TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 4, pp. 1591-1608, August 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

GENERALIZED HYERS-ULAM STABILITY OF UNCTIONAL EQUATIONS: A FIXED POINT APPROACH

Choonkil Park

Abstract. Using the fixed point method, we prove the generalized Hyers-Ulam stability of a cubic and quartic functional equation and of an additive and quartic functional equation in Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [41] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [41] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Gavruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [50] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach

Received October 10, 2008, accepted December 9, 2008.
Communicated by Sen-Yen Shaw.
2000 *Mathematics Subject Classification*: Primary 39B72, 47H10, 46C05, 46B03, 47Jxx.
Key words and phrases: Additive mapping, Cubic mapping, Quartic mapping, Functional equation, Fixed point, Generalized Hyers-Ulam stability.
This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2009-0070788).

space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [9] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 8, 13, 18, 19, 22-24, 26-49]).

In [36, 37], J. M. Rassias first introduced and investigated the cubic functional equation

$$f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y).$$

In [15], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

J. M. Rassias [34, 35] first introduced and investigated the quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.2}$$

and Lee et al. [16] investigated the quartic functional equation (1.2). It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d: X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Theorem 1.1. [4, 10]. *Let (X,d) be a complete generalized metric space and let $J: X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty, \qquad \forall n\ge n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$*;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

This paper is organized as follows: In Sections 2 and 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of the cubic and quartic functional equation

$$\begin{aligned} f(2x+y)+f(2x-y) = {} & 3f(x+y)+f(-x-y)+3f(x-y)+f(y-x) \\ & +18f(x)+6f(-x)-3f(y)-3f(-y) \end{aligned} \tag{1.3}$$

in Banach spaces.

In Sections 4 and 5, using the fixed point method, we prove the generalized Hyers-Ulam stability of the additive and quartic functional equation

$$\begin{aligned} f(2x+y)+f(2x-y) = {} & 2f(x+y)+2f(-x-y)+2f(x-y)+2f(y-x) \\ & +14f(x)+10f(-x)-3f(y)-3f(-y) \end{aligned} \tag{1.4}$$

in Banach spaces.

Throughout this paper, assume that X is a normed vector space with norm $||\cdot||$ and that Y is a Banach space with norm $\|\cdot\|$.

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 17, 20, 21, 25]).

2. Fixed Points and Generalized Hyers-ulam Stability of a Cubic and Quartic Functional Equation: an Even Case

One can easily show that an even mapping $f : X \to Y$ satisfies (1.3) if and only if the even mapping $f : X \to Y$ is a quartic mapping, i.e.,

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y),$$

and that an odd mapping $f : X \to Y$ satisfies (1.3) if and only if the odd mapping mapping $f : X \to Y$ is a cubic mapping, i.e.,

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x).$$

It is easy to show that the function $f(x) = ax^3 + bx^4$ satisfies the functional equation (1.3).

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) :=\ & f(2x+y) + f(2x-y) - 3f(x+y) - f(-x-y) - 3f(x-y) \\ & - f(y-x) - 18f(x) - 6f(-x) + 3f(y) + 3f(-y) \end{aligned}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in Banach spaces: an even case.

Theorem 2.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $L < 1$ such that $\varphi(x,0) \le \frac{1}{16}L\varphi(2x,0)$ for all $x \in X$ and*

$$\lim_{j\to\infty} 16^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0, \tag{2.1}$$

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \le \frac{L}{32-32L}(\varphi(x,0) + \varphi(-x,0)) \tag{2.3}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on S:

$$d(g,h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \le K(\varphi(x,0) + \varphi(-x,0)), \quad \forall x \in X\}.$$

It is easy to show that (S,d) is complete. (See the proof of Theorem 2.5 of [5].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of [4] that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

Letting $y = 0$ in (2.2), we get

$$\|2f(2x) - 24f(x) - 8f(-x)\| \leq \varphi(x, 0) \tag{2.4}$$

for all $x \in X$. Replacing x by $-x$ in (2.4), we get

$$\|2f(-2x) - 24f(-x) - 8f(x)\| \leq \varphi(-x, 0) \tag{2.5}$$

for all $x \in X$. Let $g(x) := f(x) + f(-x)$ for all $x \in X$. Then $g : X \to Y$ is an even mapping. It follows from (2.4) and (2.5) that

$$\|2g(2x) - 32g(x)\| \leq \varphi(x, 0) + \varphi(-x, 0)$$

for all $x \in X$. So

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq \frac{1}{2}\left(\varphi\left(\frac{x}{2}, 0\right) + \varphi\left(-\frac{x}{2}, 0\right)\right) \leq \frac{L}{32}(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$. Hence $d(g, Jg) \leq \frac{L}{32}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{2.6}$$

for all $x \in X$. Then $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $K \in (0, \infty)$ satisfying

$$\|g(x) - Q(x)\| \leq K(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$;

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 16^n g\left(\frac{x}{2^n}\right) = Q(x) \tag{2.7}$$

for all $x \in X$;

(3) $d(g, Q) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, Q) \leq \frac{L}{32 - 32L}.$$

This implies that the inequality (2.3) holds.

It follows from (2.1), (2.2) and (2.7) that

$$\|DQ(x,y)\| = \lim_{n\to\infty} 16^n \left\|Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\|$$
$$\leq \lim_{n\to\infty} 16^n \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right) = 0$$

for all $x, y \in X$. So $DQ(x,y) = 0$ for all $x, y \in X$. Since $Q : X \to Y$ is even, the mapping $Q : X \to Y$ is a quartic mapping.

Therefore, there exists a unique quartic mapping $Q : X \to Y$ satisfying (2.3), as desired. ■

Corollary 2.2. *Let $p > 4$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Df(x,y)\| \leq \theta(||x||^p + ||y||^p + ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}) \tag{2.8}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \leq \frac{\theta}{2^p - 16}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta(||x||^p + ||y||^p + ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}})$$

for all $x, y \in X$, which was introduced by J.M. Rassias et al. [49]. Then we can choose $L = 2^{4-p}$ and we get the desired result. ■

Remark 2.3. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (2.2) and

$$\lim_{j\to\infty} \frac{1}{16^j}\varphi(2^j x, 2^j y) = 0 \tag{2.9}$$

for all $x, y \in X$. By a similar method to the proof of Theorem 2.1, one can show that if there exists an $L < 1$ such that $\varphi(x, 0) \leq 16L\varphi(\frac{x}{2}, 0)$ for all $x \in X$, then there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \leq \frac{1}{32 - 32L}(\varphi(x,0) + \varphi(-x,0))$$

for all $x \in X$.

For the case $0 < p < 4$, one can obtain a similar result to Corollary 2.2: Let $0 < p < 4$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying (2.8). Then there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \leq \frac{\theta}{16 - 2^p}||x||^p$$

for all $x \in X$.

3. Fixed Points and Generalized Hyers-ulam Stability of a Cubic and Quartic Functional Equation: an Odd Case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in Banach spaces: an odd case.

Theorem 3.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (2.2) *such that there exists an $L < 1$ such that $\varphi(x, 0) \leq \frac{1}{8}L\varphi(2x, 0)$ for all $x \in X$, and*

$$\lim_{j\to\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \quad = \quad 0 \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying

$$\|f(x) - f(-x) - C(x)\| \leq \frac{L}{16 - 16L}(\varphi(x, 0) + \varphi(-x, 0)) \tag{3.2}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on S:

$$d(g, h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K(\varphi(x, 0) + \varphi(-x, 0)), \quad \forall x \in X\}.$$

It is easy to show that (S, d) is complete. (See the proof of Theorem 2.5 of [5].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of [4] that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

Letting $y = 0$ in (2.2), we get

$$\|2f(2x) - 24f(x) - 8f(-x)\| \le \varphi(x, 0) \tag{3.3}$$

for all $x \in X$. Replacing x by $-x$ in (3.3), we get

$$\|2f(-2x) - 24f(-x) - 8f(x)\| \le \varphi(-x, 0) \tag{3.4}$$

for all $x \in X$. Let $g(x) := f(x) - f(-x)$ for all $x \in X$. Then $g : X \to Y$ is an odd mapping. It follows from (3.3) and (3.4) that

$$\|2g(2x) - 16g(x)\| \le \varphi(x, 0) + \varphi(-x, 0)$$

for all $x \in X$. So

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \le \frac{1}{2}\left(\varphi\left(\frac{x}{2}, 0\right) + \varphi\left(-\frac{x}{2}, 0\right)\right) \le \frac{L}{16}(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$. Hence $d(g, Jg) \le \frac{L}{16}$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{3.5}$$

for all $x \in X$. Then $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (3.5) such that there exists a $K \in (0, \infty)$ satisfying

$$\|g(x) - C(x)\| \le K(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x) \tag{3.6}$$

for all $x \in X$;

(3) $d(g, C) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{L}{16 - 16L}.$$

This implies that the inequality (3.2) holds.

It follows from (3.1), (2.2) and (3.6) that

$$\begin{aligned} \|DC(x,y)\| &= \lim_{n\to\infty} 8^n \left\| Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} 8^n \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right) \right) = 0 \end{aligned}$$

for all $x, y \in X$. So $DC(x, y) = 0$ for all $x, y \in X$. Since $C : X \to Y$ is odd, the mapping $C : X \to Y$ is a cubic mapping.

Therefore, there exists a unique cubic mapping $C : X \to Y$ satisfying (3.2), as desired. ∎

Corollary 3.2. *Let $p > 3$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.8). *Then there exists a unique cubic mapping $C : X \to Y$ satisfying*

$$\|f(x) - f(-x) - C(x)\| \leq \frac{\theta}{2^p - 8} ||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta(||x||^p + ||y||^p + ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}})$$

for all $x, y \in X$, which was introduced by J.M. Rassias et al. [49]. Then we can choose $L = 2^{3-p}$ and we get the desired result. ∎

Combining Corollaries 2.2 and 3.2 yields the following.

Theorem 3.3. *Let $p > 4$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.8). *Then there exist a unique quartic mapping $Q : X \to Y$ and a unique cubic mapping $C : X \to Y$ satisfying*

$$\|2f(x) - Q(x) - C(x)\| \leq \left(\frac{1}{2^p - 16} + \frac{1}{2^p - 8} \right) \theta ||x||^p$$

for all $x \in X$.

Remark 3.4. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (2.2) and

$$\lim_{j\to\infty} \frac{1}{8^j}\varphi(2^j x, 2^j y) = 0$$

for all $x, y \in X$. By a similar method to the proof of Theorem 3.1, one can show that if there exists an $L < 1$ such that $\varphi(x, 0) \le 8L\varphi(\frac{x}{2}, 0)$ for all $x \in X$, then there exists a unique cubic mapping $C : X \to Y$ satisfying

$$\|f(x) - f(-x) - C(x)\| \le \frac{1}{16 - 16L}(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$.

For the case $0 < p < 3$, one can obtain a similar result to Corollary 3.2: Let $0 < p < 3$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying (2.8). Then there exists a unique cubic mapping $C : X \to Y$ satisfying

$$\|f(x) - f(-x) - C(x)\| \le \frac{\theta}{8 - 2^p}||x||^p$$

for all $x \in X$.

Combining Remarks 2.3 and 3.4 yields the following.

Theorem 3.5. *Let* $0 < p < 3$ *and* $\theta \ge 0$ *be real numbers, and let* $f : X \to Y$ *be a mapping satisfying* (2.8). *Then there exist a unique quartic mapping* $Q : X \to Y$ *and a unique cubic mapping* $C : X \to Y$ *satisfying*

$$\|2f(x) - Q(x) - C(x)\| \le \left(\frac{1}{16 - 2^p} + \frac{1}{8 - 2^p}\right)\theta||x||^p$$

for all $x \in X$.

4. Fixed Points and Generalized Hyers-ulam Stability of an Additive and Quartic Functional Equation: an Even Case

One can easily show that an even mapping $f : X \to Y$ satisfies (1.4) if and only if the even mapping $f : X \to Y$ is a quartic mapping, i.e.,

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y),$$

and that an odd mapping $f : X \to Y$ satisfies (1.4) if and only if the odd mapping mapping $f : X \to Y$ is an additive mapping, i.e.,

$$f(x + y) = f(x) + f(y).$$

It is easy to show that the function $f(x) = ax+bx^4$ satisfies the functional equation (1.4).

For a given mapping $f : X \to Y$, we define

$$Cf(x, y) := f(2x + y) + f(2x - y) - 2f(x + y) - 2f(-x - y) - 2f(x - y)$$
$$-2f(y - x) - 14f(x) - 10f(-x) + 3f(y) + 3f(-y)$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Cf(x, y) = 0$ in Banach spaces: an even case.

Theorem 4.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $L < 1$ such that $\varphi(x, 0) \le \frac{1}{16}L\varphi(2x, 0)$ for all $x \in X$, and*

(4.1) $$\lim_{j\to\infty} 16^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0,$$

(4.2) $$\|Cf(x, y)\| \le \varphi(x, y)$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \le \frac{L}{32 - 32L}(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on S:

$$d(g, h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \le K(\varphi(x, 0) + \varphi(-x, 0)), \quad \forall x \in X\}.$$

It is easy to show that (S, d) is complete. (See the proof of Theorem 2.5 of [5].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of [4] that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

Letting $y = 0$ in (4.2), we get

$$\|2f(2x) - 18f(x) - 14f(-x)\| \le \varphi(x, 0) \tag{4.3}$$

for all $x \in X$. Replacing x by $-x$ in (4.3), we get

$$\|2f(-2x) - 18f(-x) - 14f(x)\| \le \varphi(-x, 0) \tag{4.4}$$

for all $x \in X$. Let $g(x) := f(x) + f(-x)$ for all $x \in X$. Then $g : X \to Y$ is an even mapping. It follows from (4.3) and (4.4) that

$$\|2g(2x) - 32g(x)\| \le \varphi(x, 0) + \varphi(-x, 0)$$

for all $x \in X$. So

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \le \frac{1}{2}\left(\varphi\left(\frac{x}{2}, 0\right) + \varphi\left(-\frac{x}{2}, 0\right)\right) \le \frac{L}{32}(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$. Hence $d(g, Jg) \le \frac{L}{32}$.

The rest of the proof is similar to the proof of Theorem 2.1. ■

Corollary 4.2. *Let $p > 4$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Cf(x, y)\| \le \theta(||x||^p + ||y||^p + ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}) \tag{4.5}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \le \frac{\theta}{2^p - 16}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking

$$\varphi(x, y) := \theta(||x||^p + ||y||^p + ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}})$$

for all $x, y \in X$, which was introduced by J.M. Rassias et al. [49]. Then we can choose $L = 2^{4-p}$ and we get the desired result. ■

Remark 4.3. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (4.2) and

$$\lim_{j\to\infty} \frac{1}{16^j}\varphi(2^j x, 2^j y) = 0 \tag{4.6}$$

for all $x, y \in X$. By a similar method to the proof of Theorem 4.1, one can show that if there exists an $L < 1$ such that $\varphi(x, 0) \leq 16L\varphi(\frac{x}{2}, 0)$ for all $x \in X$, then there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \leq \frac{1}{32 - 32L}(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$.

For the case $0 < p < 4$, one can obtain a similar result to Corollary 4.2: Let $0 < p < 4$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying (4.5). Then there exists a unique quartic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \leq \frac{\theta}{16 - 2^p}||x||^p$$

for all $x \in X$.

5. Fixed Points and Generalized Hyers-ulam Stability of an Additive and Quartic Functional Equation: an Odd Case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Cf(x, y) = 0$ in Banach spaces: an odd case.

Theorem 5.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (4.2) *such that there exists an $L < 1$ such that $\varphi(x, 0) \leq \frac{1}{2}L\varphi(2x, 0)$ for all $x \in X$, and*

$$\lim_{j\to\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0 \tag{5.1}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - f(-x) - A(x)\| \leq \frac{L}{4 - 4L}(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on S:

$$d(g,h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \le K(\varphi(x,0) + \varphi(-x,0)), \quad \forall x \in X\}.$$

It is easy to show that (S,d) is complete. (See the proof of Theorem 2.5 of [5]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of [4] that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

Letting $y = 0$ in (4.2), we get

$$\|2f(2x) - 18f(x) - 14f(-x)\| \le \varphi(x,0) \tag{5.2}$$

for all $x \in X$. Replacing x by $-x$ in (5.2), we get

$$\|2f(-2x) - 18f(-x) - 14f(x)\| \le \varphi(-x,0) \tag{5.3}$$

for all $x \in X$. Let $g(x) := f(x) - f(-x)$ for all $x \in X$. Then $g : X \to Y$ is an odd mapping. It follows from (5.2) and (5.3) that

$$\|2g(2x) - 4g(x)\| \le \varphi(x,0) + \varphi(-x,0)$$

for all $x \in X$. So

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| \le \frac{1}{2}\left(\varphi\left(\frac{x}{2},0\right) + \varphi\left(-\frac{x}{2},0\right)\right) \le \frac{L}{4}(\varphi(x,0) + \varphi(-x,0))$$

for all $x \in X$. Hence $d(g, Jg) \le \frac{L}{4}$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. ∎

Corollary 5.2. *Let $p > 3$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying* (4.5)*. Then there exists a unique additive mapping $A : X \to Y$ satisfying*

$$\|f(x) - f(-x) - A(x)\| \le \frac{\theta}{2^p - 2}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.1 by taking

$$\varphi(x,y) := \theta(||x||^p + ||y||^p + ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}})$$

for all $x, y \in X$, which was introduced by J.M. Rassias et al. [49]. Then we can choose $L = 2^{1-p}$ and we get the desired result. ∎

Combining Corollaries 4.2 and 5.2 yields the following.

Theorem 5.3. *Let $p > 4$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying* (4.5)*. Then there exist a unique quartic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ satisfying*

$$\|2f(x) - Q(x) - A(x)\| \leq \left(\frac{1}{2^p - 16} + \frac{1}{2^p - 2}\right)\theta||x||^p$$

for all $x \in X$.

Remark 5.4. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ satisfying (4.2) and

$$\lim_{j\to\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) = 0$$

for all $x, y \in X$. By a similar method to the proof of Theorem 5.1, one can show that if there exists an $L < 1$ such that $\varphi(x, 0) \leq 2L\varphi(\frac{x}{2}, 0)$ for all $x \in X$, then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - f(-x) - A(x)\| \leq \frac{1}{4 - 4L}(\varphi(x,0) + \varphi(-x,0))$$

for all $x \in X$.

For the case $0 < p < 1$, one can obtain a similar result to Corollary 5.2: Let $0 < p < 1$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying (4.5). Then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - f(-x) - A(x)\| \leq \frac{\theta}{2 - 2^p}||x||^p$$

for all $x \in X$.

Combining Remarks 4.3 and 5.4 yields the following.

Theorem 5.5. *Let $0 < p < 1$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying* (4.5). *Then there exist a unique quartic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ satisfying*

$$\|2f(x) - Q(x) - A(x)\| \leq \left(\frac{1}{16 - 2^p} + \frac{1}{2 - 2^p} \right) \theta ||x||^p$$

for all $x \in X$.

References

1. M. Amyari, C. Park and M. S. Moslehian, Nearly ternary derivations, *Taiwanese J. Math.*, **11** (2007), 1417-1424.
2. T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2** (1950), 64-66.
3. D. G. Bourgin, Classes of transformations and bordering transformations, *Bull. Amer. Math. Soc.*, **57** (1951), 223-237.
4. L. Cadariu and V. Radu, Fixed points and the stability of Jensen's functional equation, *J. Inequal. Pure Appl. Math.*, **4**, no. 1, Art. ID 4 (2003).
5. L. Cadariu and V. Radu, On the stability of the Cauchy functional equation: a fixed point approach, *Grazer Math. Ber.*, **346** (2004), 43-52.
6. L. Cadariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory and Applications, **2008**, Art. ID 749392 (2008).
7. P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.*, **27** (1984), 76-86.
8. C. Y. Chou and J.-H. Tzeng, On approximate isomorphisms between Banach $*$-algebras or C^*-algebras, *Taiwanese J. Math.*, **10** (2006), 219-231.
9. S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg*, **62** (1992), 59-64.
10. J. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.*, **74** (1968), 305-309.
11. P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431-436.
12. D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.*, **27** (1941), 222-224.
13. D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
14. G. Isac and Th. M. Rassias, Stability of ψ-additive mappings: Appications to nonlinear analysis, *Internat. J. Math. Math. Sci.*, **19** (1996), 219-228.
15. K. Jun and H. Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, *J. Math. Anal. Appl.*, **274** (2002), 867-878.

16. S. Lee, S. Im and I. Hwang, Quartic functional equations, *J. Math. Anal. Appl.*, **307** (2005), 387-394.

17. M. Mirzavaziri and M. S. Moslehian, A fixed point approach to stability of a quadratic equation, *Bull. Braz. Math. Soc.*, **37** (2006), 361-376.

18. A. Najati and C. Park, On the stability of an n-dimensional functional equation originating from quadratic forms, *Taiwanese J. Math.*, **12** (2008), 1609-1624.

19. C. Park, Homomorphisms between Poisson JC^*-algebras, *Bull. Braz. Math. Soc.*, **36** (2005), 79-97.

20. C. Park, Fixed points and *Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory and Applications, **2007**, Art. ID 50175 (2007).

21. C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory and Applications, **2008**, Art. ID 493751 (2008).

22. C. Park, Y. Cho and M. Han, *Functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl., **2007**, Art. ID 41820 (2007).

23. C. Park and J. Cui, *Generalized stability of C^*-ternary quadratic mappings*, Abstract Appl. Anal., **2007**, Art. ID 23282 (2007).

24. C. Park and A. Najati, Homomorphisms and derivations in C^*-algebras, *Abstract Appl. Anal.*, **2007**, Art. ID 80630 (2007).

25. V. Radu, The fixed point alternative and the stability of functional equations, *Fixed Point Theory*, **4** (2003), 91-96.

26. J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.*, **46** (1982), 126-130.

27. J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *Bull. Sci. Math.*, **108** (1984), 445-446.

28. J. M. Rassias, On a new approximation of approximately linear mappings by linear mappings, *Discuss. Math.*, **7** (1985), 193-196.

29. J. M. Rassias, Solution of a problem of Ulam, *J. Approx. Theory*, **57** (1989), 268-273.

30. J. M. Rassias, Solution of a stability problem of Ulam, *Discuss. Math.*, **12** (1992), 95-103.

31. J. M. Rassias, *Solution of a stability problem of Ulam*, Functional Analysis, Approximation Theory and Numerical Analysis, 241-249, World Sci. Publ., River Edge, NJ, 1994.

32. J. M. Rassias, *On the stability of a multi-dimensional problem of Ulam*, Geometry, Analysis and Mechanics, 365-376, World Sci. Publ., River Edge, NJ, 1994.

33. J. M. Rassias, Complete solution of the multi-dimensional problem of Ulam, *Discuss. Math.*, **14** (1994), 101-107.

34. J. M. Rassias, Solution of the Ulam stability problem for quartic mappings, *Glas. Mat. Ser. III*, **34(54)** (1999), 243-252.

35. J. M. Rassias, Solution of the Ulam stability problem for quartic mappings, *J. Indian Math. Soc.*, **67** (2000), 169-178.
36. J. M. Rassias, Solution of the Ulam problem for cubic mappings, *An. Univ. Timişoara Ser. Mat.-Inform.*, **38** (2000), 121-132.
37. J. M. Rassias, Solution of the Ulam stability problem for cubic mappings, *Glas. Mat. Ser. III*, **36(56)** (2001), 63-72.
38. J. M. Rassias, Alternative contraction principle and Ulam stability problem, *Math. Sci. Res. J.*, **9** (2005), 190-199.
39. J. M. Rassias, Alternative contraction principle and alternative Jensen and Jensen type mappings, *Internat. J. Appl. Math. Stat.*, **4** (2006), 1-10.
40. J. M. Rassias and M. L. Rassias, Asymptotic behavior of alternative Jensen and Jensen type functional equations, *Bull. Sci. Math.*, **129** (2005), 545-558.
41. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.
42. Th. M. Rassias, Problem 16; 2, Report of the 27th International Symp. on Functional Equations, *Aequationes Math.*, **39** (1990), 292-293; 309.
43. Th. M. Rassias, On the stability of the quadratic functional equation and its applications, *Studia Univ. Babes-Bolyai*, **XLIII** (1998), 89-124.
44. Th. M. Rassias, The problem of S. M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.*, **246** (2000), 352-378.
45. Th. M. Rassias, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.*, **251** (2000), 264-284.
46. Th. M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.*, **62** (2000), 23-130.
47. Th. M. Rassias and P. Šemrl, On the Hyers-Ulam stability of linear mappings, *J. Math. Anal. Appl.*, **173** (1993), 325-338.
48. Th. M. Rassias and K. Shibata, Variational problem of some quadratic functionals in complex analysis, *J. Math. Anal. Appl.*, **228** (1998), 234-253.
49. K. Ravi, M. Arunkumar and J. M. Rassias, Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *Internat. J. Math. Stat.*, **3** (2008), 36-46.
50. F. Skof, Proprietà locali e approssimazione di operatori, *Rend. Sem. Mat. Fis. Milano*, **53** (1983), 113-129.
51. S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.

Choonkil Park
Department of Mathematics,
Research Institute for Natural Sciences,
Hanyang University,
Seoul 133-791,
Republic of Korea
E-mail: baak@hanyang.ac.kr