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EXISTENCE OF SOLUTIONS TO THE FIRST-ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. This paper is concerned with periodic boundary value problems for a kind of first order impulsive differential equations. Some new results related to the existence of solutions are obtained by the ideas involve differential inequalities and fixed point theorems.

1. Introduction

Impulsive differential equations have been becoming an important field because many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. For example, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems. Readers can see [1-3] and the references therein for details.

Nowadays, impulsive equations coupled with boundary value conditions have gained more attention for their widely practical background, such as science, engineering, medical, and technology. The problems concentrate on existence for solution, extreme solution, uniqueness, multiplicity of solution, periodic solution, etc. There are many ways to solve this kind of problems. For instance, upper- and lower- solutions coupled with monotone technique are efficient method to extreme solution, [4-10]; Krasnoselskii fixed point theorem is often used to solve multiplicity of solution, [11-14]; Coincidence degree theory is applied to obtain the existence for periodic solutions, [15-17].

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As we know, the first-order partial differential equations is very important in Physics, Chemistry, and other field. For example, chromatogram is a modern physical and chemical analysis. It can be describes as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{a_i u_i}{1+u}) = 0,$$

where $u = (u_1, u_2, \cdots, u_n)^T$, u_i denote the concentration of each component, a_i are the adsorption equilibrium constants of each component, and they satisfy $0 < a_1 < a_2 < \cdots < a_n$. In physics, there have the transport equation,

$$u_t + cu_x = 0,$$

and one dimensional burgers equation,

$$u_t + \frac{1}{2}u^2 = 0.$$

The partial differential equations can be easily changed to ordinary differential equations if the equations are linear ones, for example, by Fourier transform and Separation of variables. And it is well known, many evolution processes do exhibit impulsive effects. Motivated by the aforementioned, in this paper, we consider the following systems

$$(1) \qquad \begin{cases} x'(t) = f(t, x(t)), & t \in J, t \neq t_k; \\ \triangle x(t_k) = I_k(x(t_k)), & k = 1, 2, \cdots, m, \end{cases}$$

with the boundary value condition

$$ax(0) + x(T) = b.$$

Here $f \in C(J \times R, R), J = [0, T], 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(R, R)$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-), k = 1, 2, \cdots, m, a \in R, b \in R.$

The tool we used is Schaefer fixed point theorem and the Nonlinear Alternative, see [16,17]. For convenience, we introduce it first.

Theorem 1.1. *Let X be a normed space with $H : X \to X$ a compact mapping. If the set*

$$S = \{u \in X : u = \lambda Hu, \quad for \quad some \quad \lambda \in [0, 1)\}$$

is bounded, then H has at least one fixed point.

Theorem 1.2. *Let $T : \bar{B}_p \to J$ be a compact map and let $\lambda \in [0, 1]$. If*

$$x \neq \lambda Tx, \quad for \quad all \quad x \in \partial B_p \quad and \quad \lambda \in (0, 1),$$

then there exists at least one $x \in B_p$ such that $x = Tx$.

2. Preparations

The systems (1) is equivalent to the following problems with $M \in R$,

$$\left\{ \begin{aligned} x'(t) + Mx(t) &= f(t, x(t)) + Mx(t), \quad t \in J, t \neq t_k; \\ \triangle x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \cdots, m, \end{aligned} \right. \tag{2}$$

with

$$ax(0) + x(T) = b.$$

Following the equivalent relative, we claim that the solution of system (1) is also the solution of system (2). Hence, we invert our problem to system (2).

Lemma 2.1. *Assume* $ae^{MT} + 1 \neq 0$, *then* $x \in E$ *is a solution of (2) if and only if* $x \in E_0$ *is a solution of the impulsive integral equation*

$$\begin{aligned} x(t) &= \int_0^T g_1(t,s)[f(s,x(s)) + Mx(s)]ds \\ &\quad + \sum_{k=0}^{m} g_1(t,t_k) I_k(x(t_k)) + g_2(t), \ t \in J \end{aligned} \tag{3}$$

where

$$g_1(t,s) = \frac{1}{ae^{MT}+1} \left\{ \begin{array}{ll} ae^{M(T+s-t)}, & 0 \leq s \leq t \leq T; \\ -e^{M(s-t)}, & 0 \leq t < s \leq T, \end{array} \right. \quad \textit{and} \quad g_2(t) = \frac{be^{-Mt}}{a + e^{-MT}}.$$

Proof. Suppose that $x(t)$ is a solution of (2). Setting $u(t) = e^{Mt}x(t)$, then

$$u'(t) = e^{Mt}[f(t, x(t)) + Mx(t)]. \tag{4}$$

Integrating (4) from 0 to t_1, it follows

$$u(t_1) - u(0) = \int_0^{t_1} e^{Ms}[f(s, x(s)) + Mx(s)]ds.$$

Again integrating (4) from t_1 to t, where $t \in (t_1, t_2]$, then

$$\begin{aligned} u(t) &= u(t_1^+) + \int_{t_1}^{t} e^{Ms}[f(s,x(s)) + Mx(s)]ds \\ &= u(t_1) + \int_{t_1}^{t} e^{Ms}[f(s,x(s)) + Mx(s)]ds + e^{Mt_1} I_1(x(t_1)) \\ &= u(0) + \int_0^{t} e^{Ms}[f(s,x(s)) + Mx(s)]ds + e^{Mt_1} I_1(x(t_1)). \end{aligned}$$

Repeating the above procession, for $t \in J$, we have

$$u(t) = u(0) + \int_0^t e^{Ms}[f(s, x(s)) + Mx(s)]ds + \sum_{0<t_k<t} e^{Mt_k} I_k(x(t_k)).$$

Note that $u(0) = x(0)$, thus

$$e^{Mt}x(t) = x(0) + \int_0^t e^{Ms}[f(s, x(s)) + Mx(s)]ds + \sum_{0<t_k<t} e^{Mt_k} I_k x(t_k)).$$

In view of that $x(T) = b - ax(0)$, we have

$$\begin{aligned} & e^{MT}(b - ax(0)) = e^{MT}x(T) \\ = & \, x(0) + \int_0^T e^{Ms}[f(s, x(s)) + Mx(s)]ds + \sum_{0<t_k<T} e^{Mt_k} I_k(x(t_k)). \end{aligned}$$

then

$$x(0) = \frac{-be^{MT} + \int_0^T e^{Ms}[f(s, x(s)) + Mx(s)]ds + \sum\limits_{0<t_k<T} e^{Mt_k} I_k(x(t_k))}{-(1 + ae^{MT})}.$$

Then

$$\begin{aligned} & x(t) \\ = \; & e^{-Mt}\left\{ \frac{-be^{MT} + \int_0^T e^{Ms}[f(s, x(s)) + Mx(s)]ds + \sum\limits_{0<t_k<T} e^{Mt_k} I_k(x(t_k))}{-(1 + ae^{MT})} \right. \\ & \left. + \int_0^t e^{Ms}[f(s, x(s)) + Mx(s) + \sum_{0<t_k<t} e^{Mt_k} I_k(x(t_k)) \right\} \\ = \; & \frac{be^{-Mt}}{a + e^{-MT}} + \frac{\sum\limits_{0\le t_k<T} e^{M(t_k-t)} I_k(x(t_k)) - (1+ae^{MT}) \sum\limits_{0<t_k<t} e^{M(t_k-t)} I_k(x(t_k))}{-(1+ae^{MT})} \\ & + \frac{\int_0^T e^{M(s-t)}[f(s, x(s))+Mx(s)]ds - (1+ae^{MT}) \int_0^t e^{M(s-t)}[f(s, x(s))+Mx(s)]ds}{-(1+ae^{MT})} \\ = \; & \frac{be^{-Mt}}{a + e^{-MT}} + \frac{a \sum\limits_{0\le t_k<t} e^{M(T+t_k-t)} I_k(x(t_k)) - \sum\limits_{t\le t_k<T} e^{M(t_k-t)} I_k(x(t_k))}{1 + ae^{MT}} \\ & + \frac{a \int_0^t e^{M(T+s-t)}[f(s, x(s)) + Mx(s)]ds - \int_t^T e^{M(s-t)}[f(s, x(s)) + Mx(s)]ds}{1 + ae^{MT}} \\ = \; & \int_0^T g_1(t, s)[f(s, x(s)) + Mx(s)]ds + \sum_{k=1}^m g_1(t, t_k) I_k(x(t_k)) + g_2(t), \quad t \in J. \end{aligned}$$

i.e., $x(t)$ is also the solution of (3).

On the other hand, assume $x(t)$ is a solution of (3). Differentail on (3), we have

$$x'(t) + Mx(t) = f(t, x(t)) + Mx(t), \quad t \in J_k.$$

Noting that

$$g_1(0,s) = \frac{-e^{Ms}}{ae^{MT}+1}, \quad g_2(0) = \frac{be^{MT}}{ae^{MT}+1},$$
$$g_1(T,s) = \frac{ae^{Ms}}{ae^{MT}+1}, \quad g_2(T) = \frac{b}{ae^{MT}+1},$$

then by direct calculus, we can verify that $x(t)$ is a solution of (2). This completes the proof.

Consider (2) with $M = 0$, the following corollary to Lemma 2.1 is obtained.

Corollary 2.2. *$x \in E$ is a solution of (1) if and only if $y \in E_0$ is a solution of the impulsive integral equation*

$$x(t) = \int_0^T g(t,s)f(s,x(s))ds + \sum_{k=0}^{m} g(t,t_k)I_k(x(t_k)) + \frac{b}{1+a}, \ t \in J$$

where

$$g(t,s) = \frac{1}{a+1}\begin{cases} a, & 0 \le s \le t \le T; \\ -1, & 0 \le t < s \le T. \end{cases}$$

Denote a operator $A, A^* : PC(J; R^n \to PC(J; R^n)$ as

$$Ax(t) = \int_0^T g_1(t,s)[f(s,x(s)) - Mx(s)]ds + \sum_{k=0}^{m} g_1(t,t_k)I_k(x(t_k)) + g_2(t),$$

and

$$A^*x(t) = \int_0^T g(t,s)f(s,x(s))ds + \sum_{k=0}^{m} g(t,t_k)I_k(x(t_k)) + \frac{b}{1+a}, \ t \in J$$

then we can immediately get the following results.

Lemma 2.3. *Suppose g_1, g, g_2 are defined as above two proposition. Then*

(1) *If A has a fixed point x^*, it is also a solution to (2). Moreover, it is also a solution of (1).*

(2) *If A^* has a fixed point x^{**}, it is also a solution to the systems (1).*

With the continuity of f and $I_k, k = 1, 2, \cdots, m$ on J, we have the following Lemma, or readers can see Lemma 3.2 in [18].

Lemma 2.4. *If operators A, A^* are defined as above, then they are both compact maps.*

3. Main Results

Now, we are in the position to establish some new existence results for systems (1).

Denote

$$\bar{a} = \max\{|\tfrac{1}{a+1}|, |\tfrac{a}{a+1}|, a \neq -1\}; a^* = \max\{|\tfrac{ae^{MT}}{ae^{MT}+1}|, |\tfrac{ae^{MT}}{ae^{MT}+1}|, ae^{MT} \neq -1\}.$$

Theorem 3.1. *Assume that $|a| \leq 1$. If there exist non-negative constants α, K, β, L such that*

$$\|f(t,x)\| \leq 2\alpha\langle x, f(t,x)\rangle + K, \quad (t,x) \in J_k \times R^n, \tag{4}$$

$$\|I_k(x)\| \leq \beta\|x\| + L, \quad for \quad all \quad x \in R^n, k = 1, 2, \cdots, m, \tag{5}$$

$$1 - \bar{a}m\beta > 0, \tag{6}$$

then the systems (1) has at least one solution.

Proof. In order to use Theorem 1.1, we need to the set S is bounded. That is to show all potential solutions to

$$x = \lambda A^* x, \quad \lambda \in [0,1] \tag{7}$$

are bounded a priori, with the bound being independent of λ.

Let $x(t)$ be a solution to (7), obviously, $x(t)$ is also a solution to

$$\begin{aligned} x' &= \lambda f(t,x), \quad t \in J_k, \\ \Delta x(t_k) &= I_k(x(t_k)), t = t_k, k = 1, 2, \cdots, m, \\ ax(0) + x(T) &= b. \end{aligned}$$

Then for each $t \in [0,T]$,

$$\begin{aligned} \|x(t)\| &= \lambda\|A^*x\| \\ &= \|\int_0^T \lambda g(t,s) f(s,x(s))ds + \sum_{k=0}^{m} \lambda g(t,t_k) I_k(x(t_k)) + \frac{\lambda b}{1+a}\| \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^T g(t,s)\|\lambda f(s,x(s))\|ds + \sum_{k=0}^{m} \lambda g(t,t_k)\|I_k(x(t_k))\| + \frac{\lambda b}{1+a} \\
&\le \bar{a}\int_0^T [2\alpha\langle x, \lambda f(t,x)\rangle + K]ds + \bar{a}\sum_{k=0}^{m} \lambda[\beta\|x(t_k)\| + L] + |\frac{\lambda b}{1+a}| \\
&= \int_0^T \bar{a}[\alpha x, x' + K]ds + \sum_{k=0}^{m} \bar{a}[\beta\|x(t_k)\| + L] + |\frac{\lambda b}{1+a}| \\
&= \bar{a}[\alpha(\|x(T)\|^2 - \|x(0)\|^2) + KT + \beta\sum_{k=0}^{m} \|x(t_k)\| + L] + |\frac{b}{1+a}| \\
&\le \bar{a}\{[\alpha(|b| + (|a|-1)\|x(0)\|^2] + KT + \beta\sum_{k=0}^{m} \|x(t_k)\| + L\} + |\frac{b}{1+a}|.
\end{aligned}
$$

By taking

$$\sup_{t\in J}\|x(t)\| \le \frac{\bar{a}(\alpha|b| + KT + L) + |\frac{b}{1+a}|}{1 - \bar{a}m\beta},$$

we see that all the conditions in Theorem 1.1 are hold, thus system (1) has at least one solution.

Similarly, we get Theorem 3.2.

Theorem 3.2. *Assume that* $|a| \ge 1$. *If there exist non-negative constants* α, K, β, L *such that*

$$\|f(t,x)\| \le -2\alpha\langle x, f(t,x)\rangle + K, \quad (t,x) \in J_k \times R^n,$$

and (5),(6)hold, then the systems (1) has at least one solution.

Corollary 3.3. Let $b = 0, I_k = 0, k = 1, 2, \cdots, m$, then Theorem 3.1. reduces to Theorem 2.2. in [19].

Corollary 3.4. *Let* $b = 0, I_k = 0, k = 1, 2, \cdots, m$, *then Theorem 3.2. reduces to Theorem 2.3. in [19].*

Let $a = 1, b = 0$, then system (1) reduces to anti-periodic boundary value problems(ABVP),

$$
(9) \qquad \begin{cases} x'(t) &= f(t,x(t)), \quad t \in J, t \neq t_k; \\ \triangle x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \cdots, m, \end{cases}
$$

with the boundary value condition

$$x(0) = -x(T).$$

And for problem (9), we have the following corollaries.

Theorem 3.5. *Assume that* $|a| \leq 1$. *If there exist non-negative constants* α, K, β, L *such that*

$$\|f(t,x)-x\| \leq 2\alpha\langle x, f(t,x)\rangle + K, \quad (t,x) \in J_k \times R^n, \tag{10}$$

$$\|I_k(x)\| \leq \beta\|x\| + L, \quad for \quad all \quad x \in R^n, k = 1,2,\cdots,m, \tag{11}$$

and

$$1 - a^* m\beta > 0, \tag{12}$$

then the systems (1) has at least one solution.

Proof. Choose $M = -1$ in Lemma 2.1, then the solution of (2) with $M = -1$ is equivalent to

$$x(t) = \int_0^T g_{11}(t,s)[f(s,x(s))-x(s)]ds + \sum_{k=0}^{m} g_{11}(t,t_k)I_k(x(t_k)) + g_{12}(t), \ t \in J,$$

where

$$g_{11}(t,s) = \frac{1}{ae^{-T}+1}\begin{cases} ae^{-(T+s-t)}, & 0 \leq s \leq t \leq T; \\ -e^{-(s-t)}, & 0 \leq t < s \leq T, \end{cases} \quad \text{and} \quad g_{12}(t) = \frac{be^t}{a+e^T}.$$

Let

$$B_p = \{x \in C([0,T];R^n) | \max_{t\in[0,T]} \|x(t)\| < P\},$$

$$P = \bar{g}_1[2\alpha|b| + KT + \beta\sum_{k=0}^{m}\|x(t_k)\| + L] + \bar{g}_2 + 1.$$

We show that $A : \bar{B_P} \to C([0,T]); R^n)$ satisfies

$$x \neq \lambda Ax, \quad for \quad all \quad x \in \partial B_P \quad and \quad all \quad \lambda \in (0,1).$$

Note that $x = \lambda Ax$ is equivalent to the family of

$$\begin{aligned} x' - x &= \lambda[f(t,x)-x], \quad t \in J_k, \\ \Delta x(t_k) &= \lambda I_k(x(t_k)), t = t_k, k = 1,2,\cdots,m, \\ ax(0) + x(T) &= b. \end{aligned}$$

All solutions to $x = \lambda Ax$ must satisfy

$$\begin{aligned}
&\|x(t)\| = \lambda\|Ax\| \\
&= \| \int_0^T \lambda g_{11}(t,s)[f(s,x(s)) - x(s)]ds + \sum_{k=0}^{m} \lambda g_{11}(t,t_k) I_k(x(t_k)) + \lambda g_{12}\| \\
&\quad \int_0^T g_{11}(t,s)\|\lambda[f(s,x(s)) - x(s)]\|ds + \sum_{k=0}^{m} \lambda g_{11}(t,t_k)\|I_k(x(t_k))\| + \lambda\|g_{12}\| \\
&\le \bar{g}_1 \int_0^T [2\alpha\langle x, \lambda f(t,x)\rangle + K]ds + \bar{a}\sum_{k=0}^{m} \lambda[\beta\|x(t_k)\| + L] + \bar{g}_2 \\
&= \int_0^T \bar{g}_1[2\alpha\langle x, x'\rangle + K]ds + \sum_{k=0}^{m} \bar{a}[\beta\|x(t_k)\| + L] + \bar{g}_2 \\
&= \bar{g}_1[2\alpha(\|x(T)\|^2 - \|x(0)\|^2) + KT + \beta\sum_{k=0}^{m}\|x(t_k)\| + L] + \bar{g}_2 \\
&\le \bar{g}_1[2\alpha(\|x(T)\|^2 - a\|x(0)\|^2) + KT + \beta\sum_{k=0}^{m}\|x(t_k)\| + L] + \bar{g}_2 \\
&\le \bar{g}_1[2\alpha|b| + KT + \beta\sum_{k=0}^{m}\|x(t_k)\| + L] + \bar{g}_2 \\
&< P.
\end{aligned}$$

Combine Lemma 2.4 with Theorem 1.2, system (1) has at least one solution.

Similarly, we get Theorem 3.7.

Theorem 3.4. *Assume that* $|a| \ge 1$. *If there exist non-negative constants* α, K, β, L *such that*

$$\|f(t,x) + x\| \le -2\alpha\langle x, f(t,x)\rangle + K, \quad (t,x) \in J_k \times R^n,$$

and $(11), (12)$ *hold, then the systems* (1) *has at least one solution.*

Let $a = 1, b = 0$, then (1) becomes the so-called periodic boundary value problem(PBVP),

$$(14) \qquad \begin{cases} x'(t) = f(t,x(t)), & t \in J, t \ne t_k; \\ \triangle x(t_k) = I_k(x(t_k)), & k = 1,2,\cdots,m, \end{cases}$$

with the boundary value condition

$$x(0) = x(T).$$

which has been studied in [20]. Since (14) is a special case of (1), our result are more generalized, the main results in [20] are our following corollary.

Corollary 3.8. *If there exist non-negative constants* α, K, β, L, *such that inequalities* $(10), (11)$ *or* $(14), (15)$ *hold, and*

$$1 - m\beta > 0,$$

then (14) *has at least one solution.*

4. Examples

In this section, we shall give some examples to highlight the above results.

Example 4.1. Consider the following ABVPs,

$$(15) \qquad \begin{cases} x'(t) = t([x(t)]^3 + 1), \quad t \in [0,1], t \neq t_1; \\ \triangle x(t_1) = \dfrac{1}{3}(x(t_1)); \end{cases}$$

with the boundary value condition

$$x(0) = -x(T).$$

Choose $\alpha = \frac{1}{2}, K = 3, \beta = \frac{1}{3}$, and $L = 0$, then one see that all of the conditions of Corollary 3.5. hold, so (15) has at least one solution.

Example 4.2. ([10]). Consider the following PBVPs,

$$(16) \qquad \begin{cases} x'(t) = x^3(t) + x(t) + 1, \quad t \in [0,10], t \neq t_1; \\ \triangle x(t_1) = \dfrac{1}{2}(x(t_1)); \end{cases}$$

with the boundary value condition

$$x(0) = x(10).$$

By Corollary 3.8. hold, we can also verify that (16) has at least one solution.

Example 4.3. Consider the following BVPs,

$$(17) \qquad \begin{cases} x'(t) & = \dfrac{1}{2}x(t) + 1, \quad t \in [0, \dfrac{1}{3}], t \neq \dfrac{1}{5}; \\ \triangle x(\frac{1}{5}) & = \dfrac{1}{2}; \end{cases}$$

with the boundary value condition

$$\frac{1}{2}x(0) + x(\frac{1}{3}) = 3.$$

Here $f(t, x) = \frac{1}{2}x(t) + 1, I_k = \frac{1}{2}, a = \frac{1}{2}, m = 1$, so that $\bar{1} = \frac{2}{3}$.

Choose $\alpha = \frac{1}{2}, K = 2$, we have

$$\begin{aligned} 2\alpha\langle x, f(t,x)\rangle + K &= x^2 + 2x + 2 = (x + \frac{3}{4})^2 + \frac{7}{16} + \frac{1}{2}x + 1 \\ &= \frac{1}{2}x + 1 = f(t,x). \end{aligned}$$

Then let $\beta = \frac{1}{3}, L = 0$, we have

$$\beta\|x\| + L = \frac{1}{3}\|x\| + 1 \geq \frac{1}{2} = I_k,$$

and

$$1 - \bar{a}m\beta = 1 - \frac{21}{33} = \frac{7}{9} > 0,$$

thus all the conditions in Theorem 3.1. hold, so (17) has at least one solution.

Example 4.4. Consider the following BVPs,

$$(18) \qquad \begin{cases} x'(t) = -x^3(t) + \frac{1}{2}x(t), \quad t \in [0,1], t \neq \frac{1}{5}; \\ \triangle x\Big(\frac{1}{5}\Big) = \frac{1}{2}; \end{cases}$$

with the boundary value condition

$$2x(0) + x(1) = 5.$$

Here $f(t,x) = -x^3(t) + \frac{1}{2}x(t), I_k = \frac{1}{2}, a = 2, m = 1$, so that $\bar{1} = \frac{2}{3}$, and $|f(t,x) + x| = |-x^3(t) + \frac{3}{2}| \leq |x^3(t)| + \frac{3}{2}|x(t)|$.

Choose $\alpha = 1, K = 3$, we have

$$\begin{aligned} &-2\alpha\langle x, f(t,x)\rangle + 3 - \left(|x^3(t)| + \frac{3}{2}|x(t)|\right) \\ &= 2\alpha x^4 - \alpha x + 3 - \left(|x^3(t)| + \frac{3}{2}|x(t)|\right) \\ &= \left(x^2 - \frac{1}{2}x(t) - \frac{3}{2}\right)^2 + x^4 + \frac{7}{4}x^2 + \frac{1}{4} \geq 0, \end{aligned}$$

so

$$-2\alpha\langle x, f(t,x)\rangle + 3 \geq |x^3(t)| + \frac{3}{2}|x(t)| \geq |f(t,x) + x|.$$

Still let $\beta = \frac{1}{3}, L = 0$, we have

$$\beta\|x\| + L = \frac{1}{3}\|x\| + 1 \geq \frac{1}{2} = I_k,$$

and

$$1 - \bar{a}m\beta = 1 - \frac{21}{33} = \frac{7}{9} > 0,$$

thus all the conditions in Theorem 3.7. hold, so (18) has at least one solution.

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