$$\left(\frac{1}{|B|}\int_B W(x)^q dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B W(x)dx\right)$$

for every ball B in $\mathbb{G}$, where C is independent of B. We use R to denote the Riesz transforms $\nabla A^{-1/2}$ (associated to the Schrӧdinger operator A).

In the special case $\mathbb{G} = \mathbb{R}^n$, Shen [8] proved the L^p boundedness of the Riesz transform R.

Theorem A. *Suppose that $W \in B_q$ for some $d/2 \leq q \leq d$. Then for $1 < p \leq p_0$,*

$$\|R(f)\|_p \leq C_p \|f\|_p, \quad \forall\ f \in L^p(\mathbb{R}^n)$$

where $1/p_0 = 1/q - 1/d$.

In 1999, H-Q, Li [6] extended this result to general nilpotent Lie groups. More precisely, he showed the following

Theorem B. *Suppose that $W \in B_q$ for some $D/2 \leq q \leq D$. Then for $1 < p \leq p_0$,*

$$\|R(f)\|_{L^p(\mathbb{G})} \leq C_p \|f\|_{L^p(\mathbb{G})}, \quad \forall\ f \in L^p(\mathbb{G})$$

where $1/p_0 = 1/q - 1/D$.

Remark 1.1. Note that the L^2-boundedness of R is always true under the conditions in Theorems B since $p_0 \geq 2$.

The doubling conditions (1.3) implies that the nilpotent groups $\mathbb{G}$ is a space of homogeneous type in the sense of Coifman and Weiss [1]. Thus the function spaces such as Hardy spaces H^1, BMO, VMO are well defined on $\mathbb{G}$ (see [1]). In this paper, we consider the endpoint case and we want to show the $H^1(\mathbb{G})$ boundedness of R.

Theorem 1.1. Suppose that $W \in B_q$ for some $D/2 \leq q \leq D$. Then R is bounded from $H^1(\mathbb{G})$ to itself. By duality, its adjoint $R^* := -A^{-1/2}\nabla$ is bounded from $BMO(\mathbb{G})$ to itself.

Remark 1.2. On $\mathbb{R}^n$, if $W \equiv 0$, then R is just the classical Riesz transform. It is well known that, the classical Riesz transform is bounded from $H^1(\mathbb{R}^n)$ to itself. Thus our results extend this result to the case that W belongs to the reverse Hölder class even in the classical setting $\mathbb{G} = \mathbb{R}^n$.

2. Some Lemmas

We first introduce the truncated operator of Riesz transform R. For $0 < \epsilon < 1$, the truncated operator R^ϵ is defined by

$$R^\epsilon f(x) = \frac{1}{\sqrt{\pi}} \int_\epsilon^{1/\epsilon} \nabla e^{-tA} f(x) \frac{dt}{\sqrt{t}}, \qquad \forall\, f \in C_0^\infty(\mathbb{G}).$$

Lemma 2.1. For all $f \in C_0^\infty(\mathbb{G})$, $\lim_{\epsilon \to 0} R^\epsilon f = Rf \quad in\ L^2(\mathbb{G})$.

Proof. From $\langle Af, f\rangle \geq \langle Wf, f\rangle = \|W^{1/2} f\|_{L^2}^2$, it follows that

$$\sigma(A) \subset \mathbb{R}_+,$$

where $\sigma(A)$ denotes the spectrum of A and $\mathbb{R}_+$ denotes all nonnegative real numbers. Fix $\mu \in (0, \pi/2)$ and set $\Gamma_\mu = \{z \in \mathbb{C} : |\arg z| < \mu\}$, define

$$\psi_\epsilon(z) = \frac{1}{\sqrt{\pi}} \int_\epsilon^{1/\epsilon} e^{-tz} z^{1/2} \frac{dt}{\sqrt{t}}, \quad z \in \Gamma_\mu \ \text{ and } \ \epsilon > 0.$$

For any function $g \in \mathbb{D}(A^{1/2})$ (the domain of $A^{1/2}$) and $\epsilon > 0$, define

$$u_\epsilon = \frac{1}{\sqrt{\pi}} \int_\epsilon^{1/\epsilon} e^{-tA} A^{1/2} g \frac{dt}{\sqrt{t}},$$

so that $u_\epsilon = \psi_\epsilon(A) g$. Observe that $\lim_{\epsilon \to 0} \psi_\epsilon(z) = 1$ uniformly on all compact subsets of Γ_μ. By H^∞ functional calculus for A (see [7] and [11]), we therefore have

$$\lim_{\epsilon \to 0} \|A^{1/2} u_\epsilon - A^{1/2} g\|_{L^2(\mathbb{G})} = 0.$$

By the L^2 boundedness of the operator $\nabla A^{-1/2}$ (see Remark 1.1), we get

$$\lim_{\epsilon\to 0}\|\nabla u_\epsilon - \nabla g\|_{L^2(\mathbb{G})} = \lim_{\epsilon\to 0}\|\nabla A^{-1/2}A^{1/2}u_\epsilon - \nabla A^{-1/2}A^{1/2}g\|_{L^2(\mathbb{G})}$$

$$\leq C\lim_{\epsilon\to 0}\|A^{1/2}u_\epsilon - A^{1/2}g\|_{L^2(\mathbb{G})} = 0.$$

By (2.1) and taking $g = A^{-1/2}f$, we obtain

$$\lim_{\epsilon\to 0}\left\|\frac{1}{\sqrt{\pi}}\int_\epsilon^{1/\epsilon}\nabla e^{-tA}f\frac{dt}{\sqrt{t}} - \nabla A^{-1/2}f\right\|_{L^2(\mathbb{G})} = 0. \qquad \blacksquare$$

Denote by $\tilde{p_t}(x,y)$ the heat kernel on $\mathbb{G}$, then there exists positive constants $c,\ c_1$ such that (see [10] p.48)

$$|X^I\tilde{p_t}(x,y)| \leq ct^{-|I|/2}V(\sqrt{t})^{-1}\exp\left(-c_1\frac{\rho(x,y)^2}{t}\right), \tag{2.1}$$

where X^I denotes the operator $X_1^{i_1}\ldots X_m^{i_m}$ for $I = (i_1,\ldots,i_m)$. Let $p_t(x,y)$ be the kernel of Schrödinger heat semigroup e^{-tA}. Since W is nonnegative, Trotter's formula implies that

$$0 < p_t(x,y) \leq \widetilde{p}_t(x,y) \leq cV(\sqrt{t})^{-1}\ \exp\left(-c_1\frac{\rho(x,y)^2}{t}\right) \tag{2.2}$$

for all $x,y\in\mathbb{G}$, $t>0$.

Lemma 2.2. For all $\gamma, t > 0$ and $s \geq 0$,

$$\int_{\rho(x,y)\geq s^{1/2}} e^{-2\gamma\rho(x,y)^2/t}dx \leq C_\gamma V(\sqrt{t})e^{-\gamma s/t} \quad \forall\ y\in\mathbb{G}.$$

Proof. First note that

$$\int_{\rho(x,y)\geq s^{1/2}} e^{-2\gamma\rho(x,y)^2/t}dx \leq e^{-\gamma s/t}\int_{\mathbb{G}} e^{-\gamma\rho(x,y)^2/t}dx := e^{-\gamma s/t}I.$$

By (1.2), we have

$$I = \sum_{k=0}^{\infty}\int_{kt^{1/2}\leq\rho(x,y)<(k+1)t^{1/2}} e^{-\gamma\rho(x,y)^2/t}dx$$

$$\leq C\sum_{k=0}^{\infty}(k+1)^D e^{-\gamma k^2}V(\sqrt{t}) \leq C_\gamma V(\sqrt{t}).$$

Inserting this estimate into the first one yields the desired conclusion. $\blacksquare$

The following lemma is a consequence of (2.2) and Lemma 2.2.

Lemma 2.3. For all $\gamma \in (0,\ 2c_1)$,

$$\int_{\mathbb{G}} |p_t(x,y)|^2 e^{\gamma\rho(x,y)^2/t}dx \le C_\gamma V(\sqrt{t})^{-1}, \quad \forall\ y \in \mathbb{G},\ t>0.$$

Now we give estimates for $\partial_t p_t(x,y)$ and $\nabla_x p_t(x,y)$. Here and hereafter we (sometimes) use ∂_t to denotes the operator $\frac{\partial}{\partial t}$ for simplicity.

Lemma 2.4. (i) There are constants C and c_2 such that,

$$\left|\frac{\partial}{\partial t}p_t(x,y)\right| \le \frac{C}{t}V(\sqrt{t})^{-1}e^{-c_2\rho(x,y)^2/t}.$$

(ii) The gradient of Schrödinger heat kernel vanishes at infinity; that is,

$$\lim_{\rho(x,0)\to\infty} |\nabla_x p_t(x,y)| = 0. \tag{2.3}$$

Proof. The conclusion (i) can be proved by using the same argument as [2], Proposition 4. We only prove (ii). Fix $t>0$ and $z_0 \in \mathbb{G}$. Choose x_0 such that $\rho(x_0,z_0)>10$. Take a cutoff function $\eta \in C_0^\infty(B(x_0,2C_0))$($C_0$ is a constant only depends on $\mathbb{G}$) such that $\eta=1$ on $B(x_0,3/2)$, and

$$|\nabla\eta| + |\nabla^2\eta| \le C, \tag{2.4}$$

see Lemma 3.2 in [6]. Since $(-\Delta + W + \partial_t)\,p_t(x,z_0)=0$ for $x \in B(x_0,2C_0)$ (indeed for all x away form z_0),

$$\begin{aligned} p_t(x,z_0)\eta(x) =& (-\Delta+\partial_t)^{-1}[(-\Delta+\partial_t)p_t\eta](x,z_0)\\ =& \int_{\mathbb{G}} \widetilde{p_t}(x,y)(-\Delta_y+\partial_t)(p_t(y,z_0)\eta(y))dy\\ =& \int_{\mathbb{G}} \widetilde{p_t}(x,y)[-W(y)p_t(y,z_0)\eta(y) - \Delta\eta(y)p_t(y,z_0)]dy\\ & -2\int_{\mathbb{G}} \widetilde{p_t}(x,y)(\nabla_y\eta)(y)\cdot(\nabla_y p_t)(y,z_0)dy. \end{aligned} \tag{2.5}$$

Integrating by parts shows

$$\begin{aligned} & -2\int_{\mathbb{G}} \widetilde{p_t}(x,y)(\nabla_y\eta)(y)\cdot(\nabla_y p_t)(y,z_0)dy\\ =& 2\int_{\mathbb{G}} \nabla_y\widetilde{p_t}(x,y)\cdot(\nabla_y\eta)(y)p_t(y,z_0)dy + 2\int_{\mathbb{G}} \widetilde{p_t}(x,y)\Delta\eta(y)p_t(y,z_0)dy. \end{aligned} \tag{2.6}$$

By inserting (2.6) into (2.5), we get

$$\begin{aligned} p_t(x,z_0)\eta(x) =& \int_{\mathbb{G}} \widetilde{p_t}(x,y)[-W(y)p_t(y,z_0)\eta(y) + \Delta\eta(y)p_t(y,z_0)]dy\\ & + 2\int_{\mathbb{G}} \nabla_y\widetilde{p_t}(x,y)\cdot(\nabla_y\eta)(y)p_t(y,z_0)dy. \end{aligned}$$

For $x \in B(x_0, 1)$, by the choice of η we get

$$\begin{aligned}|\nabla_x p_t(x, z_0)| \le C\Big[&\int_{B(x_0,2C_0)} |\nabla_x \widetilde{p}_t(x, y)|\, W(y)\, p_t(y, z_0)dy \\ &+ \int_{B(x_0,2C_0)} |\nabla_x \widetilde{p}_t(x, y)|\, p_t(y, z_0)dy \\ &+ \int_{B(x_0,2C_0)} |\nabla_x \nabla_y \widetilde{p}_t(x, y)|\, p_t(y, z_0)|dy\Big] \\ :=\, & C(I_1 + I_2 + I_3),\end{aligned}$$

where C is constant in (2.4). When $\rho(x_0, 0) \to \infty$, using (2.1), we can easily get $I_2,\ I_3 \to 0$ since $p_t(y, z_0) \to 0$.

It thus remains to show $I_1 \to 0$ as $\rho(x_0, 0)$ goes to infinity. It is convenient to use the following auxiliary function

$$\varrho(x) := \sup_{r>0} \left\{ r : \frac{r^2}{V(r)} \int_{B(x,r)} W(y)dy \le 1 \right\}.$$

When $\rho(x_0, 0)$ is sufficiently large, we have

$$B(x_0, 2C_0) \subset B(z_0, 2\rho(x_0, z_0)), \tag{2.7}$$

and

$$2\rho(x_0, z_0) > r_0 := \varrho(z_0).$$

From $W \in B_q$ for $q > D/2 > 1$, we know that $W(y)dy$ is a doubling measure (see [6], p.158). Thus, there exists $C_1 > 1$ such that for any $r > 0$

$$\int_{B(x_0,2r)} W(y)dy \le C_1 \int_{B(x_0,r)} W(y)dy.$$

From this and $r_0^2 V(r_0)^{-1} \int_{B(z_0,r_0)} W(y)dy \le 1$, it follows that

$$\begin{aligned}\int_{B(z_0,2\rho(x_0,z_0))} W(y)dy &\le C_1^{\log_2\left(\frac{2\rho(x_0,z_0)}{r_0}+1\right)} \int_{B(z_0,r_0)} W(y)dy \\ &\le C r_0^{D-2} \left(\frac{2\rho(x_0, z_0)}{r_0} + 1\right)^{\log_2 C_1}.\end{aligned} \tag{2.8}$$

Hence, by (2.1), (2.7) and (2.8), we obtain

$$\begin{aligned}I_1 &\le C_t \sup_{y\in B(x_0,2C_0)} e^{-c\rho(y,z_0)^2/t} \int_{B(x_0,2C_0)} W(y)dy \\ &\le C e^{-c\rho(x_0,z_0)^2/t} \int_{B(z_0,2\rho(x_0,z_0))} W(y)dy\end{aligned}$$

$$\leq Cr_0^{D-2}\, e^{-c\rho(x_0,z_0)^2/t} \left(\frac{2\rho(x_0,z_0)}{r_0} + 1 \right)^{\log_2 C_0}$$
$$\to 0, \qquad \text{when } \rho(x_0,0) \to \infty.$$

Thus the proof of this lemma is finished. ∎

Lemma 2.5. For any γ, $0 < \gamma < \min\{2c_1, 2c_2\}$, and all $y \in \mathbb{G}$ and $t > 0$, we have

$$\int_{\mathbb{G}} \left(|\nabla_x p_t(x,y)|^2 + W(x) p_t(x,y)^2 \right) e^{\gamma\rho(x,y)^2/t} dx \leq \frac{C}{tV(\sqrt{t})}.$$

Proof. By (2.3) and integration by parts, we get

$$\begin{aligned}
I(t,y) :=& \int_{\mathbb{G}} \left(|\nabla_x p_t(x,y)|^2 + W(x) p_t(x,y)^2 \right) e^{\gamma\rho(x,y)^2/t} dx \\
=& \int_{\mathbb{G}} |\nabla_x p_t(x,y)|^2 e^{\gamma\rho(x,y)^2/t} dx + \int_{\mathbb{G}} W(x) p_t(x,y)^2 e^{\gamma\rho(x,y)^2/t} dx \\
\leq& \left| \int_{\mathbb{G}} p_t(x,y)(-\Delta + W) p_t(x,y) e^{\gamma\rho(x,y)^2/t} dx \right| \\
&+ \left| \int_{\mathbb{G}} p_t(x,y) \nabla_x (p_t(x,y)) \cdot \nabla_x (e^{\gamma\rho(x,y)^2/t}) dx \right| \\
:=& I_1(t,y) + I_2(t,y).
\end{aligned}$$

By (2.2) and Lemmas 2.4 and 2.2, we get

$$\begin{aligned}
I_1(t,y) =& \left| \int_{\mathbb{G}} p_t(x,y)(\partial_t p_t(x,y)) e^{\gamma\rho(x,y)^2/t} dx \right| \\
\leq& \frac{C_\gamma}{tV(\sqrt{t})^2} \int_{\mathbb{G}} e^{-(c_1+c_2-\gamma)\rho(x,y)^2/t} dx \\
\leq& \frac{C}{tV(\sqrt{t})}.
\end{aligned}$$

On the other hand, notice that $\gamma < 2c_1$, we may choose γ satisfying $\gamma < \gamma' < c_1 + (\gamma/2)$. For any Lipschitz function f with respect to ρ with Lipschitz constant C, the distribution $X_i f$ is a locally integrable function and $\sum_1^k |X_i f|^2 \leq C^2$ a.e. (see [10], for instant). In particular,

$$|\nabla \rho| \leq 1 \quad \text{a.e.} \tag{2.9}$$

Thus we have

$$\begin{aligned}
I_2(t,y) \leq& \int_{\mathbb{G}} p_t(x,y) |\nabla_x p_t(x,y)|\, e^{\gamma\rho(x,y)^2/t} 2\gamma\rho(x,y)/t dx \\
\leq& \frac{C_\gamma}{\sqrt{t}} \int_{\mathbb{G}} p_t(x,y) |\nabla_x p_t(x,y)| e^{\gamma'\rho(x,y)^2/t} dx.
\end{aligned}$$

Let $\eta = \gamma/2$ and $\xi = \gamma' - \gamma/2$. Then using Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned} I_2(t,y) &\le \frac{C}{\sqrt{t}} \left(\int_{\mathbb{G}} |p_t(x,y)|^2 e^{2\xi\rho(x,y)^2/t} dx \right)^{1/2} \left(\int_{\mathbb{G}} |\nabla_x p_t(x,y)|^2 e^{2\eta\rho(x,y)^2/t} dx \right)^{1/2} \\ &\le C_\gamma \frac{1}{\sqrt{t} V^{1/2}(\sqrt{t})} \left(\int_{\mathbb{G}} |\nabla_x p_t(x,y)|^2 e^{\gamma\rho(x,y)^2/t} dx \right)^{1/2} \\ &\le C_\gamma \frac{1}{\sqrt{t} V^{1/2}(\sqrt{t})} \sqrt{I(t,y)}. \end{aligned}$$

Hence

$$I(t,y) \le C_\gamma \left(\frac{1}{tV(\sqrt{t})} + \frac{\sqrt{I(t,y)}}{t^{1/2} V(\sqrt{t})^{1/2}} \right).$$

From the above inequality, we get $I(t,y) \le \frac{C}{tV(\sqrt{t})}$. Thus, we finish the proof of Lemma 2.5. ■

Let $q_t(x,y) = p_t(x,y) - p_t(x,y_0)$. The following lemma can be proved by using the same argument in [3] thus we omit the proof here.

Lemma 2.6. There exist $\tau, c_3 > 0$ such that for all $\rho(y,y_0) \le \sqrt{t}$,

$$|q_t(x,y)| \le C \left(\frac{\rho(y,y_0)}{\sqrt{t}} \right)^\tau V(\sqrt{t})^{-1} \exp\left(-\frac{c_3 \rho(x,y)^2}{t} \right).$$

Applying Lemmas 2.6 and 2.2, we can easily deduce

Lemma 2.7. If $\rho(y,y_0) \le \sqrt{t}$, then for any $0 < \alpha < 2c_3$, there exists $C_\alpha > 0$ such that

$$\int_{\mathbb{G}} |q_t(x,y)|^2 e^{\alpha\rho(x,y)^2/t} \, dx \le \left(\frac{\rho(y,y_0)}{\sqrt{t}} \right)^{2\tau} \frac{C_\alpha}{V(\sqrt{t})}.$$

Lemma 2.8. If $\rho(y_0,y) \le \sqrt{t}$, then for any $0 < \alpha < \min\{2c_3, 1/2\}$, there exists $C'_\alpha > 0$ such that

$$\int_{\mathbb{G}} \left(|\nabla_x q_t(x,y)|^2 + W(x) q_t(x,y)^2 \right) e^{\alpha\rho(x,y)^2/t} \, dx \le \frac{1}{t} \left(\frac{\rho(y,y_0)}{\sqrt{t}} \right)^{2\tau} \frac{C_\alpha}{V(\sqrt{t})}.$$

Proof. Denote $\xi(x,y,t) = \alpha\rho(x,y)^2/t$. By (2.9), it is easy to check for almost every $x \in \mathbb{G}$,

$$\frac{\partial \xi}{\partial t} + \frac{1}{4\alpha} |\nabla_x \xi|^2 \le 0.$$

Set

$$f(y,t)=\int_{\mathbb{G}}\left[|\nabla_x q_t(x,y)|^2+W(x)q_t^2(x,y)\right]\ e^{\xi(x,y,t)}\ dx. \tag{2.10}$$

For simplicity, the variables x, y, t will be omitted and ∇ always denotes ∇_x in the proof below. By (2.3), integrating by parts yields,

$$f(y,t)=\int qe^{\xi}Aq-\int q\nabla q\cdot\nabla(e^{\xi}).$$

The Cauchy-Schwarz inequality shows that

$$f(y,t)\le\left(\int q^2e^{\xi}\right)^{1/2}\left[\left(\int e^{\xi}(Aq)^2\right)^{1/2}+\left(\int e^{\xi}(\nabla q\cdot\nabla\xi)^2\right)^{1/2}\right]. \tag{2.11}$$

On the other hand, computing the time derivative of f in (2.10), we get

$$\begin{aligned}
\partial_t f&=2\int e^{\xi}\nabla(\frac{\partial q}{\partial t})\cdot\nabla q+\int e^{\xi}\left(|\nabla q|^2+Wq^2\right)\frac{\partial\xi}{\partial t}+2\int\frac{\partial q}{\partial t}Wqe^{\xi}\\
&\le-2\int e^{\xi}\nabla(Aq)\cdot\nabla q-\frac{1}{4\alpha}\int e^{\xi}(|\nabla q|^2+Wq^2)|\nabla\xi|^2-2\int AqWqe^{\xi}\\
&=-2\int e^{\xi}(Aq)^2+2\int Aqe^{\xi}\nabla\xi\cdot\nabla q-\frac{1}{4\alpha}\int e^{\xi}(|\nabla q|^2+Wq^2)|\nabla\xi|^2\\
&\le-2\int e^{\xi}(Aq)^2+2\left(\int e^{\xi}(Aq)^2\right)^{1/2}\left(\int e^{\xi}|\nabla q|^2|\nabla\xi|^2\right)^{1/2}\\
&\quad-\frac{1}{4\alpha}\int e^{\xi}|\nabla q|^2|\nabla\xi|^2,
\end{aligned} \tag{2.12}$$

where in the third line we use Lemma 2.4 and integration by parts and in the last inequality we use the fact

$$-\frac{1}{4\alpha}\int Wq^2e^{\xi}|\nabla\xi|^2\le 0.$$

By (2.11) and (2.12), we have for $0<c<2$

$$\begin{aligned}
\partial_t f+c\frac{f^2}{\int e^{\xi}q^2}\le&(-2+c)\int e^{\xi}(Aq)^2+(2+2c)\left(\int e^{\xi}(Aq)^2\right)^{1/2}\left(\int e^{\xi}|\nabla q|^2|\nabla\xi|^2\right)^{1/2}\\
&+\left(c-\frac{1}{4\alpha}\right)\int e^{\xi}|\nabla q|^2|\nabla\xi|^2.
\end{aligned}$$

If we choose $c=(2-4\alpha)/(1+16\alpha)$, then it is easy to check that

$$\partial_t f + c\frac{f^2}{\int e^{\xi} q^2} \leq 0. \tag{2.13}$$

Denote

$$\phi(t) = \left(\frac{\rho(y, y_0)}{\sqrt{t}}\right)^{2\tau} \frac{C_\alpha}{V(\sqrt{t})}.$$

By (2.13) and Lemma 2.7, we have

$$\frac{\partial_t f}{f^2} \leq -\frac{c}{\int e^{\xi} q^2} \leq -\frac{c}{\phi(t)}. \tag{2.14}$$

An integration on $[0, t]$ in the two sides of (2.14) implies that

$$f(y, t) \leq \frac{1}{c\int_0^t \frac{du}{\phi(u)}}.$$

Since

$$\int_0^t \frac{du}{\phi(u)} \geq C_\alpha^{-1} \int_{t/2}^t \left(\frac{u}{\rho(y, y_0)^2}\right)^{\tau} V(\sqrt{u})\, du \geq C'_\alpha \frac{t}{2} \left(\frac{t}{\rho(y, y_0)^2}\right)^{\tau} V(\sqrt{t}),$$

we finally get that

$$f(y, t) \leq \frac{C''_\alpha}{t} \left(\frac{\rho(y, y_0)^2}{t}\right)^{\tau} \frac{1}{V(\sqrt{t})}.$$

Thus we complete the proof of Lemma 2.8. ■

Now we give the definition of $H^1(\mathbb{G})$ atom.

Definition 2.1. A complex-valued function a defined on $\mathbb{G}$ is said to be an $H^1(\mathbb{G})$ atom, if it is supported on a ball B in $\mathbb{G}$ and satisfies

$$\int_{\mathbb{G}} a(x)\, dx = 0 \quad \text{and} \quad \|a\|_{L^2(\mathbb{G})} \leq |B|^{-1/2}.$$

Remark 2.1. Obviously, if a is an $H^1(\mathbb{G})$ atom, then $\|a\|_{L^1(\mathbb{G})} \leq 1$.

The following conclusion will be used in the proof of our main theorem.

Theorem 2.2. For any $H^1(\mathbb{G})$ atom a and $0 < \epsilon < 1$,

$$\int_{\mathbb{G}} R^{\epsilon} a(x)\, dx = 0.$$

Proof. Suppose that $\operatorname{supp} a \subset B(y_0, r)$. For $k \geq 2$ sufficiently large, we can choose $\phi_k \in C_c^\infty(\mathbb{G})$ such that $0 \leq \phi_k \leq 1$ and

$$\phi_k(x) = \begin{cases} 1, & \text{for } \; x \in B(y_0, k-1) \\ 0, & \text{for } \; x \notin B(y_0, C_0(k+1)), \end{cases}$$

where $C_0 \geq 1$ is a constant and

$$\|\nabla \phi_k\|_\infty \leq C$$

for some absolute constant $C > 0$, see [6], Lemma 3.2. Denote

$$I = \int_{\mathbb{G}} |a(y)| \int_\epsilon^{1/\epsilon} \int_{\mathbb{G}} |\nabla_x p_t(x,y)| \, |\phi_k(x)| \, dx \, \frac{dt}{\sqrt{t}} \, dy.$$

By Hölder's inequality and Lemma 2.5, taking $0 < \delta < \min\{2c_1, 2c_2\}$ we have

$$\begin{aligned}
\int_{\mathbb{G}} |\nabla_x p_t(x,y)| \, |\phi_k(x)| \, dx \leq & C \left(\int_{\mathbb{G}} |\nabla_x p_t(x,y)|^2 \, dx \right)^{1/2} V(C_0(k+1))^{1/2} \\
\leq & C \left(\int_{\mathbb{G}} e^{\delta \rho(x,y)^2/t} |\nabla_x p_t(x,y)|^2 \, dx \right)^{1/2} V(C_0(k+1))^{1/2} \\
\leq & \frac{C}{\sqrt{t}} \left(\frac{V(C_0(k+1))}{V(\sqrt{t})} \right)^{1/2} \\
\leq & \frac{C}{\sqrt{t}} \left(\frac{k+1}{\sqrt{t}} \right)^{D/2},
\end{aligned}$$

where in the last inequality, we use (1.1) for a sufficiently large k. Hence, we get

$$\begin{aligned}
I \leq & C \int_{B(y_0,r)} |a(y)| \int_\epsilon^{1/\epsilon} \left(\frac{k+1}{\sqrt{t}} \right)^{D/2} \frac{dt}{t} \, dy \\
\leq & C \left(\frac{k+1}{\sqrt{\epsilon}} \right)^{D/2} \int_{B(y_0,r)} |a(y)| \int_\epsilon^{1/\epsilon} \frac{dt}{t} \, dy < \infty.
\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{G}} \phi_k(x) R^\epsilon a(x) dx = & \frac{1}{\sqrt{\pi}} \int_{\mathbb{G}} \phi_k(x) \int_\epsilon^{1/\epsilon} \int_{\mathbb{G}} \nabla_x p_t(x,y) a(y) dy \, \frac{dt}{\sqrt{t}} \, dx \\
= & \frac{1}{\sqrt{\pi}} \int_{\mathbb{G}} a(y) \int_\epsilon^{1/\epsilon} \int_{\mathbb{G}} \nabla_x p_t(x,y) \phi_k(x) \, dx \, \frac{dt}{\sqrt{t}} \, dy \\
= & -\frac{1}{\sqrt{\pi}} \int_{\mathbb{G}} a(y) \int_\epsilon^{1/\epsilon} \int_{\mathbb{G}} p_t(x,y) \nabla_x \phi_k(x) \, dx \, \frac{dt}{\sqrt{t}} \, dy.
\end{aligned}$$

Notice that

$$\int_{\mathbb{G}} R^{\epsilon}a(x)dx = \lim_{k\to\infty}\int_{\mathbb{G}} \phi_k(x)R^{\epsilon}a(x)dx,$$

applying dominated convergence theorem, Theorem 2.2 therefore is a consequence of

$$\lim_{k\to\infty}\int_{\mathbb{G}} |a(y)| \int_{\epsilon}^{1/\epsilon}\int_{\mathbb{G}} |\nabla_x \phi_k(x)| p_t(x,y)\,dx\,\frac{dt}{\sqrt{t}}\,dy = 0 \tag{2.15}$$

and

$$R^{\epsilon}a \in L^1(\mathbb{G}). \tag{2.16}$$

To show (2.15), if denotes

$$I_k = \int_{\mathbb{G}} |a(y)| \int_{\epsilon}^{1/\epsilon}\int_{\mathbb{G}} |\nabla_x \phi_k(x)| p_t(x,y)\,dx\,\frac{dt}{\sqrt{t}}\,dy,$$

then by the choice of ϕ_k and (2.2), using the volume growth condition (1.1), we have

$$\begin{aligned}
I_k \leq & C\int_{B(y_0,r)} |a(y)| \int_{\epsilon}^{1/\epsilon}\int_{B(y_0,C_0(k+1))\backslash B(y_0,k-1)} p_t(x,y)\,dx\,\frac{dt}{\sqrt{t}}\,dy \\
\leq & C\int_{B(y_0,r)} |a(y)| \int_{\epsilon}^{1/\epsilon}\int_{B(y_0,C_0(k+1))\backslash B(y_0,k-1)} V(\sqrt{t})^{-1} e^{-c_1\rho(x,y)^2/t}\,dx\frac{dt}{\sqrt{t}}\,dy \\
\leq & C\int_{B(y_0,r)} |a(y)| \int_{\epsilon}^{1/\epsilon} \left(\frac{k+1}{\sqrt{t}}\right)^{D} e^{-c_1(k-r)^2/t}\,\frac{dt}{\sqrt{t}}\,dy \\
= & C\|a\|_{L^1(\mathbb{G})}\int_{\epsilon}^{1/\epsilon} \left(\frac{k+1}{\sqrt{t}}\right)^{D} e^{-c_1(k-r)^2/t}\,\frac{dt}{\sqrt{t}}.
\end{aligned}$$

Hence, the dominated convergence theorem yields $\lim_{k\to\infty} I_k = 0$.

As for (2.16), by the definition of R^{ϵ},

$$\begin{aligned}
R^{\epsilon}a(x) &= \frac{1}{\sqrt{\pi}}\int_{\epsilon}^{1/\epsilon} \nabla_x e^{-tA}a(x)\frac{dt}{\sqrt{t}} \\
&= \frac{1}{\sqrt{\pi}}\int_{\epsilon}^{1/\epsilon}\int_{\mathbb{G}} \nabla_x p_t(x,y)a(y)dy\frac{dt}{\sqrt{t}}.
\end{aligned} \tag{2.17}$$

By Lemma 2.5, it is not hard to see

$$\int_{\mathbb{G}} |\nabla_x p_t(x,y)|dx \leq C_{\epsilon} \tag{2.18}$$

which further implies

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{G}}\int_{\epsilon}^{1/\epsilon}|\nabla_x p_t(x,y)|\frac{dt}{\sqrt{t}}dx \le C_\epsilon, \tag{2.19}$$

then from Minkowski's inequality and (2.17), we get

$$\|R^\epsilon a\|_{L^1(\mathbb{G})} \le C_\epsilon \|a\|_{L^1(\mathbb{G})} \le C_\epsilon.$$

Thus, (2.16) holds and Theorem 2.2 follows. ■

Finally, let us recall the definition and properties of $BMO(\mathbb{G})$. A locally square integrable function ϕ is said to be a $BMO(\mathbb{G})$ function if

$$\|\phi\|^2_{BMO(\mathbb{G})} = \sup \frac{1}{|B|}\int_B |\phi(x)-\phi_B|^2 dx < +\infty, \tag{2.20}$$

where the supremun is taken over all the balls in $\mathbb{G}$. By the definition (2.20), it is easy to see that

$$|\phi_B - \phi_{2B}| \le C\|\phi\|_{BMO(\mathbb{G})}. \tag{2.21}$$

(2.21) yields, as in [4], that there exists $C>0$ such that for $\phi \in BMO$, $k \ge 1$ and all balls $B \subset \mathbb{G}$

$$\frac{1}{|2^k B|}\int_{2^k B}|\phi(x)-\phi_{2B}|^2 dx \le Ck^2\|\phi\|^2_{BMO(\mathbb{G})}. \tag{2.22}$$

3. Proof of Theorem 1.1

Let a be an $H^1(\mathbb{G})$ atom supported in $B = B(y_0, r)$. Taking $\phi \in C_c(\mathbb{G})$ (the continuous functions with compact support in $\mathbb{G}$). Without lost the generality, we may assume that $0 < \epsilon < \min\{1, r^2, r^{-2}\}$. Applying Theorem 2.2, we may write

$$\int_{\mathbb{G}} R^\epsilon a(x)\,\phi(x)\,dx = \int_{\mathbb{G}} R^\epsilon a(x)\,\left(\phi(x)-\phi_{2B}\right)\,dx.$$

Decompose $\phi - \phi_{2B}$ as

$$\phi - \phi_{2B} = (\phi-\phi_{2B})\chi_{2B} + (\phi-\phi_{2B})\chi_{(2B)^c} := \phi_1 + \phi_2.$$

Thus, we have

$$\int_{\mathbb{G}} R^\epsilon a(x)\,\phi(x)\,dx = \int_{\mathbb{G}} R^\epsilon a(x)\,\phi_1(x)\,dx + \int_{\mathbb{G}} R^\epsilon a(x)\,\phi_2(x)\,dx := E_1 + E_2.$$

By Cauchy-Schwarz inequality,

$$
\begin{aligned}
|E_1| &\le \int_{\mathbb{G}} |R^\epsilon a(x)|\,|\phi_1(x)|\,dx \le \|R^\epsilon a\|_{L^2(\mathbb{G})}\,\|(\phi-\phi_{2B})\chi_{2B}\|_{L^2(\mathbb{G})} \\
&\le \|R^\epsilon a\|_{L^2(\mathbb{G})}\,|2B|\|\phi\|_{BMO(\mathbb{G})}.
\end{aligned}
$$

We now deal with E_2.

$$
\begin{aligned}
E_2 =& \sum_{k\ge 1}\int_{2^{k+1}B\setminus 2^kB} R^\epsilon a(x)\,\phi_2(x)\,dx \\
=& \frac{1}{\sqrt{\pi}}\sum_{k\ge 1}\int_{2^{k+1}B\setminus 2^kB} \phi_2(x)\int_\epsilon^{r^2}\int_B \nabla_x p_t(x,y)\,a(y)\,dy\,\frac{dt}{\sqrt{t}}\,dx \\
&+\frac{1}{\sqrt{\pi}}\sum_{k\ge 1}\int_{2^{k+1}B\setminus 2^kB} \phi_2(x)\int_{r^2}^{1/\epsilon}\int_B \nabla_x p_t(x,y)\,a(y)\,dy\,\frac{dt}{\sqrt{t}}\,dx \\
:=& \sum_{k\ge 1} I_k + \sum_{k\ge 1} J_k.
\end{aligned}
$$

Fix $k\ge 1$. When $y\in B(y_0,r)$ and $2^k r\le \rho(x,y_0)\le 2^{k+1}r$, we have $2^{k-1}r\le \rho(x,y)\le 2^{k+2}r$. The Cauchy-Schwarz inequality, Lemma 2.5, (2.22) and (1.1) yield

$$
\begin{aligned}
&\int_{2^{k+1}B\setminus 2^kB} |\nabla_x p_t(x,y)|\,|\phi_2(x)|\,dx \\
&\le \left(\int_{\mathbb{G}} |\nabla_x p_t(x,y)|^2 e^{\gamma\rho(x,y)^2/t}\,dx\right)^{1/2}\left(\int_{2^{k+1}B\setminus 2^kB} |\phi_2(x)|^2 e^{-\gamma\rho(x,y)^2/t}\,dx\right)^{1/2} \\
&\le \frac{C}{(tV(\sqrt{t}))^{1/2}}\,e^{-\gamma(2^{k-1}r)^2/2t}\left(\int_{2^{k+2}B} |\phi(x)-\phi_{2B}|^2\,dx\right)^{1/2} \\
&\le \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}}\,e^{-\gamma(2^{k-1}r)^2/2t}\left(\frac{V(2^{k+2}r)}{V(\sqrt{t})}\right)^{1/2} \\
&\le \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}}\,e^{-\gamma(2^{k-1}r)^2/2t}\max\left\{1,\left(\frac{2^{k+2}r}{\sqrt{t}}\right)^{D/2}\right\} \\
&\le \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}}\,e^{-\beta 2^{2k}r^2/t},
\end{aligned}
$$

for $0 < \beta < \gamma/8$. Therefore,

$$\begin{aligned}
&\int_\epsilon^{r^2} \int_{2^{k+1}B\setminus 2^k B} |\nabla_x p_t(x,y)||\phi_2(x)|dx\frac{dt}{\sqrt{t}} \\
&\leq Ck \int_0^{r^2} e^{-\beta 2^{2k}r^2/t}\frac{dt}{t} \\
&\leq Ck \int_{2^{2k}}^{\infty} e^{-\beta t}\frac{dt}{t} \\
&\leq Ck2^{-2k}.
\end{aligned}$$

which yields $\sum_{k\geq 1} |I_k| \leq C\|\phi\|_{BMO(\mathbb{G})}$, since $\|a\|_{L^1(\mathbb{G})} \leq 1$.
The treatment of J_k is similar. Since a has mean value zero, we have

$$\begin{aligned}
\int_B a(y)\, \nabla_x p_t(x,y)\, dy &= \int_B a(y)\ (\nabla_x p_t(x,y) - \nabla_x p_t(x,y_0))\ dy \\
&= \int_B a(y)\, \nabla_x q_t(x,y)\, dy.
\end{aligned}$$

Thus we have

$$|J_k| \leq c \int_B |a(y)| \int_{r^2}^{1/\epsilon} \int_{2^{k+1}B\setminus 2^k B} |\nabla_x q_t(x)|\, |\phi_2(x)|\, dx\, \frac{dt}{\sqrt{t}}\, dy.$$

When $t \leq 2^{2k+4}r^2$, use Lemma 2.8 and (2.22) for $0 < 4\beta < \alpha < \min\{2c_3, 1/2\}$ we have

$$\begin{aligned}
&\int_{2^{k+1}B\setminus 2^k B} |\nabla_x q_t(x)|\, |\phi_2(x)|\, dx \\
&\leq \left(\int_{\mathbb{G}} |\nabla_x q_t(x)|^2 e^{\alpha\rho(x,y)^2/t} dx\right)^{1/2} \left(\int_{2^{k+1}B\setminus 2^k B} |\phi_2(x)|^2 e^{-\alpha\rho(x,y)^2/t}\, dx\right)^{1/2} \\
&\leq \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}}\, e^{-\alpha 2^{2k-2}r^2/t} \left(\frac{r}{\sqrt{t}}\right)^\tau \left(\frac{V(2^{k+2}r)}{V(\sqrt{t})}\right)^{1/2} \\
&\leq \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}} \left(\frac{r}{\sqrt{t}}\right)^\tau \left(\frac{2^{k+2}r}{\sqrt{t}}\right)^{D/2} e^{-\alpha 2^{2k-2}r^2/t} \\
&\leq \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}} \left(\frac{r}{\sqrt{t}}\right)^\tau e^{-\beta 2^{2k}r^2/t}.
\end{aligned}$$

When $t > 2^{2k+4}r^2$, the result still holds. Indeed,

$$\begin{aligned}
&\int_{2^{k+1}B\setminus 2^kB} |\nabla_x q_t(x)|\,|\phi_2(x)|\,dx \\
&\le \left(\int_{\mathbb{G}} |\nabla_x q_t(x)|^2 e^{\alpha\rho(x,y)^2/t} dx\right)^{1/2} \left(\int_{2^{k+1}B\setminus 2^kB} |\phi_2(x)|^2 e^{-\alpha\rho(x,y)^2/t}\,dx\right)^{1/2} \\
&\le \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}}\, e^{-\alpha 2^{2k-2}r^2/t} \left(\frac{r}{\sqrt{t}}\right)^{\tau} \left(\frac{V(2^{k+2}r)}{V(\sqrt{t})}\right)^{1/2} \\
&\le \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}}\, e^{-\alpha 2^{2k-2}r^2/t} \left(\frac{r}{\sqrt{t}}\right)^{\tau} \\
&\le \frac{C\|\phi\|_{BMO(\mathbb{G})}(k+2)}{\sqrt{t}}\, e^{-\beta 2^{2k}r^2/t} \left(\frac{r}{\sqrt{t}}\right)^{\tau}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{r^2}^{1/\epsilon} \int_{2^{k+1}B\setminus 2^kB} |\nabla_x q_t(x)|\,|\phi_2(x)|\,dx\,\frac{dt}{\sqrt{t}} \\
&\le C\|\phi\|_{BMO(\mathbb{G})}(k+2) \int_{r^2}^{\infty} e^{-\beta 2^{2k}r^2/t} \left(\frac{r}{\sqrt{t}}\right)^{\tau} \frac{dt}{t} \\
&\le C\|\phi\|_{BMO(\mathbb{G})}(k+2)2^{-k\tau} \int_0^{2^{2k}} e^{-\beta v} v^{\tau/2-1} dv \\
&\le C\|\phi\|_{BMO(\mathbb{G})}(k+2)2^{-k\tau} \int_0^{\infty} e^{-\beta v} v^{\tau/2-1} dv.
\end{aligned}$$

Thus, we have $\sum_{k\ge 1} |J_k| \le C\|\phi\|_{BMO(\mathbb{G})}$ from the above estimate and the fact that $\|a\|_{L^1(\mathbb{G})} \le 1$ (see Remark 2.1).

Summing up the above process, we prove that for all functions $\phi \in C_c(\mathbb{G})$ and $0 < \epsilon < \min\{1, r^2, r^{-2}\}$,

$$\left|\int_{\mathbb{G}} R^{\epsilon} a(x)\phi(x)\,dx\right| \le \|R^{\epsilon}a\|_{L^2(\mathbb{G})}|2B|^{1/2}\|\phi\|_{BMO(\mathbb{G})} + C\|\phi\|_{BMO(\mathbb{G})},$$

where C is independent of the atom a. Applying Lemma 2.1 and the L^2-boundedness of R (see Remark 1.1), we get

$$\left|\int_{\mathbb{G}} Ra(x)\phi(x)\,dx\right| \le \|Ra\|_{L^2(\mathbb{G})}|2B|^{1/2}\|\phi\|_{BMO(\mathbb{G})} + C\|\phi\|_{BMO(\mathbb{G})} \le C\|\phi\|_{BMO(\mathbb{G})},$$

where C is independent of the choice of atom a.

Now for $f \in H^1(\mathbb{G})$, write $f = \sum_j \lambda_k a_k$, then

$$\langle Rf, \phi\rangle = \langle R\sum_k \lambda_k a_k, \phi\rangle = \Big\langle \sum_k \lambda_k a_k, R^*\phi\Big\rangle = \sum_k \lambda_k \Big\langle a_k, R^*\phi\Big\rangle = \sum_k \lambda_k \langle Ra_k, \phi\rangle.$$

Hence,

$$|\langle Rf, \phi\rangle| \leq \sum_k |\lambda_k||\langle Ra_k, \phi\rangle| \leq C\left(\sum_k |\lambda_k|\right)\|\phi\|_{BMO(\mathbb{G})} \leq C\|f\|_{H^1(\mathbb{G})}\|\phi\|_{BMO(\mathbb{G})},$$

where C is independent of f. Note that $C_c(\mathbb{G})$ is dense in $VMO(\mathbb{G})$, and the dual spaces of $VMO(\mathbb{G})$ is $H^1(\mathbb{G})$ (see [1]), we get $Rf \in H^1(\mathbb{G})$. Therefore, Theorem 1.1 is proved.

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