

THE CROSSED PRODUCT VON NEUMANN ALGEBRAS ASSOCIATED WITH $SL_2(\mathbb{R})$

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Abstract. Let \mathcal{A} be the abelian von Neumann subalgebra $\{M_f : f \in L^\infty(\mathbb{H}, \mu_r)\}$ of $\mathcal{B}(L^2(\mathbb{H}, \mu_r))$, where \mathbb{H} is the upper half plane and the measure $d\mu_r = dx dy / y^{2-r}$. For any integers $r > 1$, the linear fractional action of $SL_2(\mathbb{R})$ on \mathbb{H} induces a continuous action α of $SL_2(\mathbb{R})$ on \mathcal{A} . It is shown that the crossed product $\mathcal{R}(\mathcal{A}, \alpha)$ of \mathcal{A} under the action α of $SL_2(\mathbb{R})$ is *-isomorphic to $\mathcal{B}(L^2(P, 2dx dy / y^{3-2r})) \overline{\otimes} \mathcal{L}_K$, where $SL_2(\mathbb{R}) = PK$ is the Iwasawa decomposition of $SL_2(\mathbb{R})$. Thus $\mathcal{R}(\mathcal{A}, \alpha)$ is of type I.

1. INTRODUCTION

The 2-order real special linear group $SL_2(\mathbb{R}) = \{g \in GL_2(\mathbb{R}) : \det(g) = 1\}$ with the unit e is a connected Lie group and a unimodular group. The set $\{\pm e\}$ is the unique normal subgroup of $SL_2(\mathbb{R})$. By means of the representations of the Lie algebra corresponding to the group $SL_2(\mathbb{R})$, Bargmann[1] has classified all irreducible unitary representations of $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm e\}$.

Let \mathbb{H} be the upper half complex plane $\{z = x + iy : y > 0\}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, define the linear fractional action(Möbius transformation) of g on \mathbb{H} as following:

$$gz = \frac{az + b}{cz + d}.$$

Note that g and $-g$ have the same action.

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For any $r \geq 0$, let $L^2(\mathbb{H}, dxdy/y^{2-r})$ be the Hilbert space consisting of the square integrable functions on \mathbb{H} with respect to the measure $dxdy/y^{2-r}$. The set \mathcal{H}_a^r of all analytic functions on \mathbb{H} forms a closed subspace of $L^2(\mathbb{H}, dxdy/y^{2-r})$. When $r = 0$, $d\mu = dxdy/y^2$ is called the hyperbolic measure on \mathbb{H} and $\mathcal{H}_a^r = \{0\}$. Moreover, the measure μ is invariant under the linear fractional action of the group $SL_2(\mathbb{R})$. For any $g \in SL_2(\mathbb{R})$, we define the operator U_g by

$$(U_g f)(z) = f(g^{-1}z),$$

where $f \in L^2(\mathbb{H}, dxdy/y^2)$. Then the mapping $g \rightarrow U_g$ is a continuous unitary representation of $SL_2(\mathbb{R})$ on the Hilbert space $L^2(\mathbb{H}, dxdy/y^2)$.

For any function $f \in L^\infty(\mathbb{H}, dxdy/y^2)$, we can define the bounded linear operator M_f acting on $L^2(\mathbb{H}, dxdy/y^2)$ as $(M_f \psi)(h) = f(h)\psi(h)$, $\psi \in L^2(\mathbb{H}, dxdy/y^2)$. All such operators form a maximal abelian *-subalgebra(MASA) $\mathcal{A}_0 = \{M_f : f \in L^\infty(\mathbb{H}, dxdy/y^2)\}$ of $\mathcal{B}(L^2(\mathbb{H}, dxdy/y^2))$ [2]. Then $\alpha_g(T) = U_g T U_g^*$ induces a continuous action α of the group $SL_2(\mathbb{R})$ on the MASA \mathcal{A}_0 . By using of the techniques of induced representation in [5], we have shown that the crossed product von Neumann algebra $\mathcal{R}(\mathcal{A}_0, \alpha)$ of \mathcal{A}_0 under the action α of $SL_2(\mathbb{R})$ is of type II[7].

When $r > 1$, \mathcal{H}_a^r is nonzero. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, choose a normal branch of $a - cz$ such that $arg(a - cz)$ takes its values in the interval $(-\pi, \pi]$. Using this branch for $(a - cz)^r = e^{r \ln(a - cz)}$, we define the representation π_r of $SL_2(\mathbb{R})$ on the Hilbert space $L^2(\mathbb{H}, dxdy/y^{2-r})$ as following:

$$(\pi_r(g)f)(z) = (a - cz)^{-r} f(g^{-1}z).$$

Then π_r is a projective unitary representation of $SL_2(\mathbb{R})$. In particular, $\pi_r|_{\mathcal{H}_a^r}$ is also a projective unitary representation of $SL_2(\mathbb{R})$ [4]. When $r > 1$ is an integer, $\pi_r|_{\mathcal{H}_a^r}$ is an irreducible unitary representation of $SL_2(\mathbb{R})$ which is a discrete series representation. Furthermore, π_r is also a unitary representation of $SL_2(\mathbb{R})$ on the Hilbert space $L^2(\mathbb{H}, \mu_r)$. Now, for any $g \in SL_2(\mathbb{R})$ and any operator T in the MASA $\mathcal{A} = \{M_f : f \in L^\infty(\mathbb{H}, \mu_r)\}$ of $\mathcal{B}(L^2(\mathbb{H}, dxdy/y^{2-r}))$, $\alpha_g(T) = \pi_r(g)T\pi_r(g)^*$ is a continuous action α of $SL_2(\mathbb{R})$ on \mathcal{A} . However, the action α is not the induced action, thus we have to modify the techniques in [5] to get the similar result in [7]. Let $\mathcal{R}(\mathcal{A}, \alpha)$ be the von Neumann algebra crossed product of \mathcal{A} under the action α of $SL_2(\mathbb{R})$. We will discuss the structure and properties of $\mathcal{R}(\mathcal{A}, \alpha)$ in this paper.

The paper is organized as follows. In the following section, we will recall the construction of the von Neumann algebra crossed product and some properties of it and discuss the generators and the properties of the group $SL_2(\mathbb{R})$ and analysis the measures on the subgroups of $SL_2(\mathbb{R})$ and the continuous action α of $SL_2(\mathbb{R})$ on the MASA \mathcal{A} . In the last section, we will use the modified techniques in [5] to discuss the structure of the crossed product von Neumann algebra $\mathcal{R}(\mathcal{A}, \alpha)$ and show that $\mathcal{R}(\mathcal{A}, \alpha)$ is of type I.

2. PRELIMINARIES

Supposing that \mathcal{M} is a von Neumann algebra acting on the Hilbert space \mathcal{H} , G is a locally compact group with unit e and α is a continuous action of G on \mathcal{M} with respect to the weak-operator topology. Fixing a left Haar measure μ on G , $L^2(G, \mu)$ or $L^2(G)$ is the Hilbert space consisting of the square integrable complex functions on G . The mapping $g \rightarrow L_g$ is the left regular representation of G .

Without special remarking, we will identify the Hilbert space $L^2(G, \mathcal{H})$ with $\mathcal{H} \otimes L^2(G, \mu)$. With $A \in \mathcal{M}$ and g in G , we define the operators $\pi(A)$ and $\lambda(g)$ acting on the Hilbert space $L^2(G, \mathcal{H})$ by

$$(\pi(A)f)(h) = \alpha_{h^{-1}}(A)f(h)$$

and

$$\lambda(g) = I \otimes L_g,$$

where $f \in L^2(G, \mathcal{H})$, $h \in G$. In fact, π is a faithful normal representation of \mathcal{M} on the Hilbert space $L^2(G, \mathcal{H})$ and λ is a continuous unitary representation of G . A straightforward calculation shows that $\lambda(g)\pi(A)\lambda(g)^* = \pi(\alpha_g(A))$.

Definition 2.1. With notations as above, we denote by $\mathcal{R}(\mathcal{M}, \alpha)$ the von Neumann subalgebra of $\mathcal{B}(L^2(G, \mathcal{H}))$ that is generated by the operators $\pi(\mathcal{M})$ and $\lambda(G)$. We refer to $\mathcal{R}(\mathcal{M}, \alpha)$ as the crossed product of \mathcal{M} under the action α of G .

Actually, $\mathcal{R}(\mathcal{M}, \alpha)$ is the weak-operator topological closure of the linear spanning of $\{\pi(A)\lambda(g), A \in \mathcal{M}, g \in G\}$. Furthermore, $\mathcal{R}(\mathcal{M}, \alpha) \subset \mathcal{M} \overline{\otimes} \mathcal{B}(L_2(G))$ [6]. Let $g \rightarrow R_g$ be the right regular representation of the group G . If there is a continuous unitary representation $g \rightarrow U_g$ of G on \mathcal{H} such that $\alpha_g(A) = U_gAU_g^*$ for any $A \in \mathcal{M}$ and $g \in G$, then we have $\mathcal{R}(\mathcal{M}, \alpha)' = \{A' \otimes I, U_g \otimes R_g : A' \in \mathcal{M}', g \in G\}$ " [6].

Supposing that \mathcal{M} and \mathcal{N} are von Neumann algebras, α and β are continuous actions of the group G on them respectively and σ is a *-isomorphism from \mathcal{M} onto \mathcal{N} such that $\sigma(\alpha_g(A)) = \beta_g(\sigma(A))$ for any $A \in \mathcal{M}$ and $g \in G$. Then we have $\mathcal{R}(\mathcal{M}, \alpha) \cong \mathcal{R}(\mathcal{N}, \beta)$ [6].

Now let us recall the basic properties of the group $SL_2(\mathbb{R})$. Every element of $SL_2(\mathbb{R})$ can be uniquely decomposed into the product of the elements $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ and $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ [3], where $x \in \mathbb{R}$, $y > 0$ and $\theta \in [0, 2\pi]$. Let P be the subgroup $\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} : y > 0, x \in \mathbb{R} \right\}$ and K be $\left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} : \theta \in [0, 2\pi] \right\}$ of $SL_2(\mathbb{R})$ respectively. Then $SL_2(\mathbb{R})$ has

the decomposition PK (Iwasawa decomposition)[3]. Scaling to a constant, the left Haar measure of the locally compact subgroup P is $dx dy/y^3$ and its right Haar measure is $dx dy/y$. The subgroup K is an abelian compact group and $cd\theta$ is its Haar measure.

From the Iwasawa decomposition, the subgroup P can be identified with the left quotient space $\Gamma = SL_2(\mathbb{R})/K$ by the following bijection:

$$\tau : p \mapsto [p].$$

The measure $2dx dy/y^3$ on P induces a $SL_2(\mathbb{R})$ -invariant measure ν on Γ . Then the measure $d\mu = 2dx dy d\theta/y^3$ is a left Haar measure of $SL_2(\mathbb{R})$. Therefore, $L^2(SL_2(\mathbb{R}), \mu) \cong L^2(P, 2dx dy/y^3) \otimes L^2(K, d\theta)$. This result doesn't hold for the measure $2dx dy/y^{3-2r}$ on the subgroup P . Fortunately, we have the following similar result. Let φ be the function on $SL_2(\mathbb{R})$ which is defined by

$$\varphi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) = y.$$

Then easy calculation shows that $\varphi(gk) = \varphi(g)$ for any $g \in SL_2(\mathbb{R})$ and $k \in K$. Furthermore, φ is continuous and positive. Thus we have the following result.

Proposition 2.2. [5]. *There is a unique quasi- $SL_2(\mathbb{R})$ -invariant Borel measure ν on the left quotient space Γ such that*

$$\int_{\Gamma} \left(\int_K f(gk) d\nu_K(k) \right) d\nu([g]) = \int_{SL_2(\mathbb{R})} f(g) \varphi(g)^{2r} d\mu(g)$$

for any $f \in C_c(SL_2(\mathbb{R}))$ which is the set of all continuous function on $SL_2(\mathbb{R})$ with compact support, where $d\mu$ is the left Haar measure $2dx dy d\theta/y^3$ of the group $SL_2(\mathbb{R})$.

According to this result, the Hilbert space $L^2(SL_2(\mathbb{R}), \varphi^{2r} d\mu)$ can be decomposed into the tensor product $L^2(\Gamma, d\nu) \otimes L^2(K, d\theta)$ which can be identified with $L^2(P, 2dx dy/y^{3-2r}) \otimes L^2(K, d\theta)$ by the mapping τ .

The subgroup P also can be identified with the upper half plane \mathbb{H} by the following bijection:

$$\sigma : P \ni \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \mapsto z = x + iy^2 \in \mathbb{H}.$$

Let's fix the measure $dx dy/y^{2-r}$ on \mathbb{H} and the measure $2dx dy/y^{3-2r}$ on the group P . Then the mapping σ is a measure-preserving transformation. Let ν be the measure on Γ induced by through proposition 2.2. Then the mapping σ and the

quotient mapping τ induce a unitary operator V from the Hilbert space $L^2(\mathbb{H}, d\mu_r)$ onto the Hilbert space $L^2(\Gamma, d\nu)$. The operator V is defined by

$$(Vf)([g]) = f(\sigma(\tau^{-1}([g])))$$

for any $g \in P$ and $f \in L^2(\mathbb{H}, dxdy/y^{2-r})$.

With $f \in L^2(\mathbb{H}, dxdy/y^{2-r})$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, where $r > 1$ is a fixed integer, we define a measurable function $\pi_r(g)f$ on \mathbb{H} by

$$(\pi_r(g)f)(z) = (a - cz)^{-r} f(g^{-1}z).$$

Simple calculation shows that $\pi_r(g)$ is a unitary operator. Furthermore, we have the following result [1, 3].

Proposition 2.3. *With respect to the strong pointwise topology, π_r is a continuous unitary representation of $SL_2(\mathbb{R})$ on $L^2(\mathbb{H}, dxdy/y^{2-r})$.*

With $F \in L^\infty(\mathbb{H}, dxdy/y^{2-r})$, the bounded linear operator M_F is defined by

$$(M_F f)(z) = F(z)f(z)$$

for any $f \in L^2(\mathbb{H}, dxdy/y^{2-r})$. Then $\mathcal{A} = \{M_F : F \in L^\infty(\mathbb{H}, dxdy/y^{2-r})\}$ is a MASA of $\mathcal{B}(L^2(\mathbb{H}, dxdy/y^{2-r}))$ [2]. Now for any $g \in SL_2(\mathbb{R})$, we define the action α_g of g on the MASA \mathcal{A} by $\alpha_g(A) = \pi_r(g)A\pi_r(g)^*$. Concerning the definition of the representation π_r of $SL_2(\mathbb{R})$, we have the following results.

Lemma 2.4. *With respect to the weak-operator topology, α is a continuous action of the group $SL_2(\mathbb{R})$ on the MASA \mathcal{A} .*

Proof. According to the properties of the unitary representation π_r , we just need to show that, for any $g \in SL_2(\mathbb{R})$ and $M_F \in \mathcal{A}$, we have $\alpha_g(M_F) \in \mathcal{A}$.

In fact, for all $M_F \in \mathcal{A}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we have

$$\pi_r(g)M_F\pi_r(g)^* = M_{F(g^{-1}\cdot)},$$

since

$$\begin{aligned} (\pi_r(g)M_F\pi_r(g)^* f)(z) &= (a - cz)^{-r} (M_F\pi_r(g)^* f)(g^{-1}z) \\ &= (a - cz)^{-r} F(g^{-1}z)(\pi_r(g)^* f)(g^{-1}z) \\ &= (a - cz)^{-r} F(g^{-1}z)(d + c(g^{-1}z))^{-r} f(z) \\ &= (M_{F(g^{-1}\cdot)} f)(z) \end{aligned}$$

for any $f \in L^2(\mathbb{H}, dxdy/y^{2-r})$. ■

According to the above result, the action α of $SL_2(\mathbb{R})$ is implemented by the unitary representation π_r . Now for the convenience, we identify the tensor product Hilbert space $L^2(\mathbb{H}, dxdy/y^{2-r}) \otimes L^2(SL_2(\mathbb{R}), \mu)$ with $L^2(SL_2(\mathbb{R}), L^2(\mathbb{H}, dxdy/y^{2-r}))$ by identifying $v \otimes \phi$ with $v\phi$, where $v \in L^2(\mathbb{H}, dxdy/y^{2-r})$ and $\phi \in L^2(SL_2(\mathbb{R}), \mu)$. With $M_F \in \mathcal{A}$ and $g \in G$, we define the operators $\Pi(M_F)$ and $\Lambda(g)$ acting on $L^2(\mathbb{H}, dxdy/y^{2-r}) \otimes L^2(SL_2(\mathbb{R}), \mu)$ by

$$(\Pi(M_F)v\phi)(h) = \phi(h)\alpha_{h^{-1}}(M_F)v$$

and

$$\Lambda(g) = I \otimes L_g.$$

Then Π is a faithful normal representation of \mathcal{A} and Λ is a continuous unitary representation of $SL_2(\mathbb{R})$ on $L^2(\mathbb{H}, dxdy/y^{2-r}) \otimes L^2(SL_2(\mathbb{R}), \mu)$. The crossed product $\mathcal{R}(\mathcal{A}, \alpha)$ of \mathcal{A} by α is the von Neumann subalgebra of $\mathcal{B}(L^2(\mathbb{H}, dxdy/y^{2-r}) \otimes L^2(G, \mu))$, which is generated by the operators $\Pi(M_F)$, $M_F \in \mathcal{A}$ and $\Lambda(g)$, $g \in G$. Furthermore, $\mathcal{R}(\mathcal{A}, \alpha)$ is the weak operator topology closure of the linear span of the set $\{\Pi(T)\Lambda(g) : T \in \mathcal{A}, g \in G\}$.

Using the unitary operator V and the bijection τ , we can define another action β of the group $SL_2(\mathbb{R})$ on the von Neumann algebra $\mathcal{C} = \{M_f : f \in L^\infty(\Gamma, \nu)\}$ which is acting on the Hilbert space $L^2(\Gamma, \nu)$, where the measure ν is induced by the measure $2dxdy/y^{3-2r}$ on P . We firstly define the operator $\tilde{\pi}_r(g) \in \mathcal{B}(L^2(\Gamma, 2dxdy/y^{3-2r}))$ by $\tilde{\pi}_r(g) = V\pi_r(g)V^*$ for any $g \in SL_2(\mathbb{R})$. Then according to the proposition 2.3, $\tilde{\pi}_r$ is a continuous unitary representation of $SL_2(\mathbb{R})$ with respect to the strong pointwise topology. With $g \in SL_2(\mathbb{R})$ and $M_f \in \mathcal{C}$, we define the β_g by

$$\beta_g(M_f) = \tilde{\pi}_r(g)M_f\tilde{\pi}_r(g)^*.$$

Lemma 2.5. *With respect to the weak-operator topology, β is a continuous action of the group $SL_2(\mathbb{R})$ on the von Neumann algebra \mathcal{C} .*

Proof. For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $f \in L^2(\Gamma, \nu)$, we have

$$\begin{aligned} & (\tilde{\pi}_r(g)f) \left(\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \right] \right) \\ &= (V\pi_r(g)V^*f) \left(\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \right] \right) \\ &= (\pi_r(g)V^*f)(x + iy^2) \end{aligned}$$

$$\begin{aligned} &= (a - c(x + iy^2))^{-r} (V^* f)(g^{-1}(x + iy^2)) \\ &= (a - c(x + iy^2))^{-r} f([\sigma^{-1}(g^{-1}(x + iy^2))]) \\ &= (a - c(x + iy^2))^{-r} f([g^{-1}p]). \end{aligned}$$

The last equation has used the computation in the Appendix 1. of [7].

Hence with $F \in L^\infty(\Gamma, \nu)$, we have

$$\beta_g(M_F) = \tilde{\pi}_r(g) M_F \tilde{\pi}_r(g)^* = M_{F([g^{-1}\cdot])}$$

since for all $f \in L^2(\Gamma, \nu)$ and $p = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \in P \subset SL_2(\mathbb{R})$,

$$\begin{aligned} (\beta_g(M_F)f)([p]) &= (\tilde{\pi}_r(g) M_F \tilde{\pi}_r(g)^* f)([p]) \\ &= (a - c(x + iy^2))^{-r} (M_F \tilde{\pi}_r(g)^* f)([\sigma^{-1}(g^{-1}(x + iy^2))]) \\ &= (a - c(x + iy^2))^{-r} F([g^{-1}p]) (\tilde{\pi}_r(g)^* f)([g^{-1}p]) \\ &= (a - c(x + iy^2))^{-r} F([g^{-1}p]) (d + c(g^{-1}(x + iy^2)))^{-r} f([p]) \\ &= (M_{F([g^{-1}\cdot])} f)([p]). \end{aligned}$$

Thus β is a continuous automorphism action of $SL_2(\mathbb{R})$ on the abelian von Neumann algebra \mathcal{C} with respect to the weak-operator topology by the continuity of the representation π_r . ■

With $M_F \in \mathcal{C}$ and $g \in SL_2(\mathbb{R})$, we define the operators $\tilde{\Pi}(M_F)$ and $\tilde{\Lambda}(g)$ acting on $L^2(\Gamma, \nu) \otimes L^2(SL_2(\mathbb{R}), \mu)$ by

$$(\tilde{\Pi}(M_F)v\phi)(h) = \phi(h)\beta_{h^{-1}}(M_F)v,$$

and

$$\tilde{\Lambda}(g) = I \otimes L_g.$$

The crossed product von Neumann algebra $\mathcal{R}(\mathcal{C}, \beta)$ is generated by the operators $\tilde{\Pi}(\mathcal{C})$ and $\tilde{\Lambda}(SL_2(\mathbb{R}))$.

3. MAIN RESULTS

Now we will analysis the structure of the crossed product $\mathcal{R}(\mathcal{A}, \alpha)$. Firstly, we have the following important result.

Lemma 3.1. *The crossed product $\mathcal{R}(\mathcal{A}, \alpha)$ is *-isomorphic to the crossed product $\mathcal{R}(\mathcal{C}, \beta)$.*

Proof. Let $\iota : \mathcal{A} \rightarrow \mathcal{C}$ be the following isomorphism which is induced by the mapping τ and σ . With $F \in L^\infty(\mathbb{H}, dx dy / y^{2-r})$ and $p = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \in P$, we define the operator $\iota(M_F) \in \mathcal{C}$ by

$$\begin{aligned} (\iota(M_F)f)([p]) &= F(\sigma(\tau^{-1}([p])))f([p]) \\ &= F(x + iy^2)f([p]). \end{aligned}$$

Then for any $g \in SL_2(\mathbb{R})$ and $M_F \in \mathcal{A}$, we have

$$\begin{aligned} (\iota(\alpha_g(M_F))f)([p]) &= (\iota(M_{F(g^{-1}\cdot)})f)([p]) \\ &= F(g^{-1}(x + iy^2))f([p]) \end{aligned}$$

and

$$\begin{aligned} (\beta_g(\iota(M_F))f)([p]) &= (\beta_g(M_{F(\sigma \circ \tau^{-1}\cdot)})f)([p]) \\ &= F(\sigma(\tau^{-1}([g^{-1}p])))f([p]) \end{aligned}$$

By using of the computation in the Appendix 1. of [7] again, we have $\iota \circ \alpha_g = \beta_g \circ \iota$. Thus $\mathcal{R}(\mathcal{A}, \alpha) \cong \mathcal{R}(\mathcal{C}, \beta)$. ■

According to this lemma, we just need to study the crossed product $\mathcal{R}(\mathcal{C}, \beta)$ to characterize the crossed product $\mathcal{R}(\mathcal{A}, \alpha)$. With $F \in L^\infty(SL_2(\mathbb{R}), \varphi^{2r} d\mu)$ and $f \in \mathcal{H} = L^2(SL_2(\mathbb{R}), \varphi^{2r} d\mu)$, we define the operator M_F acting on \mathcal{H} as following

$$(M_F f)(x) = F(x)f(x), x \in SL_2(\mathbb{R}).$$

Let \mathcal{M} be the MASA $\{M_F : F \in L^\infty(SL_2(\mathbb{R}), \varphi^{2r} d\mu)\}$. As the measure $\varphi^{2r} d\mu$ is not invariant under the left translation of the group $SL_2(\mathbb{R})$, then we have to define another action of the group $SL_2(\mathbb{R})$ on \mathcal{M} . With $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $k \in K$, we define the bounded linear operator $\widehat{\pi}_r(g)$ and \widehat{R}_k acting on $L^2(SL_2(\mathbb{R}), \varphi^{2r} d\mu)$ by

$$(\widehat{\pi}_r(g)f)(h) = \frac{1}{(a - c\sigma(\tau^{-1}([h])))^r} f(g^{-1}h)$$

and

$$(\widehat{R}_k f)(h) = f(hk).$$

Lemma 3.2.

- (1) $\widehat{\pi}_r$ and \widehat{R} are unitary representations of the groups $SL_2(\mathbb{R})$ and K respectively.
- (2) For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $k \in K$, we have

$$\widehat{\pi}_r(g)\widehat{R}_k = \widehat{R}_k\widehat{\pi}_r(g).$$

Proof. The proof of (1) is paralleling to the proof of Lemma 2.5. Here we just need to show that the second claim holds. With $f \in L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)$ and $h \in SL_2(\mathbb{R})$, by the following computation

$$\begin{aligned} (\widehat{\pi}_r(g)\widehat{R}_k f)(h) &= \frac{1}{(a - c\sigma(\tau^{-1}([h])))^r} (\widehat{R}_k f)(g^{-1}h) \\ &= \frac{1}{(a - c\sigma(\tau^{-1}([h])))^r} f(g^{-1}hk) \end{aligned}$$

and

$$\begin{aligned} (\widehat{R}_k\widehat{\pi}_r(g)f)(h) &= (\widehat{\pi}_r(g)f)(hk) \\ &= \frac{1}{(a - c\sigma(\tau^{-1}([hk])))^r} f(g^{-1}hk), \end{aligned}$$

the claim holds since $\tau^{-1}([hk]) = \tau^{-1}([h])$. ■

We will identify $\mathcal{C} = \{M_F : F \in L^\infty(\Gamma, \nu)\}$ with an abelian von Neumann subalgebra \mathcal{N} of \mathcal{M} and define another action of the group $SL_2(\mathbb{R})$ on the subalgebra \mathcal{N} . In fact,

$$\mathcal{N} = \{M_F \in \mathcal{M} : F(gk) = F(g), g \in SL_2(\mathbb{R}), k \in K\}.$$

We have the following result.

Lemma 3.3. *With notations as above, $\mathcal{N} = \mathcal{M} \cap \{\widehat{R}_k : k \in K\}'$.*

Proof. We just need to show that with $M_F \in \mathcal{M}$, $F(gk) = F(g)$ if and only if $M_F\widehat{R}_k = \widehat{R}_kM_F$.

If $F(gk) = F(k)$, then for any $f \in L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)$ and $h \in SL_2(\mathbb{R})$, we have

$$(M_F\widehat{R}_k f)(h) = F(h)f(hk) = F(hk)f(hk) = (\widehat{R}_kM_F f)(h).$$

When $M_F \widehat{R}_k = \widehat{R}_k M_F$, the claim holds since, for any $f \in L^2(SL_2(\mathbb{R}), \varphi^{2r} d\mu)$ and $h \in SL_2(\mathbb{R})$, we have

$$F(h)f(hk) = (M_F \widehat{R}_k f)(h) = (\widehat{R}_k M_F f)(h) = F(hk)f(hk). \quad \blacksquare$$

With $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we have $\widehat{\pi}_r(g)\mathcal{M}\widehat{\pi}_r(g)^* = \mathcal{M}$ and $\widehat{\pi}_r(g)\mathcal{N}\widehat{\pi}_r(g)^* = \mathcal{N}$. In fact, for any $M_F \in \mathcal{B}$, we have

$$\begin{aligned} (\widehat{\pi}_r(g)M_F\widehat{\pi}_r(g)^*f)(h) &= \frac{1}{(a - c\sigma(\tau^{-1}([h])))^r} (M_F\widehat{\pi}_r(g)^*f)(g^{-1}h) \\ &= \frac{1}{(a - c\sigma(\tau^{-1}([h])))^r} F(g^{-1}h)(\widehat{\pi}_r(g)^*f)(g^{-1}h) \\ &= \frac{F(g^{-1}h)f(h)}{(a - c\sigma(\tau^{-1}([h])))^r (d + c\sigma(\tau^{-1}([g^{-1}h])))^r} \\ &= (M_{F(g^{-1}\cdot)}f)(h). \end{aligned}$$

As $M_F \in \mathcal{N}$ or \mathcal{M} is equivalent to $M_{F(g^{-1}\cdot)} \in \mathcal{N}$ or \mathcal{M} , then the claim is true. Thus $\gamma_g(M_F) = \widehat{\pi}_r(g)M_F\widehat{\pi}_r(g)^*$ is a continuous automorphism action of $SL_2(\mathbb{R})$ on \mathcal{M} with respect to the weak operator topology and the restriction of γ on the subalgebra \mathcal{N} is also a continuous action. The crossed product $\mathcal{R}(\mathcal{M}, \gamma)$ is the von Neumann algebra generated by the bounded linear operators $\widehat{\Pi}(M_F)$, $M_F \in \mathcal{M}$ and $\widehat{\Lambda}(g)$, $g \in SL_2(\mathbb{R})$ acting on $L^2(SL_2(\mathbb{R}), \varphi^{2r} d\mu) \otimes L^2(SL_2(\mathbb{R}), d\mu)$ which is defined by

$$(\widehat{\Pi}(M_F)v\phi)(h) = \phi(h)\gamma_{h^{-1}}(M_F)v,$$

and

$$\widehat{\Lambda}(g) = I \otimes L_g,$$

respectively. Note that $\mathcal{R}(\mathcal{N}, \gamma)$ is generated by the operators $\widehat{\Pi}(M_F)$, $M_F \in \mathcal{N}$ and $\widehat{\Lambda}(g)$, $g \in SL_2(\mathbb{R})$. Now we can get the main result of this paper.

Theorem 3.4. *With notations as above, let \mathcal{L}_K be the group von Neumann algebra of K . Then the following von Neumann algebras $\mathcal{R}(\mathcal{A}, \alpha)$, $\mathcal{R}(\mathcal{C}, \beta)$, $\mathcal{R}(\mathcal{N}, \gamma)$ and $\mathcal{B}(L^2(P, 2dxdy/y^{3-2r})) \overline{\otimes} \mathcal{L}_K$ are *-isomorphic.*

Proof. By Lemma 3.1., the crossed product $\mathcal{R}(\mathcal{A}, \alpha)$ is *-isomorphic to the crossed product $\mathcal{R}(\mathcal{C}, \beta)$. Our plan to verify the rest is to show that $\mathcal{R}(\mathcal{C}, \beta) \cong \mathcal{R}(\mathcal{N}, \gamma)$ and $\mathcal{R}(\mathcal{N}, \gamma) \cong \mathcal{B}(L^2(P, 2dxdy/y^{3-2r})) \overline{\otimes} \mathcal{L}_K$.

Recall that $\mathcal{C} = \{M_F : F \in L^\infty(\Gamma, \nu)\}$ where Γ is the quotient space of left coset with respect to the group K . From the mapping τ , with $M_F \in \mathcal{N}$,

$\tilde{F}([g]) \triangleq F(g)$ is well defined and $M_{\tilde{F}} \in \mathcal{C}$. Conversely, for any $\tilde{F} \in L^\infty(\Gamma, \nu)$, $F(g) \triangleq \tilde{F}([g])$ is well defined and M_F belongs to \mathcal{N} . Furthermore, we have $\|F\| = \|\tilde{F}\|$. Thus $\omega : M_F \mapsto M_{\tilde{F}}$ is an *-isomorphism from \mathcal{N} onto \mathcal{C} .

Claim 1. $\mathcal{R}(\mathcal{C}, \beta) \cong \mathcal{R}(\mathcal{N}, \gamma)$ is implemented by the mapping ω .

According to the above-mentioned results, we just need to show that the mapping ω interweave the actions β and γ of the group $SL_2(\mathbb{R})$ on \mathcal{C} and \mathcal{N} . With $M_F \in \mathcal{N}$, let $G \triangleq F(g^{-1}\cdot)$. Then for any $g, h \in SL_2(\mathbb{R})$, we have

$$\begin{aligned} (\omega(\gamma_g(M_F))f)([h]) &= (\omega(M_{F(g^{-1}\cdot)})f)([h]) \\ &= (M_{\tilde{G}}f)([h]) = \tilde{G}([h])f([h]) \\ &= G(h)f([h]) = F(g^{-1}h)f([h]) \end{aligned}$$

and

$$\begin{aligned} (\beta_g(\omega(M_F))f)([h]) &= (\beta_g(M_{\tilde{F}})f)([h]) \\ &= \tilde{F}([g^{-1}h])f([h]) = F(g^{-1}h)f([h]). \end{aligned}$$

Thus the claim 1. holds.

Now we investigate the structure of the crossed product $\mathcal{R}(\mathcal{N}, \gamma)$. As the action γ of $SL_2(\mathbb{R})$ is implemented by the unitary representation $\hat{\pi}$, the commutant $\mathcal{R}(\mathcal{N}, \gamma)'$ of $\mathcal{R}(\mathcal{N}, \gamma)$ equals to the von Neumann subalgebra $\{T' \otimes I, \hat{\pi}(g) \otimes R_g : T' \in \mathcal{N}', g \in SL_2(\mathbb{R})\}''$ of $\mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu) \otimes L^2(SL_2(\mathbb{R}), d\mu))$. Thus

$$\begin{aligned} \mathcal{R}(\mathcal{N}, \gamma) &= (\mathcal{N}' \overline{\otimes} I)' \cap \{\hat{\pi}(g) \otimes R_g : g \in SL_2(\mathbb{R})\}' \\ &= (\mathcal{N} \overline{\otimes} \mathcal{B}(L^2(SL_2(\mathbb{R}), d\mu))) \cap \{\hat{\pi}(g) \otimes R_g : g \in SL_2(\mathbb{R})\}'. \end{aligned}$$

From Lemma 3.3, we have

$$\begin{aligned} \mathcal{N} \overline{\otimes} \mathcal{B}(L^2(SL_2(\mathbb{R}), d\mu)) &= (\mathcal{M} \cap \{\hat{R}_k : k \in K\}') \overline{\otimes} \mathcal{B}(L^2(SL_2(\mathbb{R}), d\mu)) \\ &= (\mathcal{M} \overline{\otimes} \mathcal{B}(L^2(SL_2(\mathbb{R}), d\mu))) \cap \{\hat{R}_k \otimes I : k \in K\}'. \end{aligned}$$

Note that the commutant algebra of $\mathcal{R}(\mathcal{M}, \gamma)$ is the von Neumann subalgebra of $\mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu) \otimes L^2(SL_2(\mathbb{R}), d\mu))$ generated by the operators $T \otimes I$, $T \in \mathcal{M}' (= \mathcal{M}) \subset \mathcal{N}'$ and $\hat{\pi}(g) \otimes R_g$, $g \in SL_2(\mathbb{R})$. Therefore,

$$\begin{aligned} \mathcal{R}(\mathcal{M}, \gamma) &= (\mathcal{M} \overline{\otimes} CI)' \cap \{\hat{\pi}(g) \otimes R_g : g \in SL_2(\mathbb{R})\}' \\ &= (\mathcal{M} \overline{\otimes} \mathcal{B}(L^2(SL_2(\mathbb{R}), d\mu))) \cap \{\hat{\pi}(g) \otimes R_g : g \in SL_2(\mathbb{R})\}'. \end{aligned}$$

Hence,

$$\mathcal{R}(\mathcal{N}, \gamma) = \mathcal{R}(\mathcal{M}, \gamma) \cap \{\widehat{R}_k \otimes I : k \in K\}'.$$

Claim 2. $\mathcal{R}(\mathcal{M}, \gamma) \cong \mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)) \overline{\otimes} CI$.

In fact, this isomorphism is induced by a unitary operator W which is defined as follows:

$$(W\xi)(g, h) = \varphi(g)^{-r} \varphi(h)^r \xi(h, gh^{-1}),$$

where $\xi \in L^2(SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), \varphi^{2r}d\mu \times d\mu)$ and μ is the above-mentioned Haar measure on $SL_2(\mathbb{R})$. With $\xi, \eta \in L^2(SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), \varphi^{2r}d\mu \times d\mu)$, we have

$$\begin{aligned} \|W\xi\|^2 &= \int_G \int_G \varphi(g)^{-2r} \varphi(h)^{2r} |\xi(h, gh^{-1})|^2 \varphi(g)^{2r} d\mu(g) d\mu(h) \\ &= \int_G \left(\int_G |\xi(h, gh^{-1})|^2 d\mu(g) \right) \varphi(h)^{2r} d\mu(h) \\ &= \int_G \int_G |\xi(h, g)|^2 \varphi(h)^{2r} d\mu(h) d\mu(g) \\ &= \|\xi\|^2 \end{aligned}$$

and

$$\begin{aligned} \langle W\xi, \eta \rangle &= \int \int \varphi(g)^r \varphi(h)^r \xi(h, gh^{-1}) \overline{\eta(g, h)} d\mu(g) d\mu(h) \\ &= \int \int \varphi(h)^r \varphi(sh)^r \xi(h, s) \overline{\eta(sh, h)} d\mu(sh) d\mu(h) \\ &= \int \int \varphi(h)^r \varphi(sh)^r \xi(h, s) \overline{\eta(sh, h)} d\mu(s) d\mu(h) \\ &= \int \int \xi(h, s) \overline{\varphi(h)^{-r} \varphi(sh)^r \eta(sh, h)} \varphi(h)^{2r} d\mu(s) d\mu(h) \\ &= \langle \xi, W^* \eta \rangle. \end{aligned}$$

Thus $(W^*\xi)(g, h) = \varphi(g)^{-r} \varphi(hg)^r \xi(hg, g)$ and $W^*W = WW^* = I$.

With $M_F \in \mathcal{M}$ and $g \in SL_2(\mathbb{R})$, recalling that $(\widehat{\Pi}(M_F)\xi)(s, t) = F(ts)\xi(s, t)$ and $(\widehat{\Lambda}(g)\xi) = \xi(s, g^{-1}t)$ for any $\xi \in L^2(SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), \varphi^{2r}d\mu \times d\mu)$. We have

$$\begin{aligned} (W\widehat{\Pi}(M_F)W^*\xi)(g, h) &= \varphi(g)^{-r} \varphi(h)^r (\widehat{\Pi}(M_F)W^*\xi)(h, gh^{-1}) \\ &= F(g) \varphi(g)^{-r} \varphi(h)^r (W^*\xi)(h, gh^{-1}) \\ &= F(g) \xi(g, h) \\ &= ((M_F \otimes I)\xi)(g, h) \end{aligned}$$

and

$$\begin{aligned}(W\widehat{\Lambda}(g)W^*\xi)(s, t) &= \varphi(s)^{-r}\varphi(t)^r((I \otimes L_g)W^*\xi)(t, st^{-1}) \\ &= \varphi(s)^{-r}\varphi(t)^r(W^*\xi)(t, g^{-1}st^{-1}) \\ &= \varphi(s)^{-r}\varphi(g^{-1}s)^r\xi(g^{-1}s, t).\end{aligned}$$

Thus the von Neumann algebra generated by the operators $W\widehat{\Pi}(M_F)W^*$, $M_F \in \mathcal{B}$ and $W\widehat{\Lambda}(g)W^*$, $g \in SL_2(\mathbb{R})$ is $\mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)) \overline{\otimes} CI$. Then the claim 2. holds.

With $k \in K$, we have $W(\widehat{R}_k \otimes I)W^* = \widehat{R}_k \otimes R_k$, since for any $\xi \in L^2(SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), \varphi^{2r}d\mu \times d\mu)$,

$$\begin{aligned}(W(\widehat{R}_k \otimes I)W^*\xi)(g, h) &= \varphi(g)^{-r}\varphi(h)^r((\widehat{R}_k \otimes I)W^*\xi)(h, gh^{-1}) \\ &= \varphi(g)^{-r}\varphi(h)^r(W^*\xi)(hk, gh^{-1}) \\ &= \varphi(g)^{-r}\varphi(h)^r\varphi(hk)^{-r}\varphi(gk)^r\xi(gk, hk) \\ &= \xi(gk, hk) \\ &= ((\widehat{R}_k \otimes R_k)\xi)(g, h).\end{aligned}$$

Summarizing all the above facts, we have

$$W\mathcal{R}(\mathcal{N}, \gamma)W^* = (\mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)) \overline{\otimes} CI) \cap \{\widehat{R}_k \otimes R_k : k \in K\}'.$$

As K is an abelian group, thus

$$\mathcal{R}(\mathcal{N}, \gamma) \cong \mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)) \cap \{\widehat{R}_k : k \in K\}'.$$

To determine the structure of $\mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)) \cap \{\widehat{R}_k : k \in K\}'$, we recall the decomposition $\mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)) \cong \mathcal{B}(L^2(P, 2dxdy/y^{3-2r})) \overline{\otimes} \mathcal{B}(L^2(K, d\theta))$. Consider the right regular representation ρ_k of K on $L^2(K, d\theta)$. Denote by \mathcal{R}_K the von Neumann algebra generated by ρ_k , $k \in K$. Then $\mathcal{R}'_K = \mathcal{L}_K$, the abelian group von Neumann algebra generated by the left regular representation of K on $L^2(K, d\theta)$. According to the Iwasawa decomposition of $SL_2(\mathbb{R})$, under the above mentioned decomposition of $\mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu))$, the representation \widehat{R}_k of K is $I \otimes \rho_k$. Hence

$$\begin{aligned}\mathcal{R}(\mathcal{N}, \gamma) &\cong \mathcal{B}(L^2(SL_2(\mathbb{R}), \varphi^{2r}d\mu)) \cap \{\widehat{R}_k : k \in K\}' \\ &\cong (\mathcal{B}(L^2(P, 2dxdy/y^{3-2r})) \overline{\otimes} \mathcal{B}(L^2(K, d\theta))) \cap (CI \overline{\otimes} \mathcal{R}_K)' \\ &= \mathcal{B}(L^2(P, 2dxdy/y^{3-2r})) \overline{\otimes} \mathcal{L}_K.\end{aligned}$$

By summarizing all the above facts, we can get this theorem. ■

According to this result, the crossed product von Neumann algebra $\mathcal{R}(\mathcal{A}, \alpha)$ is of type I. In fact, the von Neumann algebra $\mathcal{B}(L^2(\mathcal{P}, 2dxdy/y^{3-2r}))$ is of type I and the abelian von Neumann algebra \mathcal{L}_K is also of type I, thus the claim holds since the tensor product of two von Neumann algebras of type I is still of type I.

Remark 3.5. For any $r \geq 0$, with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we define the operator $\bar{\pi}_r(g)$ acting on the Hilbert space $L^2(\mathbb{H}, dxdy/y^{2-r})$ as following:

$$(\bar{\pi}_r(g)f)(z) = |a - cz|^{-r} f(g^{-1}z).$$

Then $\bar{\pi}_r(g)$ is a unitary operator and $\bar{\pi}_r$ is a unitary representation of $SL_2(\mathbb{R})$. We define the action $\bar{\alpha}$ of $SL_2(\mathbb{R})$ on the MASA $\mathcal{A} = \{M_F : F \in L^\infty(\mathbb{H}, dxdy/y^{2-r})\}$ as

$$\bar{\alpha}_g(T) = \bar{\pi}_r(g)T\bar{\pi}_r(g)^*,$$

where $g \in SL_2(\mathbb{R})$ and $T \in \mathcal{A}$. Similaring to the crossed product $\mathcal{R}(\mathcal{A}, \alpha)$, the crossed product $\mathcal{R}(\mathcal{A}, \bar{\alpha})$ has the same result.

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