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CHARACTERIZATION OF GRAPHS WITH EQUAL DOMINATION NUMBERS AND INDEPENDENCE NUMBERS

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Abstract. The domination number $\gamma(G)$ of a graph G is the minimum cardinality among all dominating sets of G, and the independence number $\alpha(G)$ of G is the maximum cardinality among all independent sets of G. For any graph G, it is easy to see that $\gamma(G) \leq \alpha(G)$. Jou [6] has characterized trees with equal domination numbers and independence numbers. In this paper, we extend the result and present a characterization of connected unicyclic graphs with equal domination numbers and independence numbers.

1. INTRODUCTION

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, we refer to $V(G)$ and $E(G)$ as the vertex set and the edge set, respectively. The cardinality of $V(G)$ is called the *order* of G, denoted by $|G|$. The (*open*) *neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent to x in G, and the *close neighborhood* $N_G[x]$ is $N_G(x) \cup \{x\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \cup_{x \in A} N_G(x)$ and $N_G[A] = \cup_{x \in A} N_G[x]$. The *degree* $deg_G(x)$ of a vertex x is the cardinality of $N_G(x)$. A vertex x is said to be a *leaf* if $deg_G(x) = 1$. Two distinct vertices u and v are called *duplicated* if $N_G(u) = N_G(v)$. The *n-path* is the graph P_n with vertex set $\{v_1, v_2, \cdots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n\}$. The *n-cycle* is the graph C_n with vertex set $\{v_1, v_2, \cdots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1\}$. The *induced subgraph* $\langle A \rangle_G$ induced by $A \subseteq V(G)$ is the graph with vertex set A and the edge set $E(\langle A \rangle_G) = \{uv \in E(G) : u \in A \text{ and } v \in A\}$. For a subset $A \subseteq V(G)$, the *deletion of* A *from* G is the graph $G - A$ by removing all vertices in A and all edges incident to these vertices. A *forest* is a graph with no cycles, and a *tree* is a connected forest. Suppose that u and v are duplicated vertices in a tree, then they are both leaves. For notation and terminology in graphs we follow [1] in general.

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If S and A are vertex subsets of a graph G, then the set S is said to dominate the set A if $A \subseteq N_G[S]$. A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G. If S is a dominating set of G with cardinality $\gamma(G)$, we call S a γ*-set* of G. A set I of vertices in a graph G is an *independent set* of G if no two vertices of I are adjacent in G. The *independence number* $\alpha(G)$ of G is the maximum cardinality among all independent sets of G. If I is an independent set of G with cardinality $\alpha(G)$, we call I an α*-set* of G. For any graph G, it is easy to see that $\gamma(G) \leq \alpha(G)$.

For a graph G, let $\widehat{G}$ be the graph with vertex set $V(\widehat{G}) = V(G) \cup \{\hat{v} : v \in V(G)\}$ and the edge set $E(\widehat{G}) = E(G) \cup \{v\hat{v} : v \in V(G)\}$. A *unicyclic graph* is a graph containing exactly one cycle. For two vertices u and v in a connected graph G, the *distance* $d_G(u, v)$ *between* u *and* v is the length of a shortest u-v path in G. Suppose that G is a connected unicyclic graph containing cycle C. If v is a vertex of G, then the *distance between v and C* is denoted by $d_G(v, C) = \min\{d_G(v, w) : w \in V(C)\}$. The *tail of* G is denoted by $tail(G) = \max\{d_G(x, C) : x \in V(G)\}$.

Over the past few years, several studies have been made on domination or independence [2-8]. Jou [6] has characterized trees with equal domination numbers and independence numbers.

Theorem 1. ([6]). *If* T *is a tree of order* $n \geq 2$*, then* $\gamma(T) = \alpha(T)$ *if and only if* $T = \widehat{H}$ *for some tree* H *of order* $n/2$*.*

In this paper, our aim is to extend the result and present a characterization of connected unicyclic graphs with equal domination numbers and independence numbers.

2. Preliminary

A vertex v of G is a *support vertex* if it is adjacent to a leaf in G. Let $L(G)$ and $U(G)$ denote the set of leaves and support vertices, respectively, of G. Let $A(G) = U(G) \cup L(G)$. If G is a connected unicyclic graph containing cycle C, we denote $M(G)$ the set of the vertices lying on C with degree 2.

We first make some straightforward lemmas.

Lemma 1. ([7]). *Let* G *be a connected graph of order* $n \geq 3$*. Then there exist an* α*-set* I *of* G *with* $L(G) \subseteq I$ *and a* γ*-set* S *of* G *with* $U(G) \subseteq S$*.*

Lemma 2. *Let* G *be a graph with components* $H_1, H_2, \cdots, H_k$*. Then the following all hold.*

(1) $\alpha(G) = \sum_{i=1}^{k} \alpha(H_i)$ *and* $\gamma(G) = \sum_{i=1}^{k} \gamma(H_i)$*.*

(2) $\alpha(G) = \gamma(G)$ *if and only if* $\alpha(H_i) = \gamma(H_i)$ *for every* i*.*

(3) *If G is a forest satisfying $\alpha(G) = \gamma(G)$, then $G = \widehat{H}$ for some forest H of order $|G|/2$.*

Suppose that the leaves $x_1, x_2, \cdots, x_k$ are adjacent to y in G, where $k \geq 2$. Let $G' = G - \{y, x_1, x_2, \cdots, x_k\}$. By Lemma 1, we have that $\alpha(G) \geq k + \alpha(G')$ and $\gamma(G) \leq 1 + \gamma(G')$. Thus, $\gamma(G) \leq 1 + \gamma(G') \leq (k-1) + \alpha(G') \leq \alpha(G) - 1$.

Lemma 3. *If G is a connected graph of order $n \geq 3$ satisfying $\gamma(G) = \alpha(G)$, then G has no duplicated leaves and $|L(G)| = |U(G)|$.*

Lemma 4. *Suppose that G is a connected graph of order $n \geq 3$ satisfying $\alpha(G) = \gamma(G)$. Let $x \in L(G)$, and let $G' = G - N_G[x]$. Then we have that $\alpha(G') = \gamma(G')$ and $\alpha(G - A(G)) = \gamma(G - A(G))$.*

Proof. By Lemma 1, we have that $\alpha(G) \geq 1 + \alpha(G')$ and $\gamma(G) \leq 1 + \gamma(G')$. Hence, $\gamma(G) = \alpha(G) \geq 1 + \alpha(G') \geq 1 + \gamma(G') \geq \gamma(G)$, so all inequalities are equalities. Thus we obtain that $\alpha(G') = \gamma(G')$. Moreover, $\alpha(G - A(G)) = \gamma(G - A(G))$ by repeatedly using the result above. ∎

Lemma 5. *For $n \geq 3$, $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

3. Characterization

Our aim in this section is to give a constructive characterization for the connected unicyclic graphs G satisfying $\alpha(G) = \gamma(G)$.

Theorem 2. *If C_n is a cycle of order $n \geq 3$ satisfying $\alpha(C_n) = \gamma(C_n)$, then $n = 3, 4, 5$ or 7 .*

Theorem 3. *Suppose that G is a connected unicyclic graph containing cycle C such that $\alpha(G) = \gamma(G)$. If there exists a leaf x having $d_G(x, C) = 1$, then $G = \widehat{H}$ for some connected unicyclic graph H of order $|G|/2$.*

Proof. Let $y \in N_G(x)$. So y is lying on C. By Lemma 4, the deletion $G' = G - N_G[x]$ is a forest of order $n - 2$ satisfying $\alpha(G') = \gamma(G')$, where $n = |G|$. By Lemma 2(3), the deletion $G' = \widehat{F}$ for some forest F of order $\frac{n}{2} - 1$. By Lemma 1, we have that $\alpha(G) = |L(G')| + 1 = |F| + 1$.

Suppose to the contrary that there exists a vertex $\hat{z} \in N_G(y)$, where $z \in V(F)$ and $z \notin L(G)$. Then z is a neighbor of some vertex $w \in V(F) \cup \{y\}$. Thus the set $\{y, w\}$ dominates the set $\{x, y, \hat{z}, z\}$, and $S = (V(F) - \{z\}) \cup \{y\}$ is a dominating set of G with cardinality $|S| = |F|$. Hence we have that $\alpha(G) = \gamma(G) \leq |S| = |F| = \alpha(G) - 1$, this is a contradiction. So we obtain that $N_G(y) \subseteq V(F) \cup \{x\}$. Let $H = \langle V(F) \cup \{y\} \rangle_G$. Then we can see that $G = \widehat{H}$ for some connected unicyclic graph H of order $|G|/2$. ∎

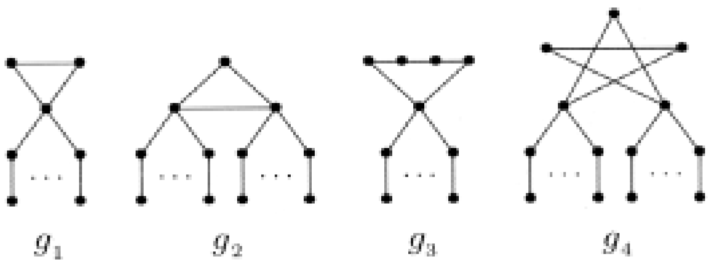

Fig. 1. The families g_1, g_2, g_3 and g_4.

In order to characterization the connected unicyclic graphs G satisfying $\alpha(G) = \gamma(G)$, we introduce four families $g_1,\ g_2,\ g_3$ or g_4 of graphs (see Figure 1).

Theorem 4. *Suppose G is a connected unicyclic graph containing cycle C such that $d_G(x, C) = 2$ for every leaf x of G. Then $\alpha(G) = \gamma(G)$ implies that $G \in g_1,\ g_2,\ g_3$ or g_4.*

Proof. By Lemma 3, we got that $|L(G)| = |U(G)|$. Let $W \subseteq V(C)$ dominate the set $M(G)$ such that $|W|$ is as small as possible. By Lemma 1, we have that $\alpha(G) = \alpha(C) + |L(G)|$ and $\gamma(G) = |W| + |U(G)|$. Thus we obtain that $\alpha(G) = \gamma(G) = |W| + |U(G)| \leq \gamma(C) + |U(G)| \leq \alpha(C) + |L(G)| = \alpha(G)$, so all inequalities are equalities. Thus we have that $|W| = \gamma(C) = \alpha(C)$. By Theorem 2, we got that $G_1 = G - A(G) = C_3, C_4, C_5$ or C_7. Suppose that $G_1 = C_4$. Then we have that $\alpha(C_4) = \gamma(C_4) = 2$ and $|W| = 1$, this is a contradiction. Suppose that $G_1 = C_7$. Then we have that $\alpha(C_7) = \gamma(C_7) = 3$ and $|W| \leq 2$, this is a contradiction. Consequently, we obtain that $G_1 = C_3$ or C_5. ■

Case 1. $G_1 = C_3$. Then $|W| = \gamma(C_3) = \alpha(C_3) = 1$. This implies that $G \in g_1$ or g_2.

Case 2. $G_1 = C_5$. Then $|W| = \gamma(C_5) = \alpha(C_5) = 2$. By the hypothesis of W, we got that $|M(G)| \geq 3$, $\langle M(G) \rangle_G \neq 3P_1$ and $\langle M(G) \rangle_G \neq P_3$. So we obtain that $\langle M(G) \rangle_G = P_4$ or $P_2 \cup P_1$, it implies that $G \in g_3$ or g_4. ■

Let F be a forest, and let C be a cycle of order 3 or 5. For $i = 1, 2, 3$ and 4, $\widetilde{g}(i, \widehat{F})$ is the collection of the connected unicyclic graphs with vertex set $V(C) \cup V(\widehat{F})$, which are obtained from C by attaching some vertices of F to the vertices of $\overline{M(g_i)} = V(C) - M(g_i)$ (see Figure 2).

Theorem 5. *Suppose G is a connected unicyclic graph of order n containing cycle C such that $d_G(x, C) \geq 2$ for every leaf $x \in L(G)$. Then $\alpha(G) = \gamma(G)$ if and only if $G \in \widetilde{g}(i, \widehat{F})$ for some forest F, where $i = 1, 2, 3$ or 4.*

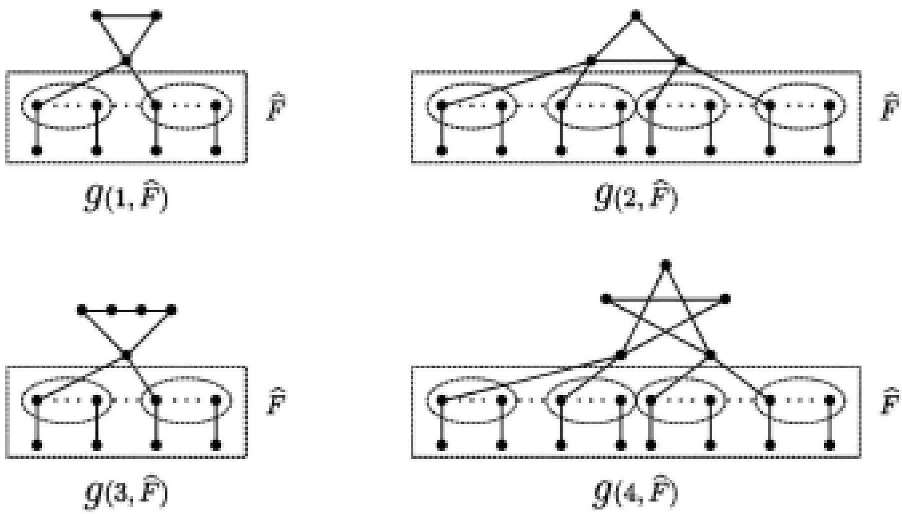

Fig. 2. The families $\widetilde{g}(1, \widehat{F})$, $\widetilde{g}(2, \widehat{F})$, $\widetilde{g}(3, \widehat{F})$, and $\widetilde{g}(4, \widehat{F})$.

Proof. First of all, we will prove the sufficiency. Suppose that $G \in \widetilde{g}(i, \widehat{F})$ for some forest F, where $i = 1, 2, 3$ or 4. Let $W \subseteq V(C)$ dominate the set $M(G)$ such that $|W|$ is as small as possible. Note that $|W| = \gamma(C) = \alpha(C)$. By Lemma 1, we have that $\alpha(G) = |F| + \alpha(C)$ and $\gamma(G) = |F| + |W| = |F| + \gamma(C)$. So we obtain that $\alpha(G) = |F| + \alpha(C) = |F| + \gamma(C) = \gamma(G)$.

We shall prove by induction on n that $\alpha(G) = \gamma(G)$ implies $G \in \widetilde{g}(i, \widehat{F})$ for some forest F, where $i = 1, 2, 3$ or 4. By Theorem 4, it's true if $d_G(x, C) = 2$ for every leaf x of G. So we assume that it's true for all $n' < n$. Suppose now $tail(G) \geq 3$. Let x_0 be a leaf adjacent to y in G such that $d_G(x_0, C) = tail(G) \geq 3$. Note that $|L(G)| = |U(G)|$. Then we can see that $deg_G(y) = 2$, say $N_G(y) = \{x_0, z\}$. By Lemmas 4 and 3, the deletion $G' = G - N_G[x_0]$ is a connected unicyclic graph satisfying $\alpha(G') = \gamma(G')$ such that $|L(G')| = |U(G')|$. Suppose to the contrary that there exists a leaf $x_1 \in L(G')$ having $d_{G'}(x_1, C) = 1$. By Theorem 3, $G' = \widehat{H}$ for some connected unicyclic graph H of order $\frac{n}{2} - 1$. Let $w \in V(C)$ be a vertex such that $\hat{w} \neq z$. Then we can see that $\hat{w} \in L(G)$ and $d_G(\hat{w}, C) = d_G(\hat{w}, w) = 1$, contradicting our assumption that $d_G(x, C) \geq 2$ for every leaf $x \in L(G)$. Hence, we obtain that $d_{G'}(x', C) \geq 2$ for every leaf $x' \in L(G')$. By induction hypothesis, $G' \in \widetilde{g}(i, \widehat{F_1})$ for some forest F_1, where $i = 1, 2, 3$ or 4.

Suppose to the contrary that $z \in L(\widehat{F_1})$, say $z = \hat{v}$ for some $v \in V(F_1)$. Let u be another neighbor of v in G', where $u \in V(F_1) \cup \overline{M(g_i)}$. Then there exists a γ-set S' of G' containing both vertices v and u. Then $S = (S' - \{v\}) \cup \{y\}$ is a dominating set of G with cardinality $|S| = |S'| = \gamma(G') = \alpha(G') = \alpha(G) - 1 = \gamma(G) - 1$, this is a contradiction. So $N_G(y) \cap L(\widehat{F_1}) = \emptyset$, this implies that $z \in V(F_1) \cup \overline{M(g_i)}$ or

$z \in M(g_i)$. Let $F = \langle F_1 \cup \{y\} \rangle$. We consider two cases.

Case 1. $z \in V(F_1) \cup \overline{M(g_i)}$. Then $G \in \widetilde{g}(i, \widehat{F})$ for some forest F, where $i = 1, 2, 3$ or 4.

Case 2. $z \in M(g_i)$. Since $\gamma(G') = \alpha(G') = \alpha(G) - 1 = \gamma(G) - 1$, we can see that $\gamma(M(G')) = \gamma(M(G))$. This implies that $G' \in \widetilde{g}(1, \widehat{F_1})$ or $\widetilde{g}(3, \widehat{F_1})$. Hence, $G \in \widetilde{g}(2, \widehat{F})$ or $\widetilde{g}(4, \widehat{F})$ for some forest F. ■

Theorem 6. *Let G be a connected unicyclic graph of order $n \geq 3$. Then $\alpha(G) = \gamma(G)$ if and only if one of the following holds.*

(1) $G = C_n$ *for* $n = 3, 4, 5$ *or* 7.

(2) $G = \widehat{H}$ *for some connected unicyclic graph* H *of order* $|G|/2$.

(3) $G \in \widetilde{g}(i, \widehat{F})$ *for some forest* F*, where* $i = 1, 2, 3$ *or* 4.

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