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DERIVATIVES OF BERNSTEIN OPERATORS AND SMOOTHNESS WITH JACOBI WEIGHTS

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Abstract. Using the modulus of smoothness with *Jacobi* weights $\omega^2_{\varphi^\lambda}(f,t)_\omega$, the relationship between the derivatives *Bernstein* operators and the smoothness of the function its approximated in the weighted approximation is characterized, an equivalent theorem between *Bernstein* operators and the modulus of smoothness with *Jacobi* weights is established. The corresponding results without weights are generalized. In addition, we obtain the direct theorem in the approximation with *Jacobi* weights by *Bernstein* operators.

1. Introduction and Main Results

Let $C[0,1]$ be the set of continuous functions on $[0,1]$, then the *Bernstein* operators on are given by

$$B_n(f;x)=\sum_{k=0}^{n}P_{n,k}(x)f(\frac{k}{n}),$$

where $P_{n,k}(x)=C_n^k x^k(1-x)^{n-k}, x\in[0,1]$, and $f\in C[0,1]$, $n\in N$.

It was shown by Ditzian. Z. [1] in 1985 that if $0<\alpha<2$ then

$$\omega_2(f,t)=O(t^\alpha)\Longleftrightarrow\mid B_n''(f,x)\mid\leq M\left\{\min\left[n^2,\frac{n}{x(1-x)}\right]\right\}^{\frac{2-\alpha}{2}}$$

In 1992, Zhou, D. X. [5] extend the result of Ditzian [1] to higher orders of smoothness. In addition, the close connection between the derivatives of the *Bernstein* type operators and the smoothness of function which has been investigated by Z. Ditzian, V. Totik, K. G. Ivanov and some other mathematicians [2, 4].

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In [3], the author obtained the equivalent theorem between *Bernstein* operators and the modulus of smoothness(without weights), but in the weighted norm $\sup_{x\in[0,1]} \mid \omega(x)f(x) \mid$, *Bernstein* operators does not converge [5], so it is not simple generalization of the results of approximation without weights [3]. Using the norm

$$\|f\|_\omega = \max_{x\in[0,1]} \mid \omega(x)f(x) \mid + \mid f(0) \mid + \mid f(1) \mid, \tag{1.1}$$

where $\omega(x) = x^a(1-x)^b$, $0 < a, b < 1$ is Jacobi weights, Zhou, D. X. [5] showed the boundness of *Bernstein* operators with the Jacobi weights.

Since we only consider the *Bernstein* operators from now on, let us suppose that $\varphi^2(x) = x(1-x)$. First we give some notations,

$$C_0 = \{f \in C[0,1], f(0) = f(1) = 0\}.$$

To characterize functions in terms of theirs behaviors of various moduli of smoothness, we assume throughout the paper that $0 \le \lambda \le 1$.

We recall the weighted K-functional given by [4]

$$K_{\varphi^\lambda}(f,t^2)_\omega = \inf_{g\in D}\left\{\|f-g\|_\omega + t^2\left\|\varphi^{2\lambda}g^{"}\right\|_\omega\right\}, \tag{1.2}$$

where $D = \{g \in C[0,1] : g' \in A.C.loc, and \parallel \varphi^{2\lambda}g'' \parallel_\omega < \infty\}$ is a weighted Sobolev space.

And modulus of smoothness with *Jacobi* weights [4] is defined by

$$\omega^2_{\varphi^\lambda}(f,t)_\omega = \sup_{0<h\le t}\left\|\Delta^2_{h\varphi^\lambda}f(x)\right\|_\omega, \tag{1.3}$$

where $\Delta^2_{h\varphi^\lambda}f(x) = f(x+h\varphi^\lambda) - 2f(x) + f(x-h\varphi^\lambda)$.

Remark 1. By [5], Since all polynomials are included in D, and obvious, through the equation (1), we have $\|f\|_\omega \le 3 \parallel f \parallel_\infty (\parallel f \parallel_\infty = \sup_{x\in[0,1]} |f(x)|)$, So D is dense in the $(C[0,1], \|\cdot\|_\omega)$. Hence $K_{\varphi^\lambda}(f,t^2)_\omega \to 0 (t \to 0)$.

For $f \in C_0$, from [4], there exists a positive constant C such that

$$C^{-1}\omega^2_{\varphi^\lambda}(f,t)_\omega \le K_{\varphi^\lambda}(f,t^2)_\omega \le C\omega^2_{\varphi^\lambda}(f,t)_\omega \tag{1.4}$$

Throughout the paper, the letter C, appearing in various formulas, denotes a positive constant independent of n , x and f. Its value may be different at different occurrences, even within the same formula.

In the present paper, we characterized relationship between the derivatives of *Bernstein* operators and function its approximated using the modulus of smoothness with *Jacobi* weights $\omega^2_{\varphi^\lambda}(f,t)_\omega$ in C_0, which extend the corresponding results in the approximation without weights. The main results can be stated as follows.

Theorem 1. *For $f \in C_0$, we have*

$$\left|\omega(x)\varphi^{2\lambda}(x)B_n^{''}(f;x)\right| \leq C((n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^{-2}\omega^2_{\varphi^\lambda}(f,n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))_\omega$$
$$\leq C\delta_n^{-1}(x)\omega^2_{\varphi^\lambda}(f,\delta_n^{\frac{1}{2}}(x))_\omega$$

where $\delta_n(x) = \frac{\varphi^{2-2\lambda}(x)}{n}\max\{n^{-1}\varphi^{-2}(x),1\}$.

Theorem 2. *For $f \in C_0$, $0<\alpha<2$, we have*

$$\left|\omega(x)\varphi^{2\lambda}(x)B_n^{''}(f;x)\right| = O(\delta_n^{\frac{\alpha-2}{2}})$$

is equivalent to

$$\omega^2_{\varphi^\lambda}(f,t)_\omega = O(t^\alpha).$$

Remark 2. The above equivalence relation without weighs have been improved in [7], that is, the quantity $\delta_n(x)$ in Theorem 2 can be replaced by $\frac{\varphi^2(x)}{n}$ when $\lambda=0$ and $a=b=0$. We conjecture that for Jacobi weights, there holds
$\left|\omega(x)\varphi^{2\lambda}(x)B_n^{''}(f;x)\right| = O((\frac{\varphi^2(x)}{n})^{\frac{\alpha-2}{2}}) \Longleftrightarrow \omega^2_{\varphi^\lambda}(f,t)_\omega = O(t^\alpha)$.
In fact, the proof of Theorem 1 shows that the direct part holds true.

2. Some Lemmas

To prove our main results, we need the following interesting results. Among these results, Lemma 1 and Lemma 2 can be found [5, 6], and then we will give Lemma 3 and Lemma 4.

Lemma 1. ([6]). *For* $0<t<\frac{1}{16}$, $\frac{t}{2}<x<1-\frac{t}{2}$, *and* $0 \leq \beta \leq 2$, *we have*

$$\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\varphi^{-\beta}(x+u+v)\mathrm{d}u\mathrm{d}v \leq Ct^2\varphi^{-\beta}(x).$$

Lemma 2. ([5]). For $f \in C_{[0,1]}$, we have

$$\left|\omega(x)\varphi^2(x)B_n^{''}(f;x)\right| \leq Cn\left\|f\right\|_\omega.$$

Lemma 3. *For $f \in D$, we have*

$$\left|\omega(x)\varphi^{2\lambda}(x)B_n^{''}(f;x)\right| \leq C\left\|\varphi^{2\lambda}f^{''}\right\|_\omega.$$

Proof. In the norm (1.1), by (see [5, Proposition 2.3]),

$$\|B_n f\|_\omega \leq 2^{1+a+b} \parallel f \parallel_\omega .$$

And from the property of *Bernstein* operators, we can easily get

$$B_n(f, x) = B_n(f_1, 1-x); \qquad B_n(f, 1-x) = B_n(f_1, x),$$

where $f_1(x) = f(1-x)$. So we only need to estimate $\left\|\omega\varphi^{2\lambda} B_n^{"} f\right\|_{C[0,\frac{1}{2}]}$ as follows.

$$\begin{aligned}
&\left|\omega(x)\varphi^{2\lambda}(x)B_n^{"}f\right| \\
&= |\omega(x)\varphi^{2\lambda}(x)n(n-1)\sum_{k=0}^{n-2}\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f^{"}(\frac{k}{n}+u+v)\mathrm{d}u\mathrm{d}vP_{n-2,k}(x)| \\
&= |\omega(x)\varphi^{2\lambda}(x)n(n-1)\sum_{k=1}^{n-3}\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f^{"}(\frac{k}{n}+u+v)\mathrm{d}u\mathrm{d}vP_{n-2,k}(x)| \\
&\quad +|\omega(x)\varphi^{2\lambda}(x)n(n-1)\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f^{"}(\frac{k}{n}+u+v)\mathrm{d}u\mathrm{d}vP_{n-2,0}(x)| \\
&\quad +|\omega(x)\varphi^{2\lambda}(x)n(n-1)\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f^{"}(\frac{k}{n}+u+v)\mathrm{d}u\mathrm{d}vP_{n-2,n-2}(x)| \\
&\triangleq I + J + K.
\end{aligned}$$

By the *Hölder* Inequality, we obtain

$$\begin{aligned}
I =&|\omega(x)\varphi^{2\lambda}(x)n(n-1)\sum_{k=1}^{n-3}\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f^{"}(\frac{k}{n}+u+v)\mathrm{d}u\mathrm{d}vP_{n-2,k}(x)| \\
\leq&\omega(x)\varphi^{2\lambda}(x)\sum_{k=1}^{n-3}(\frac{k}{n})^{-\lambda-a}(1-\frac{k+2}{n})^{-\lambda-b}\mathrm{d}u\mathrm{d}vP_{n-2,k}(x)\left\|\varphi^{2\lambda}f^{"}\right\|_\omega \\
\leq&2^{3\lambda+a+b}\omega(x)\varphi^{2\lambda}(x)\sum_{k=1}^{n-3}[(\frac{n-2}{k+1})^{\lambda+a}+(\frac{n-2}{n-1-k})^{\lambda+b}]P_{n-2,k}(x)\left\|\varphi^{2\lambda}f^{"}\right\|_\omega
\end{aligned}$$

$$\begin{aligned}
\leq&2^{3\lambda+a+b}\omega(x)\varphi^{2\lambda}(x)\left\{\left[\sum_{k=1}^{n-3}\left(\frac{n-2}{k+1}\right)^2P_{n-2,k}(x)\right]^{\frac{\lambda+a}{2}}\right. \\
&\left.+\left[\sum_{k=1}^{n-3}\left(\frac{n-2}{n-1-k}\right)^2P_{n-2,k}(x)\right]^{\frac{\lambda+b}{2}}\right\}\left\|\varphi^{2\lambda}f^{"}\right\|_\omega .
\end{aligned}$$

Making use of the following results of [8]

$$\sum_{k=0}^{n}(\frac{n}{k+1})^2 P_{n,k}(x) \le 2x^{-2}, \quad \sum_{k=0}^{n}(\frac{n}{n-k+1})^2 P_{n,k}(x) \le 2(1-x)^{-2},$$

we obtain that $I \le C \left\|\varphi^{2\lambda} f"\right\|_\omega$ and

$$\begin{aligned} J &= |\omega(x)\varphi^{2\lambda}(x)n(n-1)\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f"(u+v)\mathrm{d}u\mathrm{d}v P_{n-2,0}(x)| \\ &\le n^2\omega(x)\varphi^{2\lambda}(x)\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}(u+v)^{-\lambda-a}(1-u-v)^{-\lambda-b}\mathrm{d}u\mathrm{d}v P_{n-2,0}(x)\left\|\varphi^{2\lambda} f"\right\|_\omega. \end{aligned}$$

The following we will divide into three parts to estimate J.

Case 1. For $\lambda + a < 1$, we have

$$\begin{aligned} J &\le 2^{\lambda+b} n^2\omega(x)\varphi^{2\lambda}(x)|\int_0^{\frac{1}{n}}(u+1)^{1-\lambda-a}\mathrm{d}u P_{n-2,0}(x)|\left\|\varphi^{2\lambda} f"\right\|_\omega \\ &\le Cn^2 x^{\lambda+a}(1-x)^{n-2+\lambda+b}\left\|\varphi^{2\lambda} f"\right\|_\omega \\ &\le C\left\|\varphi^{2\lambda} f"\right\|_\omega \end{aligned}$$

Case 2. For $\lambda + a > 1$, we have

$$\begin{aligned} J &\le Cn^2\omega(x)\varphi^{2\lambda}(x)|\int_0^{\frac{1}{n}} u^{1-\lambda-a}\mathrm{d}u P_{n-2,0}(x)|\left\|\varphi^{2\lambda} f"\right\|_\omega \\ &\le Cn^{\lambda+a} x^{\lambda+a}(1-x)^{n-2+\lambda+b}\left\|\varphi^{2\lambda} f"\right\|_\omega \\ &\le C\left\|\varphi^{2\lambda} f"\right\|_\omega \end{aligned}$$

Case 3. For $\lambda + a = 1$, we can easily obtain by directly computing

$$\begin{aligned} J &\le 2^{\lambda+b} n^2\omega(x)\varphi^{2\lambda}(x)|\int_0^{\frac{1}{n}}(\ln(u+\frac{1}{n})-\ln u)\mathrm{d}u P_{n-2,0}(x)|\left\|\varphi^{2\lambda} f"\right\|_\omega \\ &\le Cn\omega(x)\varphi^{2\lambda}(x)P_{n-2,0}(x)\left\|\varphi^{2\lambda} f"\right\|_\omega \\ &\le Cnx(1-x)^{n-2+\lambda+b}\left\|\varphi^{2\lambda} f"\right\|_\omega \\ &\le C\left\|\varphi^{2\lambda} f"\right\|_\omega. \end{aligned}$$

Similarly, we can also estimate the bound of K and the proof of lemma 3 is complete.

Lemma 4. *Let* $\omega(x) = x^a(1-x)^b$*,* $0 < a, b < 1$*. Then for* $f \in C[0,1]$*,*

$$|\omega(x)(B_n(f,x) - f(x))| \le C\omega^2_{\varphi^\lambda}(f, \delta_n^{\frac{1}{2}}(x))_\omega.$$

Proof. By (1.2) and (1.4), let $g = g_n = g_{n,x,\lambda}$, for the fixed x and λ, we have

$$\|f - g_n\|_\omega \le C\omega^2_{\varphi^\lambda}(f, \delta_n^{\frac{1}{2}}(x))_\omega \tag{2.1}$$

and

$$\delta_n(x) \left\|\varphi^{2\lambda} g_n''\right\|_\omega \le C\omega^2_{\varphi^\lambda}(f, \delta_n^{\frac{1}{2}}(x))_\omega. \tag{2.2}$$

Thus,

$$\begin{aligned}
&|\omega(x)(B_n(f,x) - f(x))| \\
\le &|\omega(x)(B_n(f - g_n, x) - (f(x) - g_n(x))| + |\omega(x)(B_n(g_n, x) - g_n(x))| \\
\le &C\omega^2_{\varphi^\lambda}(f, \delta_n^{\frac{1}{2}}(x))_\omega + |\omega(x)(B_n(g_n, x) - g_n(x))|.
\end{aligned}$$

By [4], we have

$$\begin{aligned}
&|\omega(x)(B_n(g_n, x) - g_n(x))| \\
\le &\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} |\omega(x) \int_{\frac{k}{n}}^{x} (x - v) g_n''(v) \mathrm{d}v| \\
\le &\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} \omega(x) \frac{|x - \frac{k}{n}|}{\varphi^{2\lambda}(x)} |\int_{\frac{k}{n}}^{x} \varphi^{2\lambda}(v)\omega(v) g_n''(v) \omega^{-1}(v) \mathrm{d}v| \\
\le &\sum_{k=1}^{n-1} P_{n,k}(x)\omega(x) \frac{|x - \frac{k}{n}|}{\varphi^{2\lambda}(x)} |\int_{\frac{k}{n}}^{x} \omega^{-1}(v)\mathrm{d}v| \left\|\varphi^{2\lambda} g_n''\right\|_\omega \\
&+ \omega(x)|P_{n,0} \int_0^x (x - v) g_n''(v)\mathrm{d}v| + \omega(x)|P_{n,n} \int_1^x (x - v) g_n''(v)\mathrm{d}v| \\
\le &\sum_{k=1}^{n-1} P_{n,k}(x)\omega(x) \frac{|x - \frac{k}{n}|^2}{\varphi^{2\lambda}(x)} \left(\frac{1}{\omega(\frac{k}{n})} + \frac{1}{\omega(x)}\right) \left\|\varphi^{2\lambda} g_n''\right\|_\omega \\
&+ \omega(x)|P_{n,0} \int_0^x (x - v) g_n''(v)\mathrm{d}v| + \omega(x)|P_{n,n} \int_1^x (x - v) g_n''(v)\mathrm{dv}| \\
\triangleq &I + J + K.
\end{aligned}$$

For I, by the *Cauchy-Schwarz* Inequality, we obtain

$$\begin{aligned}
I &\le \left\{ \frac{\omega(x)}{\varphi^{2\lambda}(x)} \left[\left(\sum_{k=1}^{n-1} P_{n,k}(x) |x - \frac{k}{n}|^4 \right) \left(\sum_{k=1}^{n-1} P_{n,k}(x) \frac{1}{\omega^2(\frac{k}{n})} \right) \right]^{\frac{1}{2}} \right. \\
&\quad \left. + \frac{\varphi^{2-2\lambda}(x)}{n} \right\} \left\| \varphi^{2\lambda} g_n^{"} \right\|_\omega \\
&\le \left\{ \frac{\omega(x)}{\varphi^{2\lambda}(x)} \left[\frac{\varphi(x)}{n} \left(\frac{1}{n} + \varphi^2(x) \right)^{\frac{1}{2}} \omega^{-1}(x) \right] + \frac{\varphi^{2-2\lambda}(x)}{n} \right\} \left\| \varphi^{2\lambda} g_n^{"} \right\|_\omega \\
&\le \left[\frac{\varphi^{1-2\lambda}(x)}{n} \left(\frac{1}{n} + \varphi^2(x) \right)^{\frac{1}{2}} + \frac{\varphi^{2-2\lambda}(x)}{n} \right] \left\| \varphi^{2\lambda} g_n^{"} \right\|_\omega \\
&\le C\delta_n(x) \left\| \varphi^{2\lambda} g_n^{"} \right\|_\omega .
\end{aligned}$$

In the same way, for J we have

$$\begin{aligned}
J &\le \omega(x) \left| (1-x)^n \int_0^x (x-v) g_n^{"}(v) \mathrm{d}v \right| \\
&\le \omega(x) \frac{x(1-x)^n}{\varphi^{2\lambda}(x)} \left\| \varphi^{2\lambda} g_n^{"} \right\|_\omega \left| \int_0^x \omega^{-1}(v) \mathrm{d}v \right| \\
&\le \omega(x) \frac{x(1-x)^n}{\varphi^{2\lambda}(x)} \left\| \varphi^{2\lambda} g_n^{"} \right\|_\omega \left| \int_0^x v^{-a} \mathrm{d}v \right| (1-x)^{-b} \\
&\le \frac{1}{1-a} \varphi^{-2\lambda}(x) x^2 (1-x)^n \left\| \varphi^{2\lambda} g_n^{"} \right\|_\omega \\
&\le C \frac{\varphi^{2-2\lambda}(x)}{n} \left\| \varphi^{2\lambda} g_n^{"} \right\|_\omega .
\end{aligned}$$

Similarly, we can also estimate the bound of K.

The proof of the Lemma 4 is completed.

Remark 3. Lemma 4 is the direct theorem in the approximation with *Jacobi* weights by *Bernstein* operators in $C[0,1]$, which extend the result in the approximation without weights.

3. Proof of Main Results

Proof of Theorem 1. For $f \in C_0$, by Lemma 2, we have

$$\left| \omega(x) \varphi^{2\lambda}(x) B_n^{"}(f;x) \right| \le Cn\varphi^{2-2\lambda} \|f\|_\omega \le C(n^{-\frac{1}{2}} \varphi^{1-\lambda}(x))^{-2} \|f\|_\omega . \quad (3.1)$$

For all $g \in D$, by (3.7) and Lemma 3, we have

$$\begin{aligned}&\left|\omega(x)\varphi^{2\lambda}(x)B_n^{"}(f;x)\right|\\ \leq&\omega(x)\varphi^{2\lambda}(x)\left|B_n^{"}(f-g;x)\right|+\omega(x)\varphi^{2\lambda}(x)\left|B_n^{"}(g;x)\right|\\ \leq&C(n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^{-2}(\|f-g\|_\omega+(n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^2\left\|\varphi^{2\lambda}g^{"}\right\|_\omega).\end{aligned}$$

Thus, by (1.4) and the definition of $\delta_n(x)$, Theorem 1 holds.

Proof of Theorem 2. By Theorem 1 we only need to prove the inverse part. Let $0<t\varphi^\lambda(x)<\frac{1}{8}$ and $2t\varphi^\lambda(x)<x<1-2t\varphi^\lambda(x)$, then by Lemma 1, Lemma 4 and the *Hölder* Inequality, we have

$$\begin{aligned}&|\omega(x)\Delta^2_{t\varphi^\lambda}f(x)|\\ \leq&|\omega(x)\Delta^2_{t\varphi^\lambda}(B_n(f,x)-f(x))|+|\omega(x)\Delta^2_{t\varphi^\lambda}B_n(f,x)|\\ \leq&C|\omega(x)\sum_{j=0}^{2}C_2^j\omega^{-1}(x+(1-j)t\varphi^\lambda(x))\omega^2_{t\varphi^\lambda}(f,n^{-\frac{1}{2}}\varphi^{1-\lambda}(x+(1-j)t\varphi^\lambda(x))_\omega|\\ &+|\omega(x)\int_{-\frac{t\varphi^\lambda}{2}}^{\frac{t\varphi^\lambda}{2}}\int_{-\frac{t\varphi^\lambda}{2}}^{\frac{t\varphi^\lambda}{2}}|B_n^{"}(f,x+u+v)\mathrm{d}u\mathrm{d}v||\\ \leq&C\omega^2_{t\varphi^\lambda}(f,\delta_n^{\frac{1}{2}}(x))_\omega+n^{1-\frac{\alpha}{2}}\int_{-\frac{t\varphi^\lambda}{2}}^{\frac{t\varphi^\lambda}{2}}\int_{-\frac{t\varphi^\lambda}{2}}^{\frac{t\varphi^\lambda}{2}}\varphi^{\alpha-2-\alpha\lambda}(x+u+v)\mathrm{d}u\mathrm{d}v\\ \leq&C\omega^2_{t\varphi^\lambda}(f,\delta_n^{\frac{1}{2}}(x))_\omega+Cn^{1-\frac{\alpha}{2}}\left[\int_{-\frac{t\varphi^\lambda}{2}}^{\frac{t\varphi^\lambda}{2}}\int_{-\frac{t\varphi^\lambda}{2}}^{\frac{t\varphi^\lambda}{2}}\varphi^{-2}(x+u+v)\mathrm{d}u\mathrm{d}v\right]^{1-\frac{\alpha}{2}}\\ &\cdot\left[\int_{-\frac{t\varphi^\lambda}{2}}^{\frac{t\varphi^\lambda}{2}}\int_{-\frac{t\varphi^\lambda}{2}}^{\frac{t\varphi^\lambda}{2}}\varphi^{-2\lambda}(x+u+v)\mathrm{d}u\mathrm{d}v\right]^{\frac{\alpha}{2}}\\ \leq&C\omega^2_{t\varphi^\lambda}(f,\delta_n^{\frac{1}{2}}(x))_\omega+Cn^{1-\frac{\alpha}{2}}[(t\varphi^\lambda(x))^2\varphi^{-2}(x)]^{1-\frac{\alpha}{2}}[(t\varphi^\lambda(x))^2\varphi^{-2\lambda}(x)]^{\frac{\alpha}{2}}\\ \leq&C[\omega^2_{t\varphi^\lambda}(f,\delta_n^{\frac{1}{2}}(x))_\omega+t^2(n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^{\alpha-2}]\\ \leq&C(\omega^2_{t\varphi^\lambda}(f,\delta_n^{\frac{1}{2}}(x))_\omega+t^2\delta_n^{\frac{\alpha-2}{2}}).\end{aligned}$$

Let $\delta=\delta_n(x)$, $A=(2C+1)^{\frac{1}{\alpha}}$ and $\delta=t/A$. By the induction we have that for $k\in N$

$$\begin{aligned}\omega^2_{\varphi^\lambda}(f,t)_\omega &\le C(\omega^2_{\varphi^\lambda}(f;\frac{t}{A})_\omega + t^\alpha A^{2-\alpha})\\ &\le \cdots\\ &\le 2^k C^k[\omega^2_{\varphi^\lambda}(f;tA^{-k})_\omega + t^\alpha A^2\sum_{l=1}^{k}(2CA^{-\alpha})^l]\\ &\le 4t^2(2CA^{-2})^k\|f\|_\infty + t^\alpha A^2 2C/(A^\alpha - 2C).\end{aligned}$$

Thus, letting $k\to\infty$ we obtain

$$\omega^2_{\varphi^\lambda}(f;t)_\omega \le Ct^\alpha.$$

The proof is completed.

Remark 4. In the proof of Theorem 2, we can easily see the assumption $f\in C_0$ can be replaced by $f\in C[0,1]$ in the inverse part, but not in the direct part.

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