TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 4, pp. 1517-1536, August 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

SOME EXISTENCE THEOREMS FOR FUNCTIONAL EQUATIONS AND SYSTEM OF FUNCTIONAL EQUATIONS ARISING IN DYNAMIC PROGRAMMING

Zeqing Liu, Jeong Sheok Ume* and Shin Min Kang

Abstract. In this paper we study the existence, uniqueness and iterative approximation of solutions for a few classes of functional equations arising in dynamic programming of multistage decision processes. By using of monotone iterative technique, we also establish the existence and iterative approximation of coincidence solutions for certain kinds of system of functional equations. The results presented in this paper extend, improve and unify the results due to Bhakta and Mitra [7], Chang [10] and Liu [12] and others.

1. Introduction and Preliminaries

As stated in Bellman and Lee [3], Chang [10] and Chang and Ma [11], the basic forms of the functional equations and the system of functional equations of dynamic programming are as follows:

$$f(x) = \sup_{y\in D} H(x, y, f(T(x, y))), \forall x \in S, \tag{1}$$

$$\begin{cases} f(x) = \sup_{y\in D}\{u(x, y) + G(x, y, g(T(x, y)))\}, \forall x \in S, \\ g(x) = \sup_{y\in D}\{u(x, y) + F(x, y, f(T(x, y)))\}, \forall x \in S. \end{cases} \tag{2}$$

Bellman [5, 6] first established the existence of solutions for some classes of functional equations arising in dynamic programming by using famous Banach fixed-point theorem. Bellman and Roosta [4] constructed an approximation solution for a class of the infinite-stage equation arising in dynamic programming. By utilizing various fixed-point theorems, Baskaran and Subrahmanyam [1], Belbas [2], Bhakta

Received January 16, 2008, accepted November 4, 2008.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 49L20, 49L99, 90C39.
Key words and phrases: Dynamic programming, Functional equation, System of functional equations, Solution, Coincidence solutions, Fixed point, Nonexpansive mapping.
*Corresponding author.

and Mitra [7], Bhakta and Choudhury [8], Chang [10], Liu [12-14], Liu and Ume [15] and others obtained the existence, uniqueness and iterative approximation of solutions for the functional equations (1) or the system of functional equations (2). By using monotone iterative technique, Chang [10], Chang and Ma [11] and Liu [14] established the existence and iterative approximation of coincidence solutions for the system of functional equations (2).

In this paper we introduce and study the following more general functional equations and system of functional equations arising in dynamic programming of multistage decision processes:

$$\begin{aligned} f(x) &= \sup_{y\in D} \text{opt}\{p(x,y), A(x,y,f(a(x,y))), q(x,y) \\ &\qquad + B(x,y,f(b(x,y)))\}, \forall x \in S, \end{aligned} \tag{3}$$

$$f(x) = \sup_{y\in D} \text{opt}\{p(x,y), f(a(x,y)), q(x,y) + f(b(x,y))\}, \forall x \in S, \tag{4}$$

$$\begin{cases} f(x) = \sup\limits_{y\in D} \text{opt}\{p(x,y), A(x,y,g(a(x,y))), q(x,y) \\ \qquad\qquad + B(x,y,g(b(x,y)))\}, \forall x \in S, \\ g(x) = \sup\limits_{y\in D} \text{opt}\{u(x,y), C(x,y,f(c(x,y))), v(x,y) \\ \qquad\qquad + H(x,y,f(h(x,y)))\}, \forall x \in S, \end{cases} \tag{5}$$

where opt denotes the sup or inf. Under certain conditions, we establish the existence, uniqueness and iterative approximation of solutions for the functional equations (3) and (4). On the other hand, we also prove the existence and iterative approximation of coincidence solutions for the system of functional equations (5) by using monotone iterative technique. Our results extend, improve and unify the corresponding results due to Bhakta and Mitra [7], Chang [10] and Liu [14] and others.

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and I denotes the identity mapping on $\mathbb{R}^+$. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be real Banach spaces, let $S \subset X$ be the state space, let $D \subset Y$ be the decision space. Let $B(S)$ be the set of all real-valued bounded functions on S, and $BB(S)$ denotes the set of all real-valued functions on S that are bounded on bounded subsets of S. According to the ordinary addition of functions and scalar multiplication and endowing a norm $\|f\|_1 = \sup_{x\in S} |f(x)|$ for $f \in B(S)$, then $(B(S), \|\cdot\|_1)$ is a Banach space. For any $k \geq 1$ and $f, g \in BB(S)$, let

$$d_k(f,g) = \sup\{|f(x) - g(x)| : x \in \overline{B}(0,k)\},$$

$$d(f,g)=\sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\frac{d_k(f,g)}{1+d_k(f,g)},$$

where $\overline{B}(0,k)=\{x : x\in S \text{ and } \|x\|\le k\}$, and $\{d_k\}_{k\ge1}$ is a countable family of pseudometrics on $BB(S)$. A sequence $\{f_n\}_{n\ge0}$ in $BB(S)$ is said to converge to a point $f\in BB(S)$ if $\lim_{n\to\infty} d_k(f_n,f)=0$ for each $k\ge1$, and to be a Cauchy sequence if $\lim_{n,m\to\infty} d_k(f_m,f_n)=0$ for each $k\ge1$. It is easy to verify that $(BB(S),d)$ is a complete metric space. A metric space (M,ρ) is said to be metrically convex if for each $(x,y)\in M$, there is a $z\ne x,y$ for which $\rho(x,y)=\rho(x,z)+\rho(z,y)$. Clearly any Banach space is metrically convex. Define

$$\Phi_1=\{\varphi:\varphi:\mathbb{R}^+\to\mathbb{R}^+ \text{ satisfies } \varphi(t)<t,\ \forall t>0\},$$

$$\Phi_2=\{\varphi:\varphi:\mathbb{R}^+\to\mathbb{R}^+ \text{ is nondecreasing and } \sum_{n=0}^{\infty}\varphi^n(t)<t,\ \forall t>0\},$$

$$\Phi_3=\{(\varphi,\psi):\varphi \text{ and } \psi:\mathbb{R}^+\to\mathbb{R}^+ \text{ are nondecreasing and} \sum_{n=0}^{\infty}\psi(\varphi^n(t))<+\infty,\forall t>0\},$$

$$\Phi_4=\{(\varphi,\psi):(\varphi,\psi)\in\Phi_3 \text{ and } \psi(t)>0,\ \forall t>0\},$$

$$\Phi_5=\{\varphi:\varphi\in\Phi_1 \text{ is nondecreasing and continuous on the right }\}.$$

Lemma 1.1.

(i) *Φ_5 is a proper subset of Φ_1;*

(ii) *If $(\varphi,\psi)\in\Phi_4$, then $\varphi(t)<t$ and $\lim_{n\to\infty}\varphi^n(t)=0,\ \forall t>0$.*

Proof. *(i) It is clear that $\Phi_5\subseteq\Phi_1$. Note that the mapping $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ defined by*

$$\varphi(t)=\begin{cases}\dfrac{t}{1+t} & \text{for } t\in[0,1]\\[2ex] \dfrac{t}{1+t^2} & \text{for } t>1\end{cases}$$

satisfies that $\varphi\in\Phi_1$ and $\varphi\notin\Phi_5$ since $\varphi(1)=\frac{1}{2}>\frac{2}{5}=\varphi(2)$.

(iii) *First of all we show that*

$$\varphi(t)<t,\ \forall t>0. \tag{6}$$

Suppose that $\varphi(t)\ge t$ for some $t>0$. Notice that $(\varphi,\psi)\in\Phi_4$ implies that $\psi(t)>0$. Since φ and ψ are nondecreasing and $\sum_{n=0}^{\infty}\psi(\varphi^n(t))<\infty$, it follows that

$$0<\psi(t)\le\psi(\varphi(t))\le\psi(\varphi^2(t))\le\cdots\le\psi(\varphi^n(t))\to0 \text{ as } n\to\infty,$$

that is, $0<\psi(t)\le0$, which is a contradiction. Thus (6) holds.

We now show that

$$\lim_{n\to\infty} \varphi^n(t) = 0,\ \forall t > 0. \tag{7}$$

It follows from (6) that for each $t > 0$,

$$0 \le \varphi^{n+1}(t) \le \varphi^n(t) \le \cdots \le \varphi(t) < t,\ \forall n \ge 1,$$

which implies that $\lim_{n\to\infty} \varphi^n(t) = a \ge 0$ and

$$0 \le \psi(a) \le \psi(\varphi^n(t)) \ \to 0 \text{ as } n \to \infty,$$

which yields that $\psi(a) = 0$ and $a = 0$. That is, (7) holds. This completes the proof.

Remark 1.1. In case $\psi(t) = Mt,\ \forall t \in \mathbb{R}^+$, where M is a positive constant, and $\varphi \in \Phi_2$, then $(\varphi, \psi) \in \Phi_4$.

Lemma 1.2. (See Ref. 9.) *Suppose that (M, ρ) is a completely metrically convex metric space and that $f : M \to M$ satisfies*

$$\rho(f(x), f(y)) \le \varphi(\rho(x, y)),\ \forall x, y \in M,$$

where $\varphi : \overline{P} \to \mathbb{R}^+$ satisfies $\varphi(t) < t,\ \forall t > 0$, $P = \{\rho(x, y) : x, y \in M\}$ and $\overline{P}$ denotes the closure of P. Then f has a unique fixed point $u \in M$ and $\lim_{n\to\infty} f^n(x) = u$ for each $x \in M$.

Lemma 1.3. *Let A be a set, p and $q : A \to \mathbb{R}$ be mappings such that* $\text{opt}_{y\in A} p(y)$, $\text{opt}_{y\in A} q(y)$ *and* $\sup_{y\in A} |p(y) - q(y)|$ *are bounded. Then*

$$|\text{opt}_{y\in A} p(y) - \text{opt}_{y\in A} q(y)| \le \sup_{y\in A} |p(y) - q(y)|.$$

Proof. Note that

$$p(y) \le q(y) + \sup_{y\in A} |p(y) - q(y)|,\ \forall y \in A, \tag{8}$$

which implies that

$$\begin{aligned} \text{opt}_{y\in A} p(y) &\le \text{opt}_{y\in A}\{q(y) + \sup_{y\in A} |p(y) - q(y)|\} \\ &= \text{opt}_{y\in A} q(y) + \sup_{y\in A} |p(y) - q(y)|. \end{aligned} \tag{9}$$

Similarly we also have

$$\text{opt}_{y\in A} q(y) \le \text{opt}_{y\in A} p(y) + \sup_{y\in A} |q(y) - p(y)|. \tag{10}$$

Thus (8) follows from (9) and (10). This completes the proof.

Lemma 1.4. (See Ref. 15). *Let* a, b, c *and* d *be in* $\mathbb{R}$. *Then*

$$|\text{opt } \{a, b\} - \text{opt } \{c, d\}| \le \max\{|a - c|, |b - d|\}.$$

2. Existence, Uniqueness and Iterative Approximation of Solutions for a Few Classes of Functional Equations

Theorem 2.1. *Let* $a, b : S \times D \to S$, $p, q : S \times D \to \mathbb{R}$, A *and* $B : S \times D \times \mathbb{R} \to \mathbb{R}$ *satisfy the following conditions:*

($C1$) p, q, A *and* B *are bounded;*

($C2$) *there exists some* $\varphi \in \Phi_1$ *with*

$$\begin{aligned} &\max\{|A(x, y, u(a(x, y))) - A(x, y, v(a(x, y)))|, \\ &\qquad |B(x, y, u(b(x, y))) - B(x, y, v(b(x, y)))|\} \\ &\le \varphi(\|u - v\|_1),\ \forall (x, y, u, v) \in S \times T \times B(S) \times B(S). \end{aligned}$$

Then the functional equation (3) has a unique solution $w \in B(S)$ *and* $\{H^n z\}_{n\ge1}$ *converges to* w *for any* $z \in B(S)$, *where*

$$\begin{aligned} Hz(x) = &\sup_{y\in D} \text{opt}\{p(x, y), A(x, y, z(a(x, y))), q(x, y) \\ &+B(x, y, z(b(x, y)))\}, \forall x \in S. \end{aligned} \tag{11}$$

Proof. It is easy to verify that (C1) and (11) imply that H is a self-mapping on $B(S)$. Thus H has a unique fixed point $w \in B(S)$ if and only if the functional equation (3) has a unique solution $w \in B(S)$. Set

$$\begin{aligned} C(x, y, z) = &\text{opt}\{p(x, y), A(x, y, z(a(x, y))), q(x, y) \\ &+ B(x, y, z(b(x, y)))\},\ \forall (x, y, z) \in S \times D \times B(S). \end{aligned} \tag{12}$$

Let x be an arbitrary element in S. For any $u, v \in B(S)$ and $\varepsilon > 0$, by virtue of (11) and (12) we deduce that there exist $y, h \in D$ with

$$Hu(x) < C(x, y, u) + \varepsilon, \tag{13}$$

$$Hv(x) < C(x, h, v) + \varepsilon, \tag{14}$$

$$Hu(x) \geq C(x,h,u), \tag{15}$$

$$Hv(x) \geq C(x,y,v). \tag{16}$$

Using (12), (13), (16), (C2) and Lemma 1.4, we derive that

$$\begin{aligned} Hu(x) - Hv(x) <& C(x,y,u) - C(x,y,v) + \varepsilon \\ \leq& \max\{|A(x,y,u(a(x,y))) - A(x,y,v(a(x,y)))|, \\ & |B(x,y,u(b(x,y))) - B(x,y,v(b(x,y)))|\} + \varepsilon \\ \leq& \varphi(\|u-v\|_1) + \varepsilon. \end{aligned} \tag{17}$$

In view of (12), (14), (15) , (C2) and Lemma 1.4, we obtain that

$$\begin{aligned} Hu(x) - Hv(x) >& C(x,h,u) - C(x,h,v) - \varepsilon \\ \geq& -\max\{|A(x,h,u(a(x,y))) - A(x,h,v(a(x,y)))|, \\ & |B(x,h,u(b(x,y))) - B(x,h,v(b(x,y)))|\} - \varepsilon \\ \geq& -\varphi(\|u-v\|_1) - \varepsilon. \end{aligned} \tag{18}$$

(17) and (18) mean that

$$\|Hu-Hv\|_1 = \sup_{x\in D} |Hu(x) - Hv(x)| \leq \varphi(\|u-v\|_1) + \varepsilon, \forall \varepsilon > 0, \forall u, v \in B(S). \tag{19}$$

Letting $\varepsilon \to 0$ in (19), we conclude that

$$\|Hu - Hv\|_1 \leq \varphi(\|u-v\|_1),\ \forall u, v \in B(S). \tag{20}$$

Therefore the conclusion of Theorem 2.1 follows immediately from (20) and Lemma 1.2. This completes the proof.

Theorem 2.2. *Let $a, b : S \times D \to S$, p and $q : S \times D \to \mathbb{R}$ be mappings. If there exists $(\varphi, \psi) \in \Phi_4$ satisfying*

(*C3*) $\max\{|p(x,y)|, |q(x,y)|\} \leq \psi(\|x\|), \forall (x,y) \in S \times D$;

(*C4*) $\max\{\|a(x,y)\|, \|b(x,y)\|\} \leq \varphi(\|x\|), \forall (x,y) \in S \times D$,

then the functional equation (4) possesses a solution $w \in BB(S)$ that satisfies the following conditions:

(*C5*) $\lim_{n\to\infty} w_n = w$, *where $\{w_n\}_{n\geq 0}$ is defined by*

$$w_0(x) = \sup_{y\in D} \text{opt}\{p(x,y), q(x,y)\},\ \forall x \in S,$$

$$w_n(x) = \sup_{y \in D} \text{opt}\{p(x,y), w_{n-1}(a(x,y)), q(x,y)+w_{n-1}(b(x,y))\}, \ \forall x \in S, \ \forall n \geq 1;$$

($C6$) $\lim_{n\to\infty} w(x_n) = 0$, *where* $\{x_n\}_{n\geq 0}$ *satisfies the following*

($C7$) x_0 *is an arbitrary element in* S *and* $\{y_n\}_{n\geq 0}$ *is any sequence in* D *and*

$$x_n \in \{a(x_{n-1}, y_n), b(x_{n-1}, y_n)\}, \forall n \geq 1.$$

Furthermore, the solution w *in* $BB(S)$ *of the functional equation* (4) *is also unique with respect to conditions* ($C6$) *and* ($C7$).

Proof. For any $(x, y, z) \in S \times D \times BB(S)$, let

$$C(x,y,z) = \text{opt}\{p(x,y), z(a(x,y)), q(x,y) + z(b(x,y))\}, \tag{21}$$

$$Hz(x) = \sup_{y \in D} C(x,y,z). \tag{22}$$

First of all we assert that H is a nonexpansive mapping on $BB(S)$. Let k be a positive integer and let z be in $BB(S)$. For any $(x,y) \in \overline{B}(0,k) \times D$, (C4) and Lemma 1.1 mean that

$$\max\{\|a(x,y)\|, \|b(x,y)\|\} \leq \varphi(\|x\|) \leq \|x\| \leq k. \tag{23}$$

It follows from (23) that there exists $r(k) > 0$ satisfying

$$\max\{|z(a(x,y))|, |z(b(x,y))|\} \leq r(k), \ \forall (x,y) \in \overline{B}(0,k) \times D. \tag{24}$$

In the light of (C4), (21), (24) and Lemma 1.3, we know that

$$\begin{aligned} |C(x,y,z)| &\leq \sup\{|p(x,y)|, |z(a(x,y))|, |q(x,y)| + |z(b(x,y))|\} \\ &\leq \sup\{\psi(\|x\|), r(k), \psi(\|x\|) + r(k)\} \\ &\leq \psi(k) + r(k), \ \forall (x,y) \in \overline{B}(0,k) \times D. \end{aligned} \tag{25}$$

By virtue of (22), (25) and Lemma 1.3, we deduce that

$$|Hz(x)| \leq \sup_{y \in D} |C(x,y,z)| \leq \psi(k) + r(k), \ \forall x \in \overline{B}(0,k),$$

which implies that H is bounded on bounded subsets of S. Hence H maps $BB(S)$ into itself. For any $\varepsilon > 0$ and $(x,u,v) \in \overline{B}(0,k) \times BB(S) \times BB(S)$, where k is a positive integer, there exist $y, h \in D$ satisfying (13)-(16). It follows from (C4), Lemmas 1.1 and 1.3 and (13)-(16) that

$$\begin{aligned}|Hu(x)-Hv(x)| &\le \max\{|C(x,y,u)-C(x,y,v)|, |C(x,h,u)-C(x,h,v)|\}+\varepsilon \\ &\le \max\{|u(a(x,y))-v(a(x,y))|, |u(b(x,y))-v(b(x,y))|, \\ &\qquad |u(a(x,h))-v(a(x,h))|, |u(b(x,h))-v(b(x,h))|\}+\varepsilon \\ &\le d_k(u,v)+\varepsilon,\end{aligned}$$

which yields that

$$d_k(Hu,Hv) \le d_k(u,v)+\varepsilon. \tag{26}$$

Letting $\varepsilon \to 0$ in (26), we get that

$$d_k(Hu,Hv) \le d_k(u,v),\ \forall u,v \in BB(S). \tag{27}$$

(27) ensures that

$$d(Hu,Hv)=\sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\frac{d_k(Hu,Hv)}{1+d_k(Hu,Hv)}\le\sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\frac{d_k(u,v)}{1+d_k(u,v)}=d(u,v),\quad \forall u,v \in BB(S). \tag{28}$$

This gives that H is nonexpansive.

We now claim that for each $x \in S$

$$|w_n(x)| \le \sum_{i=0}^{n}\psi(\varphi^i(\|x\|)),\ \forall n \ge 0. \tag{29}$$

It follows from Lemma 1.3 and (C5) that

$$|w_0(x)| \le \sup_{y\in D}\sup\{|p(x,y)|, |q(x,y)|\} \le \psi(\|x\|),$$

that is, (29) holds for $n=0$. Suppose that (29) holds for some $n \ge 0$. Using (21), (C3), (C4) and Lemma 1.3 , we conclude that

$$\begin{aligned} &|C(x,y,w_n)| \\ &=|\mathrm{opt}\{p(x,y), w_n(a(x,y)), q(x,y)+w_n(b(x,y))\}| \\ &\le \sup\{|p(x,y)|, |w_n(a(x,y))|, |q(x,y)|+|w_n(b(x,y))|\} \\ &\le \sup\{\psi(\|x\|), \sum_{i=0}^{n}\psi(\varphi^i(\|a(x,y)\|), \psi(\|x\|)+\sum_{i=0}^{n}\psi(\varphi^i(\|b(x,y)\|))\} \\ &\le \sup\{\psi(\|x\|), \sum_{i=0}^{n}\psi(\varphi^{i+1}(\|x\|)), \psi(\|x\|)+\sum_{i=0}^{n}\psi(\varphi^{i+1}(\|x\|))\} \\ &\le \sum_{i=0}^{n+1}\psi(\varphi^i(\|x\|)),\ \forall y \in D.\end{aligned} \tag{30}$$

In view of (C5) and (30), we arrive at

$$|w_{n+1}(x)| \le |\sup_{y\in D} C(x,y,w_n)| \le \sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)).$$

Hence (29) holds for any $n \ge 0$.

We next show that $\{w_n\}_{n\ge 0}$ is a Cauchy sequence in $BB(S)$. Let k be an arbitrary positive integer and x_0 be in $\overline{B}(0,k)$. For any $\varepsilon > 0$ and $n > 0$ and $m > 0$, by (C5) we infer that there exist $y, y' \in D$ such that

$$w_{n+m}(x_0) < C(x_0, y, w_{n+m-1}) + 2^{-1}\varepsilon, \tag{31}$$

$$w_n(x_0) \ge C(x_0, y, w_{n-1}), \tag{32}$$

$$w_n(x_0) < C(x_0, y', w_{n-1}) + 2^{-1}\varepsilon, \tag{33}$$

$$w_{n+m}(x_0) \ge C(x_0, y', w_{n+m-1}). \tag{34}$$

On account of (21), (22), (31), (32) and Lemma 1.3, we know that

$$\begin{aligned} w_{n+m}(x_0) - w_n(x_0) \le& C(x_0, y, w_{n+m-1}) - C(x_0, y, w_{n-1}) + 2^{-1}\varepsilon \\ \le& \sup\{|w_{n+m-1}(a(x_0,y)) - w_{n-1}(a(x_0,y))|, \\ & |w_{n+m-1}(b(x_0,y)) - w_{n-1}(b(x_0,y))|\} + 2^{-1}\varepsilon. \end{aligned} \tag{35}$$

In terms of (21), (22), (33), (34) and Lemma 1.3, we get that

$$\begin{aligned} w_{n+m}(x_0) - w_n(x_0) \ge& C(x_0, y', w_{n+m-1}) - C(x_0, y', w_{n-1}) - 2^{-1}\varepsilon \\ \ge& -\sup\{|w_{n+m-1}(a(x_0,y')) - w_{n-1}(a(x_0,y'))|, \\ & |w_{n+m-1}(b(x_0,y')) - w_{n-1}(b(x_0,y'))|\} - 2^{-1}\varepsilon. \end{aligned} \tag{36}$$

Combining (35) and (36), we deduce that there exist $y_1 \in \{y, y'\} \subseteq D$ and $x_1 \in \{a(x_0,y_1),\, b(x_0,y_1)\}$ with

$$\begin{aligned} |w_{n+m}(x_0) - w_n(x_0)| \le& \sup\{|w_{n+m-1}(a(x_0,y)) - w_{n-1}(a(x_0,y))|, \\ & |w_{n+m-1}(b(x_0,y)) - w_{n-1}(b(x_0,y))|, \\ & |w_{n+m-1}(a(x_0,y')) - w_{n-1}(a(x_0,y'))|, \\ & |w_{n+m-1}(b(x_0,y')) - w_{n-1}(b(x_0,y'))|\} + 2^{-1}\varepsilon \\ =& |w_{n+m-1}(x_1) - w_{n-1}(x_1)| + 2^{-1}\varepsilon. \end{aligned} \tag{37}$$

Similarly there exist $y_i \in D$, $x_i \in \{a(x_{i-1}, y_i), b(x_{i-1}, y_i)\}$, $i \in \{2, 3, \cdots, n\}$ such that

$$(38)\quad \begin{aligned} |w_{n+m-1}(x_1) - w_{n-1}(x_1)| \le& |w_{n+m-2}(x_2) - w_{n-2}(x_2)| + 2^{-2}\varepsilon, \\ |w_{n+m-2}(x_2) - w_{n-2}(x_2)| \le& |w_{n+m-3}(x_3) - w_{n-3}(x_3)| + 2^{-3}\varepsilon, \\ &\vdots \\ |w_{m+1}(x_{n-1}) - w_1(x_{n-1})| \le& |w_m(x_n) - w_0(x_n)| + 2^{-n}\varepsilon. \end{aligned}$$

Since φ is nondecreasing, it follows from (C4) and Lemma 1.1 that

$$(39)\quad \begin{aligned} \|x_n\| \le& \sup\{\|a(x_{n-1}, y_n)\|, \|b(x_{n-1}, y_n)\|\} \\ \le& \varphi(\|x_{n-1}\|) \le \cdots \le \varphi^n(\|x_0\|) \le \varphi^n(k). \end{aligned}$$

Thus there exists some positive integer $j > 2$ such that

$$(40)\quad \sum_{i=j-1}^{\infty} \psi(\varphi^i(k)) < \varepsilon$$

because $\sum_{n=0}^{\infty} \psi(\varphi^n(k)) < +\infty$. It follows from (29), (37)-(40) and Lemma 1.1 that for any $n > j$ and $m \ge 1$,

$$\begin{aligned} |w_{n+m}(x_0) - w_n(x_0)| <& |w_m(x_n) - w_0(x_n)| + \varepsilon \\ \le& |w_m(x_n)| + |w_0(x_n)| + \varepsilon \\ \le& \sum_{i=0}^{m} \psi(\varphi^i(\|x_n\|)) + \psi(\|x_n\|) + \varepsilon \\ \le& \sum_{i=0}^{m} \psi(\varphi^{i+n}(k)) + \psi(\varphi^n(k)) + \varepsilon \\ \le& \sum_{i=n-1}^{n+m} \psi(\varphi^i(k)) + \varepsilon \\ <& 2\varepsilon, \end{aligned}$$

which yields that

$$d_k(w_{n+m}, w_n) = \sup\{|w_{n+m}(x) - w_n(x)| : x \in \overline{B}(0, k)\} \le 2\varepsilon,\ \forall n \ge j+1,\ m \ge 1.$$

That is, $\{w_n\}_{n\ge0}$ is a Cauchy sequence in $(BB(S), d)$ and hence it converges to some $w \in BB(S)$. Note that (28) implies that

$$d(Hw, w) \le d(Hw, Hw_n) + d(w_{n+1}, w) \le d(w, w_n) + d(w_{n+1}, w) \to 0 \text{ as } n \to \infty,$$

which means that $Hw = w$. That is, the functional equation (4) has a solution $w \in BB(S)$.

Now we show that (C6) holds. Let ε be an arbitrary positive number and set $k = [\|x_0\|] + 1$, here $[t]$ denotes the largest integer not exceeding t. Since $\lim_{i\to\infty} d_k(w_i, w) = 0$ and $\sum_{n=0}^{\infty} \psi(\varphi^n(k)) < \infty$, it follows that there exist positive integers i and l satisfying

$$(41) \qquad d_k(w_i, w) < 2^{-1}\varepsilon \text{ and } \sum_{j=n}^{\infty} \psi(\varphi^j(k)) < 2^{-1}\varepsilon, \ \forall n \geq l.$$

In view of (29), (39), (41) and Lemma 1.1, we know that for $n \geq l$

$$\begin{aligned} |w(x_n)| \leq& |w(x_n) - w_i(x_n)| + |w_i(x_n)| \\ \leq& d_k(w_i, w) + \sum_{m=0}^{i} \psi(\varphi^m(\|x_n\|)) \\ \leq& 2^{-1}\varepsilon + \sum_{m=0}^{i} \psi(\varphi^{m+n}(\|x_0\|)) \\ <& \varepsilon, \end{aligned}$$

which implies that $\lim_{n\to\infty} w(x_n) = 0$.

Finally we show that w is a unique solution of the functional equation (4) in $BB(S)$ satisfying conditions (C6) and (C7). Suppose that h is another solution of the functional equation (4) in $BB(S)$ satisfying conditions (C6) and (C7). Let ε be an arbitrary positive number and x_0 be any element in S. Since w and h are solutions of the functional equation (4), there exist $y, y' \in D$ satisfying

$$(42) \qquad w(x_0) < \text{opt}\{p(x_0, y), w(a(x_0, y)), q(x_0, y) + w(b(x_0, y))\} + 2^{-1}\varepsilon,$$

$$(43) \qquad h(x_0) < \text{opt}\{p(x_0, y'), h(a(x_0, y')), q(x_0, y') + h(b(x_0, y'))\} + 2^{-1}\varepsilon,$$

$$(44) \qquad w(x_0) \geq \text{opt}\{p(x_0, y'), w(a(x_0, y')), q(x_0, y') + w(b(x_0, y'))\},$$

$$(45) \qquad h(x_0) \geq \text{opt}\{p(x_0, y), h(a(x_0, y)), q(x_0, y) + h(b(x_0, y))\}.$$

According to Lemma 1.3 and (42)-(45), we conclude that there exist $y_1 \in \{y, y'\} \subseteq D$ and $x_1 \in \{a(x_0, y_1), b(x_0, y_1)\}$ such that

$$(46) \qquad \begin{aligned} & |w(x_0) - h(x_0)| \\ <& \sup\{|w(a(x_0, y)) - h(a(x_0, y))|, |w(b(x_0, y)) - h(b(x_0, y))|, \\ & |w(a(x_0, y')) - h(a(x_0, y'))|, |w(b(x_0, y')) - h(b(x_0, y'))|\} + 2^{-1}\varepsilon \\ =& |w(x_1) - h(x_1)| + 2^{-1}\varepsilon. \end{aligned}$$

Using the same argument as before, we can find sequences $\{y_n\}_{n\geq 1} \subseteq D$ and $\{x_n\}_{n\geq 0} \subseteq S$ with $x_n \in \{a(x_{n-1}, y_n), b(x_{n-1}, y_n)\},\ \forall n \geq 1$,

$$|w(x_{n-1}) - h(x_{n-1})| < |w(x_n) - h(x_n)| + 2^{-n}\varepsilon,\ \forall n \geq 1. \tag{47}$$

Combining (46) and (47), we deduce that

$$|w(x_0) - h(x_0)| < |w(x_n) - h(w_n)| + (1 - 2^{-n})\varepsilon,\ \forall n \geq 1. \tag{48}$$

Letting $n \to \infty$ in (48), by (C6) and (C7) we derive that

$$|w(x_0) - h(x_0)| \leq \varepsilon,$$

which implies that $w(x_0) = h(x_0)$ by letting $\varepsilon \to 0$. That is, $w = h$. This completes the proof.

As in the proofs of Theorems 2.1 and 2.2, we immediately conclude the following results.

Theorem 2.3. *Let $b : S \times D \to S$, $q : S \times D \to \mathbb{R}$ and $B : S \times D \times R \to \mathbb{R}$ satisfy the following conditions:*

(*C8*) *q and B are bounded;*

(*C9*) *there exists some $\varphi \in \Phi_1$ with*

$$|B(x, y, u(b(x, y))) - B(x, y, v(b(x, y)))|$$
$$\leq \varphi(\|u - v\|_1), \forall (x, y, u, v) \in S \times T \times B(S) \times B(S).$$

Then the functional equation

$$f(x) = \sup_{y\in D}\{q(x, y) + B(x, y, f(b(x, y)))\},\ \forall x \in S, \tag{49}$$

possesses a unique solution $w \in B(S)$ and $\{H^n z\}_{n\geq 0}$ converges to w for any $z \in B(S)$, where

$$Hz(x) = \sup_{y\in D}\{q(x, y) + B(x, y, z(b(x, y)))\},\ \forall x \in S. \tag{50}$$

Remark 2.1. Theorem 2.3 extends Theorem 2.1 of Bhakta and Mitra [7] in the following aspects:

(i) It follows from Lemma 1.1 that $\Phi_5 \subsetneqq \Phi_1$. Thus the condition imposed on φ in Theorem 2.3 is more weaker than that in Theorem 2.1 [7].

(ii) Theorem 2.1 [7] shows only the existence and uniqueness of solution for the functional equation (49). But Theorem 2.3 proves not only the existence and uniqueness of solution for the functional equation (49), but also discusses the convergence of the Picard iteration.

Theorem 2.4. *Let* $b : S \times D \to S$, $q : S \times D \to \mathbb{R}$ *and* $B : S \times D \times \mathbb{R} \to \mathbb{R}$ *satisfy* (*C8) and* (*C9). Then the functional equation*

$$f(x) = \sup_{y \in D} \text{opt}\{q(x, y), A(x, y, f(b(x, y)))\},\ \forall x \in S \tag{51}$$

possesses a unique solution $w \in B(S)$ *and* $\{H^n z\}_{n \geq 0}$ *converges to* w *for any* $z \in B(S)$*, where*

$$Hz(x) = \sup_{y \in D} \text{opt}\{q(x, y), A(x, y, z(b(x, y)))\},\ \forall x \in S. \tag{52}$$

Theorem 2.5. *Let* $b : S \times D \to S$ *and* $q : S \times D \to \mathbb{R}$ *be mappings. If there exists* $(\varphi, \psi) \in \Phi_4$ *satisfying*

(*C10*) $|q(x, y)| \leq \psi(\|x\|),\ \forall (x, y) \in S \times D$;

(*C11*) $\|b(x, y)\| \leq \varphi(\|x\|),\ \forall (x, y) \in S \times D$,

then the functional equation

$$f(x) = \sup_{y \in D}\{q(x, y) + f(b(x, y))\},\ \forall x \in S \tag{53}$$

possesses a solution $w \in B(S)$ *that satisfies the following conditions:*

(*C12*) $\lim_{n\to\infty} w_n = w$*, where* $\{w_n\}_{n\geq 0}$ *is defined by*

$$w_0(x) = \sup_{y \in D} q(x, y),\ \forall x \in S,$$

$$w_n(x) = \sup_{y \in D}\{q(x, y) + w_{n-1}(b(x, y))\},\ \forall x \in S,\ \forall n \geq 1;$$

(*C13*) $\lim_{n\to\infty} w(x_n) = 0$*, where* $\{x_n\}_{n\geq 0}$ *satisfies the following*

(*C14*) x_0 *is an arbitrary element in* S *and* $\{y_n\}_{n\geq 1}$ *is any sequence in* D *and* $x_n = b(x_{n-1}, y_n),\ \forall n \geq 1$.
Furthermore, the solution w *in* $BB(S)$ *of the functional equation* (53) *is also unique with respect to conditions* (*C13*) *and* (*C14*).

Theorem 2.6. *Let* $b : S \times D \to S$ *and* $q : S \times D \to \mathbb{R}$ *be mappings. If there exists* $(\varphi, \psi) \in \Phi_4$ *satisfying* (*C10) and* (*C11*)*, then the functional equation*

$$f(x) = \sup_{y \in D} \text{opt}\{q(x, y), f(b(x, y))\},\ \forall x \in S \tag{54}$$

possesses a solution $w \in BB(S)$ *that satisfies the following conditions:*

(*C15*) $\lim_{n\to\infty} w_n = w$, *where* $\{w_n\}_{n\geq 0}$ *is defined by*

$$w_0(x) = \sup_{y\in D} q(x,y),\ \forall x \in S,$$

$$w_n(x) = \sup_{y\in D} \text{opt}\{q(x,y), w_{n-1}(b(x,y))\},\ \forall x \in S,\ \forall n \geq 1;$$

(*C16*) $\lim_{n\to\infty} w(x_n) = 0$, *where* $\{x_n\}_{n\geq 0}$ *satisfies* (*C14*).
Furthermore, the solution w *in* $BB(S)$ *of the functional equation (54) is also unique with respect to conditions* (*C16*) *and* (*C14*).

Remark 2.2. It follows from Remark 1.1 that Theorem 2.4 of Bhakta and Mitra [7] is a special case of Theorem 2.5. The following example reveals that Theorem 2.5 properly extends the result of Bhakta and Mitra [7].

Example 2.1. Let $X = Y = \mathbb{R}$, $S = D = \mathbb{R}^+$, $b(x,y) = x(3+\sin xy)^{-1}$, $q(x,y) = 2x^4y(1+x^2y^2)^{-1}$, $\forall (x,y) \in S\times D$, $\varphi(t) = 2^{-1}t$ and $\psi(t) = t^2$, $\forall t \in \mathbb{R}^+$. It is easy to verify that the conditions of Theorem 2.5 are satisfied. It follows from Theorem 2.5 that the functional equation (53) possesses a solution $w \in BB(S)$ which satisfies conditions (C12)-(C14). But we cannot invoke Theorem 2.4 of Bhakta and Mitra [7] to show that the functional equation (53) possesses a solution in $BB(S)$ because q does not satisfy the following

(*C17*) there exists a positive constant M with

$$|q(x,y)| \leq M|x|,\ \forall (x,y) \in S\times D.$$

In fact, given $M > 0$, there exist $x_M = 1+M \in S$ and $y_M = 1 \in D$ and satisfying

$$|q(x_M,y_M)| = 2|x_M^4 y_M(1+x_M^2y_M^2)^{-1}| \geq |x_M^2| > M|x_M|,$$

which means that (C17) does not hold.

3. Existence and Iterative Approximation of Coincidence Solutions for Certain Classes of System of Functional Equations

$\overline{f}$ and $\overline{g}$ are said to be coincidence solutions of the system of functional equations (5) if they satisfy the following condition

$$(55)\qquad \begin{cases} \overline{f}(x) = \sup_{y\in D} \text{opt}\{p(x,y), A(x,y,\overline{g}(a(x,y))), q(x,y) \\ \qquad\qquad +B(x,y,\overline{g}(b(x,y)))\}, \forall x \in S, \\ \overline{g}(x) = \sup_{y\in D} \text{opt}\{u(x,y), C(x,y,\overline{f}(c(x,y))), v(x,y) \\ \qquad\qquad +H(x,y,\overline{f}(h(x,y)))\}, \forall x \in S. \end{cases}$$

Theorem 3.1. *Let* $a, b, c, h : S \times D \to S$, $p, q, u, v : S \times D \to \mathbb{R}$ *and* $A, B, C, H : S \times D \times \mathbb{R} \to \mathbb{R}$ *be mappings and let* (φ, ψ) *be in* Φ_3 *satisfying the following conditions:*

(*C18*) $\max\{|p(x,y)|, |q(x,y)|, |u(x,y)|, |v(x,y)|\} \le \psi(\|x\|),\ \forall (x,y) \in S \times D$;

(*C19*) $\max\{\|a(x,y)\|, \|b(x,y)\|, \|c(x,y)\|, \|h(x,y)\|\} \le \varphi(\|x\|), \forall (x,y) \in S \times D$;

(*C20*) $\max\{|A(x,y,z)|, |B(x,y,z)|, |C(x,y,z)|, |H(x,y,z)|\} \le |z|,\ \forall (x,y,z) \in S \times D \times \mathbb{R}$;

(*C21*) *For any fixed* $(x,y) \in S \times D$, $A(x,y,\cdot), B(x,y,\cdot), C(x,y,\cdot), H(x,y,\cdot)$ *are both left continuous and nondecreasing with respect to the third argument on* $\mathbb{R}$. *Then the system of functional equations*

$$(56)\qquad \begin{cases} f(x) = \sup_{y \in D} \sup\{p(x,y), A(x,y,g(a(x,y))), q(x,y) \\ \qquad\qquad + B(x,y,g(b(x,y)))\}, \forall x \in S, \\ g(x) = \sup_{y \in D} \sup\{u(x,y), C(x,y,f(c(x,y))), v(x,y) \\ \qquad\qquad + H(x,y,f(h(x,y)))\}, \forall x \in S \end{cases}$$

possesses coincidence solutions f *and* g *in* $BB(S)$

Proof. Put

$$(57)\qquad g_0(x) = \sup_{y \in D} u(x,y),\ \forall x \in S,$$

$$(58)\qquad \begin{aligned} g_{2n}(x) = &\sup_{y \in D} \sup\{u(x,y), C(x,y,f_{2n-1}(c(x,y))), v(x,y) \\ &+ H(x,y,f_{2n-1}(h(x,y)))\}, \forall x \in S,\ \forall n \ge 1, \end{aligned}$$

$$(59)\qquad \begin{aligned} f_{2n+1}(x) = &\sup_{y \in D} \sup\{p(x,y), A(x,y,g_{2n}(a(x,y))), q(x,y) \\ &+ B(x,y,g_{2n}(b(x,y)))\}, \forall x \in S,\ \forall n \ge 0. \end{aligned}$$

It follows from (C21) that for any $x \in S$

$$(60)\qquad g_0(x) \le g_2(x) \le \cdots \le g_{2n}(x) \le g_{2n+2}(x) \le \cdots,$$

$$(61)\qquad f_1(x) \le f_3(x) \le \cdots \le f_{2n-1}(x) \le f_{2n+1}(x) \le \cdots.$$

Let x be in S and let k be a positive integer with $x \in \overline{B}(0,k)$. Making use of Lemma 1.3, (57) and (C18), we get that

$$|g_0(x)| = |\sup_{y\in D} u(x,y)| \le \psi(\|x\|). \tag{62}$$

According to Lemma 1.3, (59), (62) and (C18)-(C20), we deduce that

$$\begin{aligned}
&|f_1(x)| \\
=&|\sup_{y\in D}\sup\{p(x,y), A(x,y,g_0(a(x,y))), q(x,y)+B(x,y,g_0(b(x,y)))\}| \\
\le&\sup_{y\in D}\sup\{|p(x,y)|, |A(x,y,g_0(a(x,y)))|, |q(x,y)|+|B(x,y,g_0(b(x,y)))|\} \\
\le&\sup_{y\in D}\sup\{\psi(\|x\|), |g_0(a(x,y))|, \psi(\|x\|)+|g_0(b(x,y))|\} \\
\le&\sup_{y\in D}\sup\{\psi(\|a(x,y)\|), \psi(\|x\|)+\psi(\|b(x,y)\|)\} \\
\le&\sum_{i=0}^{1}\psi(\varphi^i(\|x\|)).
\end{aligned} \tag{63}$$

By using the similar argument, we obtain that

$$|g_{2n}(x)| \le \sum_{i=0}^{2n}\psi(\varphi^i(\|x\|)) \le \sum_{i=0}^{\infty}\psi(\varphi^i(k)),\ \forall n\ge 0, \tag{64}$$

$$|f_{2n+1}(x)| \le \sum_{i=0}^{2n+1}\psi(\varphi^i(\|x\|)) \le \sum_{i=0}^{\infty}\psi(\varphi^i(k)),\ \forall n\ge 0. \tag{65}$$

Thus (64) and (65) ensure that $\{g_{2n}(x)\}_{n\ge 0}$ and $\{f_{2n+1}(x)\}_{n\ge 0}$ are both bounded. By virtue of (62)-(65) we conclude that

$$\lim_{n\to\infty} g_{2n}(x) = g(x),\quad \lim_{n\to\infty} f_{2n+1}(x) = f(x),\ \forall x\in\overline{B}(0,k) \tag{66}$$

and

$$\max\{|g(x)|, |f(x)|\} \le \sum_{i=0}^{\infty}\psi(\varphi^i(k)),\ \forall x\in\overline{B}(0,k). \tag{67}$$

It is easy to see that (67) yields that $g, f \in BB(S)$. Set

$$\begin{cases}
M(x) = \sup_{y\in D}\sup\{p(x,y), A(x,y,g(a(x,y))), q(x,y) \\
\qquad\qquad +B(x,y,g(b(x,y)))\}, \forall x\in S, \\
N(x) = \sup_{y\in D}\sup\{u(x,y), C(x,y,f(c(x,y))), v(x,y) \\
\qquad\qquad\qquad +H(x,y,f(h(x,y)))\}, \forall x\in S.
\end{cases} \tag{68}$$

Notice that (58)-(60) and (68) imply that for any $(x, y) \in S \times D$,

$$\sup\{u(x,y), C(x,y,f_{2n-1}(c(x,y))), v(x,y) + H(x,y,f_{2n-1}(h(x,y)))\} \le g_{2n}(x) \le N(x),\ \forall n \ge 1, \tag{69}$$

$$\sup\{p(x,y), A(x,y,g_{2n}(a(x,y))), q(x,y) + B(x,y,g_{2n}(b(x,y)))\} \le f_{2n+1}(x) \le M(x),\ \forall n \ge 0. \tag{70}$$

Letting $n \to \infty$ in (69) and (70), by (C21) and (66), we derive that

$$\sup\{u(x,y), C(x,y,f(c(x,y))), v(x,y) + H(x,y,f(h(x,y)))\} \le g(x) \le N(x),\ \forall (x,y) \in S \times D, \tag{71}$$

$$\sup\{p(x,y), A(x,y,g(a(x,y))), q(x,y) + B(x,y,g(b(x,y)))\} \le f(x) \le M(x),\ \forall (x,y) \in S \times D. \tag{72}$$

From (68), (71) and (72) we get that

$$N(x) \le g(x) \le N(x),\ M(x) \le f(x) \le M(x),\ \forall x \in S.$$

That is,

$$g(x) = N(x),\ f(x) = M(x),\ \forall x \in S.$$

This completes the proof.

By using the same proof as in Theorem 3.1, we have the following results.

Theorem 3.2. *Let $a, c : S \times D \to S$, $p, u : S \times D \to \mathbb{R}$ and $A, C : S \times D \times \mathbb{R} \to \mathbb{R}$ be mappings and let (φ, ψ) be in Φ_3 satisfying the following conditions:*

$(C22)$ $\max\{|p(x,y)|, |u(x,y)|\} \le \psi(\|x\|),\ \forall (x,y) \in S \times D$;

$(C23)$ $\max\{|a(x,y)|, |c(x,y)|\} \le \varphi(\|x\|),\ \forall (x,y) \in S \times D$;

$(C24)$ $\max\{|A(x,y,z)|, |C(x,y,z)|\} \le |z|,\ \forall (x,y,z) \in S \times D \times \mathbb{R}$;

$(C25)$ *For any fixed $(x,y) \in S \times D$, $A(x,y,\cdot), C(x,y,\cdot)$ are both left continuous and nondecreasing with respect to the third argument on $\mathbb{R}$.*
Then the system of functional equations

$$\begin{cases} f(x) = \sup_{y \in D} \sup\{p(x,y), A(x,y,g(a(x,y)))\}, \forall x \in S, \\ g(x) = \sup_{y \in D} \sup\{u(x,y), C(x,y,f(c(x,y)))\}, \forall x \in S \end{cases} \tag{73}$$

possesses coincidence solutions f and g in $BB(S)$.

Theorem 3.3. *Let* $a, c : S \times D \to S$, $p, u : S \times D \to \mathbb{R}$ *and* $A, C : S \times D \times R \to \mathbb{R}$ *be mappings and let* (φ, ψ) *be in* Φ_3 *satisfying* $(C22), (C23), (C25)$ *and* $(C26)$ $0 \le C(x, y, z) \le |z|,\ |A(x, y, z)| \le |z|, \forall (x, y, z) \in S \times D \times \mathbb{R}$.
Then the system of functional equations

$$\begin{cases} f(x) = \sup_{y \in D}\{p(x, y) + A(x, y, g(a(x, y)))\}, \forall x \in S, \\ g(x) = \sup_{y \in D}\{u(x, y) + C(x, y, f(c(x, y)))\}, \forall x \in S \end{cases} \tag{74}$$

possesses coincidence solutions f *and* g *in* $BB(S)$.

Remark 3.1.

(i) Theorem 2.3 of Bhakta and Mitra [7] is a special case of Theorem 3.3 with $\psi = I$, $a = c$, $0 \le p = u$, $A = C$ and $f = g$.

(ii) If $\psi = I$, $a = c$, $0 \le p = u$ and $0 \le A(x, y, z) \le |z|, \forall (x, y, z) \in S \times D \times \mathbb{R}$, then Theorem 3.3 reduces to Theorem 4.1 of Chang [10].

(iii) In case $\psi = I$, then Theorem 3.3 reduces to Theorem 4.2 of Liu [12].

The example below shows that Theorem 3.3 is a indeed common generalization of the results of Bhakta and Mitra [7], Chang [10] and Liu [12].

Example 3.1. Let $X = Y = \mathbb{R}$, $S = D = \mathbb{R}^+$, Define $a, c : S \times D \to S$, $p, u : S \times D \to \mathbb{R}$ and $A, C : S \times D \times \mathbb{R} \to \mathbb{R}$ as follows:

$$p(x, y) = 2x^4 y(1 + x^2 y^2)^{-1}, u(x, y) = x^2(1 + y^2)^{-1} \sin x, \forall (x, y) \in S \times D,$$

$$a(x, y) = x(4 + \sin(x + y^2))^{-1}, c(x, y) = x(3 + |\cos(x^2 y - 1)|)^{-1}, \forall (x, y) \in S \times D,$$

$$C(x, y, z) = \begin{cases} z(1 + x^2 + y^2)^{-1}, & \text{if } (x, y, z) \in S \times D \times \mathbb{R}^+, \\ 0, & \text{if } (x, y, z) \in S \times D \times (\mathbb{R} - \mathbb{R}^+), \end{cases}$$

$$A(x, y, z) = z|\sin(x^2 + y^2)|,\ \forall (x, y, z) \in S \times D \times \mathbb{R}.$$

Set $\psi(t) = t^3$ and $\varphi(t) = 3^{-1}t$, $\forall t \in \mathbb{R}^+$. Then Theorem 3.3 ensures that the system of functional equations (78) possesses coincidence solutions f and g in $w \in BB(S)$. However, the results of Bhakta and Mitra [7], Chang [10] and Liu [12] are not applicable since

$$|p(x, y)| \le |x|,\ \forall (x, y) \in S \times D$$

does not hold.

We last pose the following questions.

Question 3.1. Does the system of functional equations

$$\begin{cases} f(x) = \sup_{y \in D} \inf\{p(x,y), A(x,y,g(a(x,y)))\}, \forall x \in S, \\ g(x) = \sup_{y \in D} \inf\{u(x,y), C(x,y,f(c(x,y)))\}, \forall x \in S \end{cases}$$

possess coincidence solutions in $BB(S)$?

Question 3.2. Does the system of functional equations

$$\begin{cases} f(x) = \sup_{y \in D} \inf\{p(x,y), A(x,y,g(a(x,y))), q(x,y) \\ \qquad\qquad +B(x,y,g(b(x,y)))\}, \forall x \in S, \\ g(x) = \sup_{y \in D} \inf\{u(x,y), C(x,y,f(c(x,y))), v(x,y) \\ \qquad\qquad +H(x,y,f(h(x,y)))\}, \forall x \in S \end{cases}$$

possess coincidence solutions in $BB(S)$?

Acknowledgments

This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (2009A419) and Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00042).

References

1. R. Baskaram and P. V. Subrahmanyam, A Note on the Solution of a Class of Functional Equations, *Appl. Anal.*, **22** (1986), 235-241.

2. S. A. Belbas, Dynamic Programming and Maximum Principle for Discrete Goursat Systems, *J. Math. Anal. Appl*, **161** (1991), 57-77.

3. R. Bellman and E. S. Lee, Functional Equations Arising in Dynamic Programming, *Aequationes Math.*, **17** (1978), 1-18.

4. R. Bellman and M. Roosta, A Technique for the Reduction of Dimensionality in Dynamic Programming, *J. Math. Anal. Appl.*, **88** (1982), 543-546.

5. R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, New Jersey, 1957.

6. R. Bellman, *Methods of Nonlinear Analysis*, II, Academic Press, New York, NY, 1973.

7. P. C. Bhakta and S. Mitra, Some Existence Theorems for Functional Equations Arising in Dynamic Programming, *J. Math. Anal. Appl.*, **98** (1984), 348-362.

8. P. C. Bhakta and S. R. Choudhury, Some Existence Theorems for Functional Equations Arising in Dynamic Programming II, *J. Math. Anal. Appl.*, **131** (1988), 217-231.

9. D. W. Boyd and J. S. W. Wong, On Nonlinear Contractions, *Proc. Amer. Math. Soc.*, **20** (1969), 458-464.

10. S. S. Chang, Some Existence Theorems of Common and Coincidence Solutions for a Class of Functional Equations Arising in Dynamic Programming, *Appl. Math. Mechanics*, **12** (1991), 31-37.

11. S. S. Chang and Y. H. Ma, Coupled Fixed Points for Mixed Monotone Condensing Operators and an Existence Theorem of the Solutions for a Class of Functional Equations Arising in Dynamic Programming, *J. Math. Anal. Appl.*, **106** (1991), 468-479.

12. Z. Liu, Existence Theorems of Solutions for Certain Classes of Functional Equations Arising in Dynamic Programming, *J. Math. Anal. Appl.*, **262** (2001), 529-553.

13. Z. Liu, Coincidence Theorems for Expansive Mappings with Applications to the Solutions of Functional Equations Arising in Dynamic Programming, *Acta Scientiarum Mathematicarum (Szeged)*, **65** (1999), 359-369.

14. Z. Liu, Compatible Mappings and Fixed Points, *Acta Scientiarum Mathematicarum (Szeged)*, **65** (1999), 371-383.

15. Z. Liu and J. S. Ume, On Properties of Solutions for a class of Functional Equations Arising in Dynamic Programming, *J. Opti. Theory Appl.*, **117** (2003), 533-551.

Zeqing Liu
Department of Mathematics,
Liaoning Normal University,
Dalian, Liaoning 116029,
P. R. China
E-mail: zeqingliu@sina.com.cn

Jeong Sheok Ume
Department of Applied Mathematics,
Changwon National University,
Changwon 641-773,
Korea
E-mail: jsume@changwon.ac.kr

Shin Min Kang
Department of Mathematics and Research Institute of Natural Science,
Gyeongsang National University,
Chinju 660-701,
Korea
E-mail: smkang@nongae.gsnu.ac.kr