TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 4, pp. 1543-1570, August 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON THE EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR SOME NONLINEAR WAVE EQUATIONS OF KIRCHHOFF TYPE

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Abstract. In this work, we consider the following nonlinear problem

$$u_{tt} - M(\|\nabla u(t)\|_2^2)\Delta u - \frac{\partial}{\partial t}\Delta u = f(u),$$
$$u = 0 \ \text{ in } \Gamma_0 \times (0, T),$$
$$M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial}{\partial t}(\frac{\partial u}{\partial \nu}) = -u_t \ \text{ in } \Gamma_1 \times (0, T),$$
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),$$

in a bounded domain Ω. The existence, asymptotic behavior and nonexistence of solutions are discussed under some conditions.

1. Introduction

In this paper we shall consider the third order parabolic initial boundary value problem of the type

$$u_{tt} - M(\|\nabla u(t)\|_2^2)\Delta u - \frac{\partial}{\partial t}\Delta u = f(u), \tag{1.1}$$

$$u = 0 \ \text{ in } \Gamma_0 \times (0, T), \tag{1.2}$$

$$M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial}{\partial t}(\frac{\partial u}{\partial \nu}) = -u_t \ \text{ in } \Gamma_1 \times (0, T), \tag{1.3}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.4}$$

for any $x \in \Omega$, where $\Omega \subset R^N$ is a bounded domain in R^N, $N \geq 1$, with a C^2 boundary Γ, and (Γ_0, Γ_1) be a partition of Γ, both parts having positive measure

Received March 24, 2008, accepted November 11, 2008.
Communicated by Yingfei Yi.
2000 *Mathematics Subject Classification*: 35L70, 35L15, 65M60.
Key words and phrases: Blow-up, Exponential decay, Viscosity, Galerkin's method.
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with $\Gamma_0 \cap \Gamma_1 = \phi$, and ν be the unit normal vector pointing toward the exterior of Ω. Here $M \in C^1([0,T), R)$ is a function such that $M(s) \geq m_0 > 0$, $\forall s \geq 0$ and f is a nonlinear function like $f(u) = |u|^{p-2}\, u$, $p > 2$.

The one dimension equation (1.1) with $f = 0$ was first introduced by Kirchoff [1] in 1883 in order to describe the nonlinear vibrations of an elastic string. More precisely, the original equation is

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L (\frac{\partial u}{\partial x})^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral deflection, E the Young modulus, ρ the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistence modulus, and f the external force. There is a large literature concerned with the existence of the solutions of the initial boundary value problem with null Dirichlet boundary conditions for (1.1). Some works in a bounded domain can be found in Bernstein [2], Pohozaev [23], Lions [16], Arosio and Spagnolo [1], Nishihara [20], Ebihara Medeiros and Miranda [6] and Yamada [27]. For unbounded domain, it is investigated in Matos [17] and Bisognin [3]. On the other hand, the nonexistence of global solutions of nonlinear equations of hyperbolic type of second order was studied by many authors and the references cited therein [8, 21, 22, 25, 26].

It is interesting to observe that when $M \equiv 1$, problems without viscosity, that is $\Delta u_t = 0$, and with a boundary damping on all or part of the boundary of the spatial domains were investigated by many authors, see G. Chen [5], Quinn-Rusell [24], Lagnese [13] and Komornik and Zuazua [11]. For the nonlinear boundary damping case, we quote the papers of Zuazua [28], Lasiecka and Tataru [14], Lasiecka and Ong [15] and Komornik [12]. Recently, some articles have appeared in which M is not a constant function and with a dissipative term on all or part of the boundary of the spatial domains. Miranda and San Gil Jutucaln [36] discussed the global existence and boundary stabilization of the solution for (1.1) with $f(u) = 0$ under linear boundary damping. Cavalcanti, Soriano and Prates Filho [4] studied the global existence and the exponential decay of solutions for (1.1) with viscosity under internal force on part of the boundary. Park and Bae[2] proved the existence and uniqueness of weak solution to problem (1.1). These articles investigated the existence result and asymptotic behavior of solutions, yet the nonexistence phenomena are not studied. Motivated by these works, we consider problem (1.1)-(1.4) with external force and a special kind of boundary damping condition (1.3). Our intention here is to show the local existence result for the problem (1.1)-(1.4) and prove that the energy decays exponentially by using the perturbed energy method [28]. Moreover, the nonexistence of the global solution for (1.1)-(1.4) is obtained by direct approach [19]. In this way, we can extend the asymptotic result of [4, 9] to a boundary linear damping and the result of blow-up properties.

Our paper is organized as follows. In section 2, we give some notations, lemmas and assumptions which will be used in the paper. In section 3, we first use Galerkin's approximation to study the existence of the simpler problem (3.1)-(3.4). Then, we obtain the local existence Theorem 3.2 by using contraction mapping principle. In section4, we obtain exponential decay for solutions obtained in section 3. Finally, the blow-up properties are derived.

2. Preliminaries

In this section, we shall give some lemmas and notations that will be used later. Let

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\},$$

which endowed with the norm $\|\nabla \cdot \|_{L^2(\Omega)}$ is a Hilbert subspace of $H^1(\Omega)$ and we denote

$$\|u\|_{2,\Gamma_1}^2 = \int_{\Gamma_1} |u(x)|^2 \, d\Gamma.$$

Lemma 2.1. (Sobolev-Poincaré inequality [21]). *If* $2 \le p \le \frac{2N}{N-2}$, *then*

$$\|u\|_p \le B_1 \|\nabla u\|_2 ,$$

for $u \in H_0^1(\Omega)$ *holds with some constant* B_1, *where* $\|\cdot\|_p$ *denotes the norm of* $L^p(\Omega)$.

Lemma 2.2. ([19]). *Let* $\delta > 0$ *and* $B(t) \in C^2(0,\infty)$ *be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \ge 0. \tag{2.1}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.2}$$

then

$$B'(t) > K_0$$

for $t > 0$, *where* K_0 *is a constant*, $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ *is the smallest root of the equation*

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Lemma 2.3. ([19]). *If* $J(t)$ *is a nonincreasing function on* $[t_0,\infty)$, $t_0 \ge 0$ *and satisfies the differential inequality*

$$J'(t)^2 \ge a + bJ(t)^{2+\frac{1}{\delta}} \text{ for } t_0 \ge 0, \tag{2.3}$$

where $a > 0, b \in R$, *then there exists a finite time* T^* *such that*

$$\lim_{t \to T^{*-}} J(t) = 0$$

and the upper bound of T^* *is estimated respectively by the following cases:*

(i) *If* $b < 0$ *and* $J(t_0) < \min\left\{1, \sqrt{\frac{a}{-b}}\right\}$ *then*

$$T^* \le t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}.$$

(ii) *If* $b = 0$, *then*

$$T^* \le t_0 + \frac{J(t_0)}{\sqrt{a}}.$$

(iii) *If* $b > 0$, *then*

$$T^* \le \frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\},$$

where $c = (\frac{a}{b})^{2+\frac{1}{\delta}}$. *Now, we state the general hypothesis :*

$(\mathbf{A1})$ $f(0) = 0$ *and there is a positive constant* k_1 *such that*

$$|f(u) - f(v)| \le k_1 |u - v| \left(|u|^{p-2} + |v|^{p-2}\right),$$

where $2 < p \le \frac{2(N-1)}{N-2}$ *and* $u, v \in R$.

3. Local Existence

In this section, we will discuss the local existence of solutions for nonlinear wave equations $(1.1) - (1.4)$ by using contraction mapping principle. An important tool in the proof of local existence theorem 3.2 is the study of the following simpler problem :

(3.1) $$u_{tt} - M(\|\nabla u(t)\|_2^2)\Delta u - \frac{\partial}{\partial t}\Delta u = f_1(x,t) \text{ on } \Omega \times (0,t),$$

(3.2) $$u = 0 \text{ on } \Gamma_0 \times (0,T),$$

$$M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial}{\partial t}(\frac{\partial u}{\partial \nu}) = -u_t \text{ on } \Gamma_1 \times (0,T), \tag{3.3}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), x \in \Omega \tag{3.4}$$

Here, $T > 0$ and f_1 is a fixed forcing term on $\Omega \times (0,T)$.

Lemma 3.1. *Suppose that* (*A1*) *holds and that* $u_0 \in V$, $u_1 \in V$ *and* $f_1 \in L^2([0,T); L^2(\Omega))$. *Then the problem* $(3.1)-(3.4)$ *admits a unique solution* u *such that*

$$u \in C([0,T); V),$$
$$u_t \in C([0,T); L^2(\Omega)) \cap L^2([0,T); V),$$

and

$$u_{tt} \in L^2([0,T); L^2(\Omega)).$$

Proof. Let $(w_n)_{n\in N}$ be a basis in V and V_n be the space generated by $w_1,\cdots, w_n, n = 1,2,3,\cdots$. Let us consider

$$u_n(t) = \sum_{i=1}^{n} r_{in}(t)w_i$$

be the weak solution of the following approximate problem corresponding to $(3.1)-(3.4)$

$$\begin{aligned} &\int_\Omega u_n''(t)wdx + M(\|\nabla u_n(t)\|_2^2)\int_\Omega \nabla u_n(t)\cdot\nabla wdx \\ &+\int_\Omega \nabla u_n'(t)\cdot\nabla wdx + \int_{\Gamma_1} u_n'(t)wd\Gamma \\ =&\int_\Omega f_1(x,t)wdx, \text{ for } w \in V_n, \end{aligned} \tag{3.5}$$

with initial conditions

$$u_n(0) = u_{0n} \equiv \sum_{i=1}^{n} p_{in}w_i \to u_0 \text{ in } V, \tag{3.6}$$

and

$$u_n'(0) = u_{1n} \equiv \sum_{i=1}^{n} q_{in}w_i \to u_1 \text{ in } V, \tag{3.7}$$

where $p_{in} = \int_\Omega u_0 w_i dx$, $q_{in} = \int_\Omega u_1 w_i dx$ and $u' = \frac{\partial u}{\partial t}$.

By standard methods in differential equations, we prove the existence of solutions to $(3.5) - (3.7)$ on some interval $[0, t_n), 0 < t_n < T$. In order to extend the solution of $(3.5) - (3.7)$ to the whole interval $[0, T)$, we need the following a priori estimates.

Step 1. Setting $w = u_n'(t)$ in (3.5), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|u_n'(t)\right\|_2^2 + \hat{M}(\|\nabla u_n(t)\|_2^2)\right) + \left\|\nabla u_n'(t)\right\|_2^2 + \left\|u_n'(t)\right\|_{2,\Gamma_1}^2 \\ &\leq \frac{1}{2}\|f_1\|_2^2 + \frac{1}{2}\left\|u_n'(t)\right\|_2^2, \end{aligned} \tag{3.8}$$

where $\hat{M}(s) = \int_0^s M(r)dr$. By integrating (3.8) over $(0, t)$, we deduce

$$\begin{aligned} &\left\|u_n'(t)\right\|_2^2 + \hat{M}(\|\nabla u_n(t)\|_2^2) + 2\int_0^t \left\|\nabla u_n'(t)\right\|_2^2 dt + 2\int_0^t \left\|u_n'(t)\right\|_{2,\Gamma_1}^2 dt \\ &\leq c_1 + \int_0^t \left\|u_n'(t)\right\|_2^2 dt. \end{aligned} \tag{3.9}$$

where $c_1 = \|u_{1n}\|_2^2 + \hat{M}(\|\nabla u_{0n}\|_2^2) + \int_0^t \|f_1\|_2^2 dt$. Thus, employing Gronwall's Lemma, we see that

$$\left\|u_n'(t)\right\|_2^2 + \hat{M}(\|\nabla u_n(t)\|_2^2) + \int_0^t \left\|\nabla u_n'(t)\right\|_2^2 dt + \int_0^t \left\|u_n'(t)\right\|_{2,\Gamma_1}^2 dt \leq L_1, \tag{3.10}$$

for $t \in [0, T)$ and L_1 is a positive constant independent of $n \in N$.

Step 2. Setting $w = u_n''(t)$ in (3.5), we have

$$\begin{aligned} &\frac{1}{2}\left\|u_n''(t)\right\|_2^2 + \frac{d}{dt}\left[M(\|\nabla u_n(t)\|_2^2)\int_\Omega \nabla u_n(t) \cdot \nabla u_n'(t)dx\right] \\ &+\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla u_n'(t)\right\|_2^2 + \left\|u_n'(t)\right\|_{2,\Gamma_1}^2\right) \\ &\leq \left[\frac{d}{dt}M(\|\nabla u_n(t)\|_2^2)\right]\int_\Omega \nabla u_n(t) \cdot \nabla u_n'(t)dx \\ &+M(\|\nabla u_n(t)\|_2^2)\left\|\nabla u_n'(t)\right\|_2^2 + \frac{1}{2}\|f_1\|_2^2. \end{aligned} \tag{3.11}$$

By Hölder inequality and (3.10), we note that

$$\begin{aligned} &\left[\frac{d}{dt}M(\|\nabla u_n(t)\|_2^2)\right]\int_\Omega \nabla u_n(t)\cdot\nabla u_n'(t)dx \\ &= 2M'(\|\nabla u_n\|_2^2)\left(\int_\Omega \nabla u_n\cdot\nabla u_n' dx\right)^2 \\ &\le \frac{2M_1L_1}{m_0}\left\|\nabla u_n'(t)\right\|_2^2, \end{aligned} \tag{3.12}$$

and by (3.10) again, we also get

$$M(\|\nabla u_n(t)\|_2^2)\left\|\nabla u_n'(t)\right\|_2^2 \le M_2\left\|\nabla u_n'(t)\right\|_2^2, \tag{3.13}$$

where $M_1 = \sup\limits_{0\le t\le T}\left\{|M'(s)|\,;\ 0\le s\le \frac{L_1}{m_0}\right\}$ and $M_2 = \sup\limits_{0\le t\le T}\left\{|M(s)|\,;\ 0\le s\le \frac{L_1}{m_0}\right\}$. Thus, integrating (3.11) over $(0,t)$ and using $(3.12)-(3.13)$, we obtain

$$\begin{aligned} &\int_0^t \left\|u_n''(t)\right\|_2^2 dt + 2M(\|\nabla u_n(t)\|_2^2)\int_\Omega \nabla u_n(t)\cdot\nabla u_n'(t)dx \\ &+\left\|\nabla u_n'(t)\right\|_2^2 + \left\|u_n'(t)\right\|_{2,\Gamma_1}^2 \\ &\le c_2 + c_3\int_0^t \left\|\nabla u_n'(\tau)\right\|_2^2 d\tau, \end{aligned} \tag{3.14}$$

where $c_2 = 2M(\|\nabla u_{0n}\|_2^2)\,\|\nabla u_{0n}\|_2\,\|\nabla u_{1n}\|_2 + \|\nabla u_{1n}\|_2^2 + \|u_{1n}\|_{2,\Gamma_1}^2 + \int_0^t \|f_1\|_2^2\, dt$ and $c_3 = \left(\frac{4M_1L_1}{m_0} + 2M_2\right)$. By Hölder inequality, Young's inequality and (3.10), we observe that

$$\left\|2M(\|\nabla u_n(t)\|_2^2)\int_\Omega \nabla u_n(t)\cdot\nabla u_n'(t)dx\right\| \le c_4 + \frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2, \tag{3.15}$$

where $c_4 = \frac{2M_2^2L_1}{m_0}$. Then, from (3.14) and by (3.15), we see that

$$\begin{aligned} &\int_0^t \left\|u_n''(t)\right\|_2^2 dt + \frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2 + \left\|u_n'(t)\right\|_{2,\Gamma_1}^2 \\ &\le c_5 + c_3\int_0^t \left\|\nabla u_n'(\tau)\right\|_2^2 d\tau, \end{aligned}$$

where $c_5 = c_4 + c_2$. Hence, by Gronwall's Lemma, we have

$$\int_0^t \left\|u_n''(t)\right\|_2^2 dt + \left\|\nabla u_n'(t)\right\|_2^2 + \left\|u_n'(t)\right\|_{2,\Gamma_1}^2 \le L_2, \tag{3.16}$$

for all $t \in [0,T)$ and L_2 is a positive constant independent of $n \in N$.

Step 3. Let $n_2 \geq n_1$ be two natural numbers and consider $z_n = u_{n_2} - u_{n_1}$. Then z_n satisfies the following system :

$$z_n'' - \Delta z_n' - M(\|\nabla u_{n_2}(t)\|_2^2)\Delta u_{n_2} + M(\|\nabla u_{n_1}(t)\|_2^2)\Delta u_{n_1} = 0, \tag{3.17}$$

with boundary conditions

$$z_n = 0 \quad \text{on } \Gamma_0 \times (0,T), \tag{3.18}$$

$$\begin{aligned} &M(\|\nabla u_{n_2}(t)\|_2^2)\frac{\partial u_{n_2}}{\partial \nu} - M(\|\nabla u_{n_1}(t)\|_2^2)\frac{\partial u_{n_1}}{\partial \nu} \\ &+\frac{\partial z_n}{\partial \nu} = -z_n' \quad \text{on } \Gamma_1 \times (0,T), \end{aligned} \tag{3.19}$$

and initial conditions

$$z_n(x,0) = u_{0n_2} - u_{0n_1}, \quad z_n'(x,0) = u_{1n_2} - u_{1n_1}, \ x \in \Omega. \tag{3.20}$$

Multiplying (3.17) by $2z_n'$ and integrating it over Ω,, we get

$$\begin{aligned} &\frac{d}{dt}\left(\|z_n'(t)\|_2^2 + M(\|\nabla u_{n_2}(t)\|_2^2)\|\nabla z_n(t)\|_2^2\right) \\ &+2\|\nabla z_n'(t)\|_2^2 + 2\,\|z_n'(t)\|_{2,\Gamma_1}^2 \\ = \ & 2\int_\Omega \left[M(\|\nabla u_{n_1}(t)\|_2^2) - M(\|\nabla u_{n_2}(t)\|_2^2)\right]\nabla u_{n_1}\cdot\nabla z_n' dx \\ &+\left[\frac{d}{dt}M(\|\nabla u_{n_2}(t)\|^2)\right]\|\nabla z_n(t)\|_2^2. \end{aligned} \tag{3.21}$$

We note that (3.10) yields

$$\begin{aligned} &\left|M(\|\nabla u_{n_1}(t)\|_2^2) - M(\|\nabla u_{n_2}(t)\|_2^2)\right| \\ \leq\ & \int_{\|\nabla u_{n_1}(t)\|_2^2}^{\|\nabla u_{n_2}(t)\|_2^2} \left|M'(s)\right| ds \\ \leq\ & M_1\left(\|\nabla u_{n_2}(t)\|_2 + \|\nabla u_{n_1}(t)\|_2\right)\|\nabla z_n(t)\|_2 \\ \leq\ & c_6\|\nabla z_n(t)\|_2, \end{aligned} \tag{3.22}$$

where $c_6 = 2M_1\left(\frac{L_1}{m_0}\right)^{\frac{1}{2}}$. Then, from (3.22), (3.10) and by Hölder inequality and Young's inequality, we see that

$$\begin{aligned} &2\left[M(\|\nabla u_{n_1}(t)\|_2^2) - M(\|\nabla u_{n_2}(t)\|_2^2)\right]\int_\Omega \nabla u_{n_1}\cdot\nabla z_n' dx \\ \leq\ & c_7\|\nabla z_n(t)\|_2^2 + \|\nabla z_n'(t)\|_2^2, \end{aligned} \tag{3.23}$$

where $c_7 = \frac{c_6^2 L_1}{m_0}$.

Again from (3.10) and (3.16), we have

$$\left[\frac{d}{dt} M(\|\nabla u_{n_2}(t)\|_2^2)\right] \|\nabla z_n(t)\|_2^2 \le c_8 \|\nabla z_n(t)\|_2^2, \tag{3.24}$$

where $c_8 = 2M_1 \left(\frac{L_1 L_2}{m_0}\right)^{\frac{1}{2}}$.

Thus, by $(3.22) - (3.24)$, we have from (3.21)

$$\begin{aligned} &\frac{d}{dt}\left[\|z_n'(t)\|_2^2 + M(\|\nabla u_{n_2}(t)\|_2^2)\|\nabla z_n(t)\|_2^2\right] + \|\nabla z_n'(t)\|_2^2 \\ &+2\left\|z_n'(t)\right\|_{2,\Gamma_1}^2 \le c_9 \|\nabla z_n(t)\|_2^2, \end{aligned} \tag{3.25}$$

where $c_9 = c_7 + c_8$. By integrating (3.25) over $(0,t)$, we obtain

$$\begin{aligned} &\|z_n'(t)\|_2^2 + M(\|\nabla u_{n_2}(t)\|_2^2)\|\nabla z_n(t)\|_2^2 + \int_0^t \|\nabla z_n'(t)\|_2^2 dt + 2\int_0^t \left\|z_n'(t)\right\|_{2,\Gamma_1}^2 dt \\ &\le c_{10} + \frac{c_9}{m_0}\int_0^t M(\|\nabla u_{n_2}(t)\|_2^2)\|\nabla z_n(t)\|_2^2 dt, \end{aligned}$$

where $c_{10} = \|z_{1n}\|_2^2 + M(\|\nabla u_{1n}\|_2^2)\|\nabla z_{1n}\|_2^2$. Hence, by Gronwall's Lemma, we deduce

$$\begin{aligned} &\left\|z_n'(t)\right\|_2^2 + M(\|\nabla u_{n_2}(t)\|_2^2)\,\|\nabla z_n(t)\|_2^2 + \int_0^t \left\|\nabla z_n'(t)\right\|_2^2 dt \\ &+2\int_0^t \|z_n'(t)\|_{2,\Gamma_1}^2\, dt \le L_3, \end{aligned} \tag{3.26}$$

for all $t \in [0,T)$ and L_3 is a positive constant independent of $n \in N$. Therefore, from (3.10), (3.16) and (3.26), we see that

$$u_i \to u \quad \text{strongly} \quad \text{in} \quad C([0,T);V), \tag{3.27}$$

$$u_i' \to u' \quad \text{strongly} \quad \text{in} \quad C([0,T);L^2(\Omega)), \tag{3.28}$$

$$u_i' \to u' \quad \text{strongly} \quad \text{in} \quad L^2([0,T);V), \tag{3.29}$$

$$u_i' \to u' \quad \text{strongly} \quad \text{in} \quad L^2([0,T);L^2(\Gamma_1)), \tag{3.30}$$

$$u_i' \to u' \quad \text{weak-*} \quad \text{in} \quad L^\infty([0,T);L^2(\Gamma_1)), \tag{3.31}$$

$$u_i' \to u' \quad \text{weak-*} \quad \text{in} \quad L^\infty([0,T);V), \tag{3.32}$$

(3.33) $$u_i'' \to u'' \text{ weakly in } L^2([0,T);L^2(\Omega)).$$

By (3.27), we have

$$\|\nabla u_i\|_2 \to \|\nabla u\|_2 \text{ in } C(0,T).$$

and since $M \in C^1([0,\infty);R^+)$, we obtain

$$M(\|\nabla u_i\|_2^2) \to M(\|\nabla u\|_2^2) \text{ in } C(0,T).$$

Thus

(3.34) $$M(\|\nabla u_i\|_2^2)u_i \to M(\|\nabla u\|_2^2)u \text{ in } C([0,T);V).$$

Multiplying (3.5) by $\theta \in D(0,T)$ and integrating it over $(0,T)$, we get

(3.35) $$\begin{aligned} &\int_0^T \int_\Omega u_n'' v\theta dxdt + \int_0^T M(\|\nabla u_n\|_2^2)\int_\Omega \nabla u_n \cdot \nabla v\theta dxdt \\ &+ \int_0^T \int_\Omega \nabla u_n' \cdot \nabla v\theta dxdt + \int_0^T \int_{\Gamma_1} u_n' v\theta d\Gamma dt \\ = &\int_0^T \int_\Omega f_1 v\theta dxdt, \text{ for } v \in V. \end{aligned}$$

The convergences (3.27) – (3.34) are sufficient to pass the limit in (3.35) in order to obtain

$$\begin{aligned} &\int_0^T \int_\Omega u'' v\theta dxdt + \int_0^T M(\|\nabla u\|_2^2)\int_\Omega \nabla u \cdot \nabla v\theta dxdt \\ &+ \int_0^T \int_\Omega \nabla u' \cdot \nabla v\theta dxdt + \int_0^T \int_{\Gamma_1} u' v\theta d\Gamma dt \\ = &\int_0^T \int_\Omega f_1 v\theta dxdt, \end{aligned}$$

for all $\theta \in D(0,T)$ and $v \in V$. In particular, let $v\theta \in D((0,T)\times\Omega)$, then we see that

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u - \Delta u_t = f_1(x,t) \text{ in } D'((0,T)\times\Omega).$$

Since $u'' \in L^2([0,T);L^2(\Omega))$, we have

$$\Delta\left(M(\|\nabla u(t)\|_2^2 + u_t\right) = u'' - f_1(x,t) \in L^2([0,T);L^2(\Omega)).$$

Thus

(3.36) $$u_{tt} - M(\|\nabla u(t)\|_2^2)\Delta u - \Delta u_t = f_1(x,t) \text{ in } L^2([0,T);L^2(\Omega)).$$

From (3.36) and by Green's formula, we get

$$M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial \nu}+\frac{\partial}{\partial t}(\frac{\partial u}{\partial \nu})=-u_t \text{ in } D'([0,T);H^{-\frac{1}{2}}(\Gamma_1)),$$

but $u_t \in L^2([0,T);L^2(\Gamma_1))$, so we deduce that

$$M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial \nu}+\frac{\partial}{\partial t}(\frac{\partial u}{\partial \nu})=-u_t \text{ in } L^2([0,T);L^2(\Gamma_1)).$$

Next, we want to show the uniqueness of $(3.1)-(3.4)$. Let $u^{(1)}, u^{(2)}$ be two solutions of $(3.1)-(3.4)$. Then $z=u^{(1)}-u^{(2)}$ satisfies

$$z_{tt}-\Delta z_t+M\left(\|\nabla u^{(2)}\|_2^2\right)\Delta u^{(2)}-M\left(\|\nabla u^{(1)}\|_2^2\right)\Delta u^{(1)}=0, \tag{3.37}$$

with boundary conditions

$$z=0 \text{ on } \Gamma_0\times(0,T),$$

$$M(\|\nabla u^{(1)}(t)\|_2^2)\frac{\partial u^1}{\partial \nu}-M(\|\nabla u^{(2)}(t)\|_2^2)\frac{\partial u^{(2)}}{\partial \nu}+\frac{\partial}{\partial t}(\frac{\partial z}{\partial \nu})=-z_t \text{ on } \Gamma_1\times(0,T),$$

and initial conditions

$$z(x,0)=0, z'(x,0)=0, \quad x\in\Omega.$$

Setting $w=z'(t)$ in (3.37), then as in deriving (3.10), we see that

$$\left\|z'(t)\right\|_2^2+\hat{M}\left(\|\nabla z(t)\|_2^2\right)\le\int_0^t\left\|z'(t)\right\|_2^2 dt.$$

Thus, employing Gronwall's Lemma, we conclude that

$$\left\|z'(t)\right\|_2=\|\nabla z(t)\|_2=0 \text{ for all } t\in[0,T]. \tag{3.38}$$

Therefore, we have the uniqueness. We have just showed the existence of solutions to problem $(3.1)-(3.4)$ when the initial data are smooth. However, when

$$\{u_0,\ u_1,\ f\}\in V\times L^2(\Omega)\times L^2([0,T);L^2(\Omega))$$

there exists

$$\{u_0^\mu,\ u_1^\mu,\ f_\mu\}\in V\times V\times H^1([0,T);L^2(\Omega)),$$

such that

$$\{u_0^\mu,\ u_1^\mu,\ f_\mu\}\to\{u_0,\ u_1,\ f\} \text{ in } V\times L^2(\Omega)\times L^2([0,T);L^2(\Omega)),$$

and using the density arguments and proceeding analogous to the estimate of step $1-3$, we can find a sequence $\{u_\mu\}$ of solution to problem $(3.1)-(3.4)$ such that

$$\begin{aligned} u_\mu &\to u \text{ strongly in } C([0,T);V), \\ u'_\mu &\to u' \text{ strongly in } C([0,T);L^2(\Omega)), \\ u'_\mu &\to u' \text{ strongly in } L^2([0,T);V), \\ u'_\mu &\to u' \text{ strongly in } L^2([0,T);L^2(\Gamma_1)), \\ u''_\mu &\to u'' \text{ weakly in } L^2([0,T);L^2(\Omega)). \end{aligned}$$

The above convergences are sufficient to pass to the limit in order to obtain a weak solution of $(3.1)-(3.4)$ which satisfies

$$u_{tt} - M(\|\nabla u(t)\|_2^2)\Delta u - \Delta u_t = f_1(x,t) \text{ in } L^2([0,T);V').$$

Moreover, as the same way in [4], we also deduce

$$M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial}{\partial t}(\frac{\partial u}{\partial \nu}) = -u_t \text{ in } L^2([0,T);L^2(\Gamma_1)).$$

Now, we are ready to show the local existence of the problem $(1.1)-(1.4)$.

Theorem 3.2. *Suppose that (A1) hold, and that* $u_0 \in V$ *and* $u_1 \in L^2(\Omega)$, *then there exists a unique solution* u *of* $(1.1)-(1.4)$ *satisfying*

$$u \in C([0,T];V), u_t \in C([0,T];L^2(\Omega)) \cap L^2([0,T];V).$$

Moreover, at least one of the following statements is valid :

(i) $T=\infty$,

(ii) $e(u(t)) \equiv \|u_t\|_2^2 + \|\nabla u\|_2^2 \to \infty$ *as* $t \to T^-$.

Proof. Define the following two-parameter space :

$$X_{T,R_0} = \left\{ \begin{array}{c} v: \ v \in C([0,T);V), \\ v_t \in C([0,T);L^2(\Omega)) \cap L^2([0,T);V), \\ e(v(t)) \le R_0^2, \text{ for all } t \in [0,T), \ v(0)=u_0 \text{ and } v_t(0)=u_1. \end{array} \right\},$$

for $T>0$, $R_0>0$. Then X_{T,R_0} is a complete metric space with the distance

$$d(y,z) = \sup_{0\le t\le T} e(y(t)-z(t))^{\frac{1}{2}}. \tag{3.39}$$

where $y,\ z \in X_{T,R_0}$. Given $v \in X_{T,R_0}$, we consider the following problem

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u - \Delta u_t = f(v), \tag{3.40}$$

with boundary conditions

$$u = 0 \ \text{ on } \Gamma_0 \times (0,T), \tag{3.41}$$

$$M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial}{\partial t}(\frac{\partial u}{\partial \nu}) = -u_t \ \text{ on } \Gamma_1 \times (0,T), \tag{3.42}$$

and initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \tag{3.43}$$

First, we observe by (A1) that $f(v) \in L^2([0,T); L^2(\Omega))$. Thus, by Lemma 3.1, there exists a unique solution $u \in$ of $(3.40) - (3.43)$. We define the nonlinear mapping $Sv = u$, and then, we shall show that there exist $T > 0$ and $R_0 > 0$ such that

(i) $S : X_{T,R_0} \to X_{T,R_0}$,

(ii) S is a contraction mapping in X_{T,R_0} with respect to the metric $d(\cdot,\cdot)$ defined in (3.39).

(i) Multiplying (3.40) by $2u_t$ and then integrating it over $\Omega \times (0,t)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|u_t(t)\|_2^2 + \hat{M}(\|\nabla u(t)\|_2^2)\right) + 2\,\|\nabla u_t(t)\|_2^2 + 2\,\|u_t(t)\|_{2,\Gamma_1}^2 \\ &\le 2\int_\Omega f(v) u_t dx. \end{aligned} \tag{3.44}$$

By (A1), Hölder inequality and Poincaré inequality, we have

$$\begin{aligned} &2\int_\Omega f(v)u_t dx \\ &\le\ 2k_1 B_1^{p-1}\,\|\nabla v\|_2^{p-1}\,\|u_t\|_2 \\ &\le\ 2k_1 B_1^{p-1} R_0^{p-1} e(u(t))^{\frac{1}{2}}. \end{aligned} \tag{3.45}$$

Then integrating (3.44) and using (3.45), we obtain

$$e(u(t)) \le \eta_0^2 + 2c_{11}k_1\,(B_1 R_0)^{p-1}\int_0^t e(u)^{\frac{1}{2}} dt, \tag{3.46}$$

where $\eta_0 = c_{11}\left[\|u_1\|_2^2 + \hat{M}(\|\nabla u_0\|_2^2)\right]$ and $c_{11}^{-1} = \min(1, m_0)$. Thus, by Gronwall's Lemma, we have

$$e(u(t)) \leq \chi(u_0, u_1, R_0, T)^2, \tag{3.47}$$

for any $t \in [0, T)$ and

$$\chi(u_0, u_1, R_0, T) = \eta_0 + c_{11}k_1\,(B_1R_0)^{p-1}\,T.$$

We see that if parameters T and R_0 satisfy

$$\chi(u_0, u_1, R_0, T)^2 \leq R_0^2, \tag{3.48}$$

then S maps X_{T,R_0} into itself. Therefore, we show that S maps X_{T,R_0} into itself. Next, we will show that S is a contraction mapping with respect to the metric $d(\cdot,\cdot)$. Let $v_i \in X_{T,R_0}$ and $u^{(i)}$ be the corresponding solution to $(3.40) - (3.43)$. By the above argument, we see that $u^{(i)} \in X_{T,R_0}$, $i = 1, 2$. Setting $w(t) = (u^{(1)} - u^{(2)})(t)$, then w satisfy the following system:

$$\begin{aligned} &w_{tt} - M\left(\|\nabla u^{(1)}\|_2^2\right)\Delta w - \Delta w_t \\ &= f(v_1) - f(v_2) + \left[M\left(\|\nabla u^{(1)}\|_2^2\right) - M\left(\|\nabla u^{(2)}\|_2^2\right)\right]\Delta u^{(2)}, \end{aligned} \tag{3.49}$$

with boundary conditions

$$w = 0 \quad \text{on } \Gamma_0 \times (0, T), \tag{3.50}$$

$$\begin{aligned} &M(\|\nabla u^{(1)}(t)\|^2)\frac{\partial u^1}{\partial \nu} - M(\|\nabla u^{(2)}(t)\|^2)\frac{\partial u^{(2)}}{\partial \nu} + \frac{\partial}{\partial t}(\frac{\partial w}{\partial \nu}) \\ &= -w_t \quad \text{on } \Gamma_1 \times (0, T) \end{aligned} \tag{3.51}$$

and initial conditions

$$w(x, 0) = 0,\; w_t(x, 0) = 0,\; x \in \Omega. \tag{3.52}$$

Multiplying (3.47) by $2w_t$ and integrating it over Ω, we have

$$\begin{aligned} &\frac{d}{dt}\left[\|w_t\|_2^2 + M(\left\|\nabla u^{(1)}\right\|_2^2)\,\|\nabla w\|_2^2\right] + 2\,\|\nabla w_t\|_2^2 + 2\,\|w_t\|_{2,\Gamma_1}^2 \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.53}$$

where

$$I_1 = -2\left[M\left(\|\nabla u^{(1)}\|_2^2\right) - M\left(\|\nabla u^{(2)}\|_2^2\right)\right]\int_\Omega \nabla u^{(2)} \cdot \nabla w_t dx,$$

$$I_2 = 2\int_{\Omega}(f(v_1)-f(v_2))\,w_t dx,$$

and

$$I_3 = \left(\frac{d}{dt}M(\left\|\nabla u^{(1)}\right\|_2^2)\right)\|\nabla w\|_2^2 .$$

By Hölder inequality and Young's inequality, we observe that

$$\begin{aligned} &|I_1| \\ &\le 2M_3\left(\left\|\nabla u^{(1)}\right\|_2+\left\|\nabla u^{(2)}\right\|_2\right)\left\|\nabla u^{(1)}-\nabla u^{(2)}\right\|_2\left\|\nabla u^{(2)}\right\|_2\|\nabla w_t\|_2 \\ &\le 2M_3^2R_0^4 d(u^{(1)},u^{(2)})^2+2\|\nabla w_t\|_2^2 , \end{aligned} \tag{3.54}$$

$$|I_2| \le k_1^2B_1^{2(p-1)}R_0^{2(p-2)}d(v_1,v_2)^2+d(u^{(1)},u^{(2)})^2, \tag{3.55}$$

and

$$|I_3| \le 2M_3\left\|\nabla u^{(1)}\right\|_2\left\|\nabla u_t^{(1)}\right\|_2\|\nabla w\|_2^2 , \tag{3.56}$$

where $M_3 = \sup\limits_{0\le t\le T}\{|M'(s)|;\ 0\le s\le R_0\}$. Note that, as in the estimate Step 2, we deduce

$$\left\|\nabla u_t^{(1)}\right\|_2^2 \le c_{12}+c_{13}\int_0^t\left\|\nabla u_t^{(1)}\right\|_2^2 dt,$$

where $c_{12}=2M(\|\nabla u_0\|_2^2)\,\|\nabla u_0\|_2\,\|\nabla u_1\|_2+\|\nabla u_1\|_2^2+\|u_1\|_{2,\Gamma_1}^2+k_1B_1^{(p-1)}R_0^{(p-1)}T$ $+2M_4^2R_0^2$ and $c_{13} = 4M_3R_0^2+2M_4$, here $M_4=\sup\limits_{0\le t\le T}\{|M(s)|;\ 0\le s\le R_0\}$. Since, by Gronwall's Lemma, we have

$$\left\|\nabla u_t^{(1)}\right\|_2^2 \le c_{12}\mathrm{e}^{c_{13}T},$$

then, from (3.54), we arrive at

$$|I_3| \le \sqrt{c_{12}}M_3R_0\mathrm{e}^{\frac{c_{13}T}{2}}d(u^{(1)},u^{(2)})^2. \tag{3.57}$$

From (3.51) and by $(3.52)-(3.55)$, we obtain

$$\frac{d}{dt}\left[\|w_t\|_2^2+M(\left\|\nabla u^{(1)}\right\|_2^2)\,\|\nabla w\|_2^2\right] \le c_{14}d(v_1,v_2)^2+c_{15}d(u^{(1)},u^{(2)})^2, \tag{3.58}$$

where $c_{14} = k_1^2B_1^{2(p-1)}R_0^{2(p-2)}$ and $c_{15} = 2M_3^2R_0^4+1+\sqrt{c_{12}}M_3R_0e^{\frac{c_{13}T}{2}}$. Integrating (3.56) over $(0,t)$ and using (3.39), we have

$$d(u^{(1)},u^{(2)})^2 \le c_{16}Td(v_1,v_2)^2+c_{17}\int_0^t d(u^{(1)},u^{(2)})^2dt,$$

where $c_{16} = c_{11}c_{14}$ and $c_{17} = c_{11}c_{15}$. Thus, by Gronwall's Lemma, we see that

$$d(u^{(1)}, u^{(2)})^2 \le c_{16} T \mathrm{e}^{c_{17}T} d(v_1, v_2)^2.$$

Hence

$$d(u^{(1)}, u^{(2)}) \le C(T, R_0) d(v_1, v_2) \tag{3.59}$$

where

$$C(T, R_0) = \sqrt{c_{16}T}\, \mathrm{e}^{\frac{c_{17}T}{2}}.$$

Therefore, under inequality (3.48), S is a contraction mapping if $C(T, R_0) < 1$. Indeed, we choose R_0 sufficiently large and T sufficiently small so that (3.47) and (3.59) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

4. Global Existence

In this section, we consider the global solution and energy decay of solutions for a kind of the problem $(1.1) - (1.4)$:

$$u_{tt} - M(\|\nabla u(t)\|^2)\Delta u - \frac{\partial}{\partial t}\Delta u = |u|^{p-2}\, u, \tag{4.1}$$

$$u = 0 \quad \text{in } \Gamma_1 \times (0, T), \tag{4.2}$$

$$M(\|\nabla u(t)\|^2)\frac{\partial u}{\partial \nu} + \frac{\partial}{\partial t}(\frac{\partial u}{\partial \nu}) = -u_t \quad \text{in } \Gamma_0 \times (0, T), \tag{4.3}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{4.4}$$

where $2 < p \le \frac{2(N-1)}{N-2}$ and $M \in C'([0, \infty), R)$ is a nondecreasing function. Let

$$I_1(u(t)) \equiv I_1(t) = m_0 \|\nabla u\|_2^2 - \|u\|_p^p\,, \tag{4.5}$$

$$I_2(u(t)) \equiv I_2(t) = M(\|\nabla u\|_2^2)\, \|\nabla u\|_2^2 - \|u\|_p^p\,, \tag{4.6}$$

and

$$J(u(t)) \equiv J(t) = \frac{1}{2}\hat{M}(\|\nabla u(t)\|_2^2) - \frac{1}{p}\, \|u\|_p^p\,, \tag{4.7}$$

for $u \in V$. Let u be the solution of $(4.1)-(4.4)$, we define the energy function

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + J(t). \tag{4.8}$$

Lemma 4.1 *$E(t)$ is a nonincreasing function on $[0,T)$ and*

$$E'(t) = -\|\nabla u_t\|_2^2 - \|u_t\|_{2,\Gamma_1}^2. \tag{4.9}$$

Proof. By using Divergence theorem and $(4.1)-(4.4)$, we see that (4.9) follows at once.

Lemma 4.2. *Let u be the solution of $(4.1)-(4.4)$. Assume the conditions of Theorem 3.2 hold. If $I_1(0) > 0$ and*

$$\alpha = \frac{2pB_1^p}{(p-2)m_0}\left(\frac{2p}{m_0(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{4.10}$$

then $I_2(t) > 0$ for all $t \in [0,T)$.

Proof. Since $I_1(0) > 0$, it follows from the continuity of $u(t)$ that

$$I_1(t) \geq 0, \tag{4.11}$$

for some interval near $t = 0$. Let $t_{\max} > 0$ be a maximal time (possibly $t_{\max} = T$), when (4.11) holds on $[0, t_{\max})$. From (4.7) and (4.5), we have

$$\begin{aligned} J(t) &\geq \frac{1}{2}m_0\,\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p \\ &\geq \frac{(p-2)\,m_0}{2p}\,\|\nabla u\|_2^2 + \frac{1}{p}I_1(t),\ \ t \in [0, t_{\max}). \end{aligned} \tag{4.12}$$

By using (4.12), (4.9) and Lemma 4.1, we get

$$\|\nabla u\|_2^2 \leq \frac{2p}{(p-2)\,m_0}J(t) \leq \frac{2p}{(p-2)\,m_0}E(t) \leq \frac{2p}{(p-2)\,m_0}E(0). \tag{4.13}$$

Then, from Poincaré inequality, (4.13) and (4.10), we obtain

$$\begin{aligned} \|u\|_p^p \leq B_1^p\|\nabla u\|_2^p &\leq \frac{B_1^p}{m_0}\left(\frac{2p}{m_0(p-2)}E(0)\right)^{\frac{p-2}{2}} m_0\,\|\nabla u\|_2^2 \\ &= \frac{(p-2)\alpha}{2p}m_0\,\|\nabla u\|_2^2 < m_0\,\|\nabla u\|_2^2 \ \text{ on } [0, t_{\max}). \end{aligned} \tag{4.14}$$

Thus

$$I_1(t) = m_0 \|\nabla u\|_2^2 - \|u\|_p^p > 0 \text{ on } [0, t_{\max}). \tag{4.15}$$

This implies that we can take $t_{\max} = T$. But, from (4.5) and (4.6), we see that

$$I_2(t) \geq I_1(t),\ t \in [0, T).$$

Therefore, we have $I_2(t) > 0$, $t \in [0, T)$.

Moreover, from (4.13), we observe that

$$\|\nabla u\|_2^2 \leq \frac{2p}{(p-2)\, m_0} E(0), \tag{4.16}$$

and from (4.14) and (4.12), we also have

$$\|u_t\|_2^2 \leq 2E(t) \leq 2E(0). \tag{4.17}$$

Then, it follows by (4.16) and (4.17) that

$$e(u(t)) \leq c_1 E(0),$$

where $c_1 = \frac{2p}{(p-2)m_0} + 2$. Therefore, by Theorem 3.2, we have $T = \infty$.

Let us consider the perturbed energy

$$E_\varepsilon(t) = E(t) + \varepsilon \Psi(t) \text{ for } \varepsilon > 0, \tag{4.18}$$

where

$$\Psi(t) = \int_\Omega u_t u dx + \frac{1}{2}\left(\|\nabla u\|_2^2 + \|u\|_{2,\Gamma_1}^2\right).$$

Lemma 4.3. *There exists a positive constant* k_2 *such that*

$$|E_\varepsilon(t) - E(t)| \leq k_2 \varepsilon E(t) \text{ for } \varepsilon > 0 \text{ and } t \geq 0.$$

Proof. By Hölder inequality, Young's inequality and Poincaré inequality, we have

$$\begin{aligned}
|\Psi(t)| &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{B_1^2+1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_{2,\Gamma_1}^2 \\
&\leq E(t) + c_2 \|\nabla u\|_2^2 \\
&\leq k_2 E(t),
\end{aligned}$$

where $c_2 = \frac{\lambda^2 + \left(1+B_1^2\right)}{2}$, $k_2 = 1 + \frac{2pc_2}{m_0(p-2)}$, here λ is a positive constant such that $\|u\|_{2,\Gamma_1} \leq \lambda \|\nabla u\|_2$ for $u \in V$.

Lemma 4.4. *There exist* $k_3 = 1 - \frac{\alpha(p-1)}{p}$ *and* $\varepsilon_1 = \frac{2}{3B_1}$ *such that for* $\varepsilon \in (0, \varepsilon_1]$, *we have*

$$E'_\varepsilon(t) \leq -k_3 \varepsilon E(t) \text{ for } t \geq 0.$$

Proof. By (4.1)-(4.4) and Divergence Theorem, we have

$$\begin{aligned} \Psi'(t) &= \int_\Omega u_{tt} u dx + \int_\Omega u_t^2 dx + \int_\Omega \nabla u \cdot \nabla u_t + \int_{\Gamma_1} u_t u d\Gamma \\ &= -M(\|\nabla u\|_2^2) \|\nabla u\|_2^2 + \|u\|_p^p + \|u_t\|_2^2 . \end{aligned} \tag{4.19}$$

Then, from (4.18) and by Lemma 4.1 and (4.19), we obtain

$$\begin{aligned} E'_\varepsilon(t) &= E'(t) + \varepsilon \Psi'(t) \\ &= -\|\nabla u_t\|_2^2 - \|u_t\|_{2,\Gamma_1}^2 + \varepsilon \left[-M(\|\nabla u\|_2^2) \|\nabla u\|_2^2 + \|u\|_p^p + \|u_t\|_2^2 \right] \\ &\leq \left(-1 + \frac{3\varepsilon B_1}{2} \right) \|\nabla u_t\|_2^2 - \|u_t\|_{2,\Gamma_1}^2 - k_3 \varepsilon E(t), \end{aligned}$$

where $k_3 = 1 - \frac{\alpha(p-1)}{p} > 0$. Thus

$$E'_\varepsilon(t) \leq \left(-1 + \frac{3\varepsilon B_1}{2} \right) \|\nabla u_t\|_2^2 - k_3 \varepsilon E(t).$$

Considering $\varepsilon \in (0, \varepsilon_1]$ where $\varepsilon_1 = \frac{2}{3B_1}$, then we deduce

$$E'_\varepsilon(t) \leq -k_3 \varepsilon E(t) \text{ for } t \geq 0.$$

Hence, we have the result.

Theorem 4.5. (Global existence and Energy decay). *Assume that* $u_0 \in V$, $u_1 \in L^2(\Omega)$, $I_1(0) > 0$ *and (4.10) holds. Then there exists a unique solution* u *of* $(4.1) - (4.4)$ *satisfying*

$$u \in C([0,\infty); V), u_t \in C([0,\infty); L^2(\Omega)) \cap L^2([0,\infty); V).$$

Moreover, we have the following estimate :

$$E(t) \leq 3E(0) e^{-\frac{\varepsilon}{2} k_3 t},$$

where $t \geq 0$ *and* $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 = \min\{\frac{1}{2k_2}, \varepsilon_1\}$.

Proof. By Lemma 4.3, we have

$$(1 - k_2 \varepsilon) E(t) \leq E_\varepsilon(t) \leq (1 + k_2 \varepsilon) E(t).$$

Since $\varepsilon \leq \frac{1}{2k_2}$, then

$$\frac{1}{2}E(t) \leq E_\varepsilon(t) \leq \frac{3}{2}E(t) \leq 2E(t) \text{ for } t \geq 0,$$

and, therefore,

$$-\varepsilon k_2 E(t) \leq -\frac{\varepsilon k_2}{2} E_\varepsilon(t).$$

Hence, by Lemma 4.4 , we see that

$$E'_\varepsilon(t) \leq -\frac{\varepsilon}{2} k_3 E_\varepsilon(t).$$

Consequently,

$$E_\varepsilon(t) \leq \mathrm{e}^{-\frac{\varepsilon}{2}k_3 t} E_\varepsilon(0).$$

Thus, we get

$$E(t) \leq 3E(0)\mathrm{e}^{-\frac{\varepsilon}{2}k_3 t}, \ t \geq 0.$$

5. Blow-up Property

In this section, we will consider the blow-up property of solutions for $(1.1) - (1.4)$:

(5.1) $$u_{tt} - M(\|\nabla u(t)\|^2)\Delta u - \frac{\partial}{\partial t}\Delta u = f(u),$$

where $M \in C'([0,\infty), R)$ is a function such that $M(s) \geq m_0 > 0, \nabla s \geq 0$ and f satisfies assuaption (A1). In order to state our work, we impose further assumptions on f and M :

(A2) there exists a positive constant δ such that

(5.2) $$sf(s) \geq (2+4\delta)F(s), \text{ for all } s \in R,$$

and

(5.3) $$(2\delta+1)\overline{M}(s) \geq M(s)s, \text{ for all } s \geq 0,$$

where

(5.4) $$F(s) = \int_0^s f(r)dr \quad \text{and} \quad \hat{M}(s) = \int_0^s M(r)dr.$$

Remark.

(1) In this case, we define the energy function of the solution u of (5.1), (1.2) and (1.3) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\hat{M}(\|\nabla u(t)\|_2^2) - \int_\Omega F(u(t))dx, \tag{5.5}$$

for $t \geq 0$. Then we have

$$E(t) = E(0) - \int_0^t \|\nabla u_t(t)\|_2^2 \, dt - \int_0^t \|u_t(t)\|_{2,\Gamma_1}^2 \, dt. \tag{5.6}$$

(2) It is clear that $f(u) = |u|^{p-2} u$, $p > \max\{2, 2(\gamma+1)\}$ satisfies (A2) with $\frac{\gamma}{2} \leq \delta \leq \frac{p-2}{4}$ and $M(s) = m_0 + bs^\gamma$, satisfies (A2) for $m_0 \geq 0$, $b \geq 0$, $m_0 + b > 0$, $\gamma > 0$, $s \geq 0$.

Let u be a solution of (5.1), $(1.2)-(1.3)$ and define

$$a(t) = \|\nabla u\|_2^2 + \int_0^t \|\nabla u\|_2^2 dt + \int_0^t \|u\|_{2,\Gamma_1}^2 dt, \; t \geq 0. \tag{5.7}$$

Lemma 5.1. *Assume that (A1) and (A2) hold, then we have*

$$\begin{aligned} a''(t) - 4(\delta+1) \textstyle\int_\Omega u_t^2 dx \geq (-4-8\delta)E(0) + (4+8\delta) \displaystyle\int_0^t \|\nabla u_t\|_2^2 dt \\ +(4+8\delta) \int_0^t \|u_t\|_{2,\Gamma_1}^2 dt. \end{aligned} \tag{5.8}$$

Proof. From (5.7), we deduce

$$a'(t) = 2\int_\Omega u_t u dx + \|\nabla u\|_2^2 + \|u\|_{2,\Gamma_1}^2, \tag{5.9}$$

and

$$a''(t) = 2\int_\Omega u_t^2 dx - 2M(\|\nabla u(t)\|^2)\|\nabla u\|_2^2 + 2\int f(u)udx. \tag{5.10}$$

Then, by Lemma 4.2, we see that

$$\begin{aligned} & a''(t) - 4(\delta+1)\int_\Omega u_t^2 dx \\ = \; & (-4-8\delta)E(0) + (4+8\delta)\int_0^t \|\nabla u_t\|^2 dt + (4+8\delta)\int_0^t \|u_t\|_{2,\Gamma_1}^2 dt \\ & + \left[(2+4\delta)\hat{M}(\|\nabla u(t)\|_2^2) - 2M(\|\nabla u(t)\|^2)\|\nabla u\|_2^2\right] \\ & + \int_\Omega 2\left[f(u)u - (2+4\delta)F(u)\right] dx. \end{aligned}$$

Therefore, by (5.2)-(5.3), we have(5.8). Now, we consider three different cases on the sign of the initial energy $E(0)$.

(1) If $E(0) < 0$, then from (5.8), we have

$$a'(t) \geq a'(0) - 4(1+2\delta) E(0) t, \ \ t \geq 0.$$

Thus we get $a'(t) > \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2$ for $t > t^*$, where

$$t^* = \max \left\{ \frac{a'(0) - \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \right)}{4(1+2\delta) E(0)}, \ 0 \right\}. \tag{5.11}$$

(2) If $E(0) = 0$, then $a''(t) \geq 0$ for $t \geq 0$. Furthermore, if $a'(0) > \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2$, then $a'(t) > \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2$, $t \geq 0$

(3) For the case that $E(0) > 0$, we first note that

$$2\int_0^t \int_\Omega \nabla u \cdot \nabla u_t dx dt = \|\nabla u(t)\|_2^2 - \|\nabla u_0\|_2^2. \tag{5.12}$$

By Hölder inequality and Young's inequality, we have from (5.12)

$$\|\nabla u(t)\|_2^2 \leq \|\nabla u_0\|_2^2 + \int_0^t \|\nabla u\|_2^2 dt + \int_0^t \|\nabla u_t\|_2^2 dt. \tag{5.13}$$

Similarly, we have

$$2\int_0^t \int_{\Gamma_1} u u_t d\Gamma dt = \|u\|_{2,\Gamma_1}^2 - \|u_0\|_{2,\Gamma_1}^2,$$

and

$$\|u\|_{2,\Gamma_1}^2 \leq \|u_0\|_{2,\Gamma_1}^2 + \int_0^t \|u\|_{2,\Gamma_1}^2 dt + \int_0^t \|u_t\|_{2,\Gamma_1}^2 dt. \tag{5.14}$$

From (5.9) and by Hölder and Young's inequality again, we see that

$$a'(t) \leq \|u\|_2^2 + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_{2,\Gamma_1}^2.$$

Then, from (5.13), (5.14) and (5.7), we deduce

$$\begin{aligned} a'(t) \ &\leq \ a(t) + \|u_t\|_2^2 + \int_0^t \|\nabla u_t\|_2^2 dt + \int_0^t \|u_t\|_{2,\Gamma_1}^2 dt \\ &\quad + \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \end{aligned} \tag{5.15}$$

Hence, by (5.8) and (5.15), we obtain

$$a''(t) - 4(\delta+1)a'(t) + 4(\delta+1)a(t) + K_1 \geq 0,$$

where

$$K_1 = (4+8\delta)E(0) + 4(\delta+1)\left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2\right).$$

Let

$$b(t) = a(t) + \frac{K_1}{4(1+\delta)},\ t > 0.$$

Then $b(t)$ satisfies (2.1). By (2.2), we see that if

$$a'(0) > r_2\left[a(0) + \frac{K_1}{4(1+\delta)}\right] + \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2, \tag{5.16}$$

then $a'(t) > \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2$, $t > 0$. Consequently, we have

Lemma 5.2. *Assume that (A1)-(A2) hold and that either one of the following statements is satisfied :*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $a'(0) > \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2$,

(iii) $E(0) > 0$ *and* (5.16) *holds, then* $a'(t) > \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2$ *for* $t > t_0$, *where* $t_0 = t^*$ *is given by* (5.11) *in case* (i) *and* $t_0 = 0$ *in cases* (ii) *and* (iii).

Now, we will find the estimate for the life span of $a(t)$. Let

$$J(t) = \left[a(t) + (T_1 - t)\left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2\right)\right]^{-\delta},\ \text{for } t \in [0, T_1], \tag{5.17}$$

where $T_1 > 0$ is a certain constant which will be specified later. Then we have,

$$J'(t) = -\delta J(t)^{1+\frac{1}{\delta}}\left[a'(t) - \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2\right)\right]$$

and

$$J''(t) = -\delta J(t)^{1+\frac{2}{\delta}} V(t), \tag{5.18}$$

where

$$\begin{aligned} V(t) \;&=\; a''(t)\left[a(t) + (T_1 - t)\left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2\right)\right] \\ &\quad - (1+\delta)\left[a'(t) - \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2\right)\right]^2. \end{aligned} \tag{5.19}$$

For simplicity of calculation, we denote

$$\begin{aligned} P_1 &= \int_\Omega u^2 dx,\ P_2 = \int_0^t \|u\|_{2,\Gamma_1}^2 dt \\ W_1 &= \int_0^t \|\nabla u\|_2^2 dt,\ W_2 = \int_0^t \|\nabla u_t\|_2^2 dt \\ S_1 &= \int_\Omega u_t^2 dx,\ S_2 = \int_0^t \|u_t(t)\|_{2,\Gamma_1}^2 dt. \end{aligned}$$

By (5.9) and using (5.13), (5.14) and Hölder inequality, we have

$$\begin{aligned} a'(t) &= 2\int_\Omega u_t u dx + \|\nabla u_0\|_2^2 + 2\int_0^t\int_\Omega \nabla u\cdot\nabla u_t dx dt \\ &\quad +2\int_0^t\int_{\Gamma_1} u u_t d\Gamma dt + \|u_0\|_{2,\Gamma_1}^2 \\ &\le 2(\sqrt{S_1P_1}+\sqrt{P_2S_2}+\sqrt{W_1W_2}) + \|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 . \end{aligned} \tag{5.20}$$

By (5.8), we have

$$a''(t) \ge (-4-8\delta)E(0) + 4(1+\delta)(S_1+W_2+S_2). \tag{5.21}$$

Thus, from (5.20) and (5.21), we obtain

$$\begin{aligned} V(t) \ \ge\ & [(-4-8\delta)E(0)+4(1+\delta)(S_1+W_2+S_2)][a(t)+ \\ & (T_1-t)\left(\|\nabla u_0\|_2^2+\|u_0\|_{2,\Gamma_1}^2\right)] \\ & -4(1+\delta)(S_1+W_2+S_2)^2 . \end{aligned}$$

And by (5.17), we have

$$\begin{aligned} V(t) \ge\ & (-4-8\delta)E(0)J(t)^{-\frac{1}{\delta}} + 4(1+\delta)(S_1+W_2+S_2)(T_1-t)(\|\nabla u_0\|_2^2 \\ & +\|u_0\|_{2,\Gamma_1}^2) + 4(1+\delta)[(S_1+W_2+S_2)(P_1+W_1+P_2) \\ & -(S_1+W_2+S_2)^2]. \end{aligned}$$

By Schwarz inequality, the last term in the above inequality is nonnegative. Hence we have

$$V(t) \ge (-4-8\delta)E(0)J(t)^{-\frac{1}{\delta}},\ t\ge t_0. \tag{5.22}$$

Therefore, by (5.18) and (5.22), we get

$$J''(t) \le \delta(4+8\delta)E(0)J(t)^{1+\frac{1}{\delta}},\ t\ge t_0. \tag{5.23}$$

Note that by Lemma 5.2, $J'(t) < 0$ for $t > t_0$. Multiplying (5.23) by $J'(t)$ and integrating from t_0 to t, we get

$$J'(t)^2 \geq \alpha + \beta J(t)^{2+\frac{1}{\delta}} \text{ for } t \geq t_0,$$

where

$$\alpha = \delta^2 J(t_0)^{2+\frac{2}{\delta}} \left[\left(a'(t_0) - \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \right) \right)^2 - 8E(0)\, J(t_0)^{\frac{-1}{\delta}} \right] \tag{5.24}$$

and

$$\beta = 8\delta^2 E(0). \tag{5.25}$$

We observe that

$$\alpha > 0 \;\text{ iff }\; E(0) < \frac{\left[a'(t_0) - \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \right) \right]^2}{8(a(t_0) + (T_1 - t) \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \right))}.$$

Then by Lemma 2.3, there exists a finite time T^* such that $\lim_{t \to T^{*-}} J(t) = 0$ and the upper bound of T^* is estimated respectively according to the sign of $E(0)$. This will imply that

$$\lim_{t \to T^{*-}} \left[\|\nabla u\|_2^2 + \int_0^t \|\nabla u\|_2^2 dt + \int_0^t \|u\|_{2,\Gamma_1}^2 dt \right] = \infty. \tag{5.26}$$

Therefore, we say that u may be blow up in Ω or on the boundary Γ_1 in the above sense. In conclusion, we have

Theorem 5.3. *Assume that (A1)−(A2) hold and that either one of the following statements is satisfied :*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $a'(0) > \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \right)$,

(iii) $0 < E(0) < \frac{\left[a'(t_0) - \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \right) \right]^2}{8\left[a(t_0) + (T_1 - t) \left(\|\nabla u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 \right) \right]}$ *and* (5.16) *holds, then the solution* u *blows up at finite time* T^* *in the sense of* (5.26).

In case (i),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}. \tag{5.27}$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{\alpha}{-\beta}}\right\}$, we have

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - J(t_0)}. \tag{5.28}$$

In case (ii),

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)} \tag{5.29}$$

or

$$T^* \le t_0 + \frac{J(t_0)}{\sqrt{\alpha}}. \tag{5.30}$$

In case (iii),

$$T^* \le \frac{J(t_0)}{\sqrt{\alpha}} \tag{5.31}$$

or

$$T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}} \left\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\right\}, \tag{5.32}$$

where $c = \left(\frac{\alpha}{\beta}\right)^{2+\frac{1}{\delta}}$, here α and β are in (5.24) and (5.25) respectively. Note that in case (i), $t_0 = t^*$ is given in (5.11) and $t_0 = 0$ in case (ii) and (iii).

Remark. The choice of T_1 in (5.17) is possible under some conditions as in [25, 26].

References

1. A. Arosio and S. Spagnolo, *Global solutions of the Cauchy problem for a nonlinear hyperbolic equation*, Vol 6, Pitman, London, 1984.
2. S. Bernstein, On a class of functional partial differential equations, *Izv. Acad. Nauk SSSR. Math.*, **4** (1940), 17-26.
3. E. Bisognin, Hyperbolic parabolic equations with nonlinearity of Kirchhoff-Carrier type, *Revista Mathemática Univ. Complutense Madrid*, **8** (1995), 401-430.
4. M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano and J. S. Prates Filho, Existence and asymptotic behaviour for a degenerate Kirchhoff-Carrier model with viscosity and nonlinear boundary conditions, *Revista Matematica Complutense*, **XIV** (2001), 177-203.

5. G. Chen, A note on the boundary stabilization of the wave equation, *SIAM J. Control and Optimization*, **19** (1981), 106-113.

6. Y. Ebihara, Milla Miand Mranda and L. A. Medeiros, Local solutions for a nonlinear degenerate hyperbolic equation, *Nonlinear Anal*., **10(1)** (1986), 27-40.

7. M. Hosoya and Y. Yamada, On some nonlinear wave equations II: global existence and energy decay of solutions, *J. Fac. Sci. Univ. Toyko Sect. IA Math.*, **38** (1991), 239-250.

8. R. Ikehata, A note on the global solvability of solutions to some nonlinear wave equations with dissipative terms, *Differential and Integral Equations*, **8** (1995), 607-616.

9. Jong Yeoul Park and Jeong Ja Bae, On the existence of solutions for some nondegenerate nonlinear wave equationsof Kirchoff type, *Czechoslovak Mathe. J.*, **52 (127)** (2002), 781-795.

10. G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.

11. V. Komornik and E. Zuazua, A direct method for boundary stabilization of the wave equations, *J. Math. Pures et Appl*., **69** (1990), 33-54.

12. V. Komornik, *Exact Controability and Stabilization the Multiplier Method*, Masson, 1994.

13. J. E. Lagnese, Note on boundary stabilization of the wave equations, *SIAM J. Control and Optimization*, **26** (1988), 1250-1257.

14. I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, *Differential and Integral Equations*, **6** (1993), 507-533.

15. I. Lasiecka and J. Ong, Global solvability and uniform decays of solutions to quasilinear equation with nonlinear boundary dissipation, *Comm. in PDE*, **24** (1999), 2069-2107.

16. J. L. Lions, *On some questions in boundary value problems of mathematical physics*, Contemporary developments in continum mechanics and partial differential equations, G. M. De la Penha and L. A. Medeiros, North-Holland, London, 1978.

17. M. P. Matos, Mathematical analysis of the nonlinear model for the vibrations of a string, *Nonlinear Anal. TMA*, **17** (1991), 1125-1137.

18. M. M. Miranda and L. P. San Gil Jutuca, Existence and boundary stabilization of the solutions for the Kirchhoff equation, *Comm. in PDE*, **24** (1999), 1759-1800.

19. M. R. Li and L. Y. Tsai, Existence and nonexistence of global solutions of some systems of semilinear wave equations, *Nonlinear Anal. TMA*, **54** (2003), 1397-1415.

20. K. Nishihara and Y. Yamada, On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms, *Funkcial Ekvac*., **33** (1990), 151-159.

21. K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation, *Math. Methods in The Appl. Sci.*, **20** (1997), 151-177.

22. K. Ono, On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation, *J. Math. Anal. Appli.*, **216** (1997), 321-342.

23. S. I. Pohozaev, On a class of quasilinear hyperbolic equations, *Math. Sbornik*, **96** (1975), 145-158.

24. J. Quinn and D. L. Rusell, Asymptotic stability and energy decay for solutions of hyperbolic equations with boundary damping, *Proceedings of Royal Society of Edinburg*, **77A** (1977), 97-127.

25. Shun-Tang Wu and L. Y. Tsai, Blow-up of solutions for some nonlinear wave equations of Kirchhoff type with some dissipation, *Nonlinear Anal. TMA*, **65** (2006), 243-264.

26. Shun-Tang Wu and L. Y. Tsai, On global existence and blow-up of solutions for an integro-differential equation with strong damping, *Taiwanese Journal of Mathematics*, **10(4)** (2006), 979-1014.

27. Y. Yamada, On the decay for some nonlinear evolution equations of second order, *Nagoya Math. J.*, **73** (1970), 69-98.

28. E. Zuazua, Uniform stabilization of the wave equation by nonlinear boundary feedback, *SIAM J. Control and Optimization*, **28** (1990), 466-477.

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