

## CLASSES OF MEROMORPHIC $\alpha$ -CONVEX FUNCTIONS

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**Abstract.** For a fixed analytic univalent function  $\phi$ , the class of meromorphic univalent  $\alpha$ -convex functions with respect to  $\phi$  is introduced. A representation theorem for functions in the class, as well as a necessary and sufficient condition for functions to belong to the class are obtained. Also we obtain a sharp growth theorem and estimate on a certain coefficient functional for meromorphic starlike functions with respect to  $\phi$ . Differential subordination and superordination conditions are also obtained for the subclass of meromorphic starlike functions with respect to  $\phi$ .

### 1. INTRODUCTION

Let  $\Sigma$  denote the class of meromorphic univalent functions  $f$  defined on the punctured unit disk  $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$  having the form  $f(z) = 1/z + \sum_{k=0}^{\infty} a_k z^k$ . A function  $f \in \Sigma$  is said to be *meromorphic starlike of order  $\alpha$*  ( $0 \leq \alpha < 1$ ) if  $-\Re[zf'(z)/f(z)] > \alpha$  for all  $z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}$ . We denote by  $\Sigma^*(\alpha)$  the class of all such meromorphic starlike functions of order  $\alpha$  in  $\Delta^*$ .

Several authors [2, 3, 7, 10, 11, 14, 16, 17] have studied various subclasses of  $\Sigma^*(\alpha)$ , as well as subclasses of *meromorphic convex functions of order  $\alpha$* . The latter class is characterized by the property  $-\Re[1 + zf''(z)/f'(z)] > \alpha$ . We shall unify these functions in Definition 1.1.

First we recall the definition of subordination. For two functions  $f$  and  $g$  analytic in  $\Delta$ , we say that the function  $f(z)$  is *subordinate* to  $g(z)$  in  $\Delta$ , and write  $f \prec g$  or  $f(z) \prec g(z)$  ( $z \in \Delta$ ), if there exists a Schwarz function  $w(z)$ , analytic in  $\Delta$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in \Delta$ ), such that  $f(z) = g(w(z))$  ( $z \in \Delta$ ). In particular, if the function  $g$  is *univalent* in  $\Delta$ , the above subordination is equivalent to  $f(0) = g(0)$  and  $f(\Delta) \subset g(\Delta)$ .

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**Definition 1.1.** Let  $\phi(z)$  be an analytic univalent function in  $\Delta$  with  $\phi(0) = 1$ . Let  $\Sigma_\alpha^*(\phi)$  be the class of functions  $f \in \Sigma$  satisfying  $f(z)f'(z) \neq 0$  and

$$(1.1) \quad - \left[ (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \prec \phi(z) \quad (z \in \Delta).$$

The function  $f \in \Sigma_\alpha^*(\phi)$  is called a meromorphic  $\alpha$ -convex function with respect to  $\phi$ . (Here  $\prec$  denotes subordination between analytic functions.) We shall write  $\Sigma_0^*(\phi)$  by  $\Sigma^*(\phi)$ .

With

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1),$$

it is obvious that  $\Sigma_0^*(\phi)$  is the class of meromorphic starlike functions of order  $\alpha$ , while  $\Sigma_1^*(\phi)$  is the class of meromorphic convex functions of order  $\alpha$ . The class  $\Sigma^*(\phi)$  reduces to the class  $\Sigma(\alpha, \beta, \gamma)$  introduced by Kulkarni and Joshi [5] when

$$(1.2) \quad \phi(z) = \frac{1 + \beta(1 - 2\alpha\gamma)z}{1 + \beta(1 - 2\gamma)z} \quad (0 \leq \alpha < 1; 0 < \beta \leq 1; 1/2 \leq \gamma \leq 1).$$

Karunakaran [4] have considered a special case of the class  $\Sigma^*(\phi)$  consisting of functions  $f \in \Sigma$  for which

$$-\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (0 \leq B < 1; -B < A < B),$$

where  $w(z)$  is an analytic function in  $\Delta$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in \Delta$ ). He denoted this class by  $K_1(A, B)$ .

In this paper, a representation theorem as well as a necessary and sufficient condition for functions to belong to  $\Sigma_\alpha^*(\phi)$  is obtained. Also we obtain a sharp growth theorem and estimate for the coefficient functional  $|a_1 - \mu a_0^2|$  for functions in  $\Sigma^*(\phi)$ . Finally we investigate the subclass  $\Sigma^*(\phi)$  from the perspective of first-order differential subordination and superordination [8, 9].

## 2. A REPRESENTATION THEOREM

We first prove a representation formula for functions in the class  $\Sigma_\alpha^*(\phi)$ .

**Theorem 2.1.** A function  $f(z) \in \Sigma_\alpha^*(\phi)$  if and only if

$$[zf(z)]^{1-\alpha}[-z^2f'(z)]^\alpha = \exp \left( \int_0^z \frac{1 - \phi(w(\eta))}{\eta} d\eta \right),$$

where  $w(z)$  is analytic in  $\Delta$  satisfying  $w(0) = 0$  and  $|w(z)| \leq 1$ .

*Proof.* Let  $f(z) \in \Sigma_{\alpha}^*(\phi)$ . Then (1.1) holds and therefore there is a function  $w(z)$  analytic in  $\Delta$  with  $w(0) = 0$  and  $|w(z)| \leq 1$  such that

$$-\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] = \phi(w(z)), \quad (z \in \Delta).$$

Rewriting the above equation in the form

$$\left[(1-\alpha)\left(\frac{1}{z} + \frac{f'(z)}{f(z)}\right) + \alpha\left(\frac{2}{z} + \frac{f''(z)}{f'(z)}\right)\right] = \frac{1 - \phi(w(z))}{z}, \quad (z \in \Delta)$$

and integrating from 0 to  $z$ , we obtain the desired expression upon exponentiation. The converse follows directly by differentiation. ■

**Example 2.1.** For the function  $\phi(z)$  given by (1.2) and with  $\alpha = 0$ , we obtain [5, Theorem 1, p. 198]: Let  $f \in \Sigma$  and  $0 \leq \alpha < 1$ ,  $0 < \beta \leq 1$  and  $1/2 \leq \gamma \leq 1$ . Then  $f \in \Sigma(\alpha, \beta, \gamma)$  if and only if

$$zf(z) = \exp\left(-\int_0^z \frac{2\beta\gamma(1-\alpha)w(\eta)}{[1 + \beta(1-2\gamma)w(\eta)]\eta} d\eta\right)$$

where  $w(z)$  is analytic in  $\Delta$  satisfying  $w(0) = 0$  and  $|w(z)| \leq 1$ .

### 3. A NECESSARY AND SUFFICIENT CONDITION

We need the following subordination result.

**Lemma 3.1.** [13]. Let  $\phi$  be a convex univalent function defined on  $\Delta$  and  $\phi(0) = 1$ . Define  $F(z)$  by

$$F(z) = z \exp\left(\int_0^z \frac{\phi(\eta) - 1}{\eta} d\eta\right).$$

Let  $q(z)$  be analytic in  $\Delta$  and  $q(0) = 1$ . Then

$$(3.1) \quad 1 + \frac{zq'(z)}{q(z)} \prec \phi(z)$$

if and only if for all  $|s| \leq 1$  and  $|t| \leq 1$ ,

$$(3.2) \quad \frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}.$$

Using Lemma 3.1, we obtain the following necessary and sufficient conditions for functions to belong to  $\Sigma_{\alpha}^*(\phi)$ .

**Theorem 3.1.** Let  $\phi(z)$  and  $F(z)$  be as in Lemma 3.1. A function  $f$  belongs to  $\Sigma_\alpha^*(\phi)$  if and only if for all  $|s| \leq 1$  and  $|t| \leq 1$ ,

$$\left(\frac{sf(sz)}{tf(tz)}\right)^{1-\alpha} \left(\frac{s^2 f'(sz)}{t^2 f'(tz)}\right)^\alpha \prec \frac{sF(tz)}{tF(sz)}.$$

*Proof.* Define the function  $q(z)$  by

$$\frac{1}{q(z)} := (zf(z))^{1-\alpha} (-z^2 f'(z))^\alpha.$$

Then a computation shows that

$$1 + \frac{zq'(z)}{q(z)} = - \left[ (1-\alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right]$$

and the result now follows from Lemma 3.1. ■

**Example 3.2.** Let  $\Sigma_\alpha^*[A, B]$  be the class of all meromorphic  $\alpha$ -convex functions  $f \in \Sigma$  satisfying

$$- \left[ (1-\alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1; z \in \Delta).$$

The function  $f \in \Sigma_\alpha^*[A, B]$  if and only if for all  $|s| \leq 1$  and  $|t| \leq 1$ ,

$$\left(\frac{sf(sz)}{tf(tz)}\right)^{1-\alpha} \left(\frac{s^2 f'(sz)}{t^2 f'(tz)}\right)^\alpha \prec \begin{cases} \left(\frac{1+Btz}{1+Bs z}\right)^{(A-B)/B} & \text{if } B \neq 0 \\ e^{A(t-s)z} & \text{if } B = 0 \end{cases}.$$

#### 4. GROWTH THEOREM FOR FUNCTIONS IN $\Sigma^*(\phi)$

We need the following Lemma in the proof of Theorem 4.1.

**Lemma 4.1.** [8, Corollary 3.4h.1, p.135]. Let  $q(z)$  be univalent in  $\Delta$  and let  $\psi(z)$  be analytic in a domain containing  $q(\Delta)$ . If  $zq'(z)/\psi(q(z))$  is starlike, and

$$zp'(z)\psi(p(z)) \prec zq'(z)\psi(q(z)),$$

then  $p(z) \prec q(z)$  and  $q(z)$  is the best dominant.

Theorem 4.1 below is a special case of Theorem 3.1 if  $\phi$  is a convex univalent function. However we prove Theorem 4.1 without the convexity assumption.

**Theorem 4.1.** *Let  $\phi(z)$  be an analytic function with positive real part on  $\Delta$  with  $\phi(0) = 1$ ,  $\phi'(0) > 0$  and maps the unit disk  $\Delta$  onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let the functions  $h_{\phi n}$  ( $n = 2, 3, \dots$ ) be defined by*

$$\frac{zh'_{\phi}(z)}{h_{\phi}(z)} = \phi(z) \quad (h_{\phi}(0) = 0 = h'_{\phi}(0) - 1).$$

If  $f(z) \in \Sigma^*(\phi)$ , then

$$zf(z) \prec \frac{z}{h_{\phi}(z)}.$$

*Proof.* Define the function  $p(z)$  by

$$p(z) := zf(z) \quad (z \in \Delta).$$

Then a computation shows that

$$-\frac{zf'(z)}{f(z)} = 1 - \frac{zp'(z)}{p(z)}.$$

If  $f(z) \in \Sigma^*(\phi)$ , then

$$\frac{zp'(z)}{p(z)} \prec 1 - \phi(z).$$

Since  $\phi(z)$  is starlike in  $\Delta$ , by an application of Lemma 4.1, we obtain  $p(z) \prec q(z)$  where  $q(z)$  is given by

$$\frac{zq'(z)}{q(z)} = 1 - \frac{zh'_{\phi}(z)}{h_{\phi}(z)}$$

or  $q(z) = z/h_{\phi}(z)$ . ■

As a consequence of Theorem 4.1, we immediately obtain

**Theorem 4.2.** (Growth Theorem). *Let  $\phi(z)$  be an analytic function with positive real part on  $\Delta$  with  $\phi(0) = 1$ ,  $\phi'(0) > 0$  and maps the unit disk  $\Delta$  onto a region starlike with respect to 1 and symmetric with respect to the real axis. If  $f(z) \in \Sigma^*(\phi)$ , then*

$$[h_{\phi}(r)]^{-1} \leq |f(z)| \leq [-h_{\phi}(-r)]^{-1} \quad (|z| = r < 1).$$

For the choice  $p(z) = (1 - Az)/(1 - Bz)$ ,  $0 \leq B \leq 1$ ;  $-B < A < B$ , we obtain the following result of Karanukaran:

**Corollary 4.1.** [4]. *If  $f \in K_1(A, B)$ , then*

$$r^{-1}(1 - Br)^{(B-A)/B} \leq |f(z)| \leq r^{-1}(1 + Br)^{(B-A)/B}$$

### 5. COEFFICIENT PROBLEM FOR THE CLASS $\Sigma^*(\phi)$

Now we consider coefficient problems for the class  $\Sigma^*(\phi)$ .

**Theorem 5.1.** *Let  $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots$ . If  $f(z) = 1/z + \sum_{k=0}^{\infty} a_k z^k$  belongs to  $\Sigma^*(\phi)$ , then*

$$|a_1 - \mu a_0^2| \leq \begin{cases} \frac{1}{2}(B_1^2 - 2\mu B_1^2 - B_2) & \text{if } 2\mu B_1^2 \leq B_1^2 - B_1 - B_2 \\ \frac{1}{2}B_1 & \text{if } B_1^2 - B_1 - B_2 \leq 2\mu B_1^2 \leq B_1^2 + B_1 - B_2 \\ \frac{1}{2}(-B_1^2 + 2\mu B_1^2 + B_2) & \text{if } B_1^2 + B_1 - B_2 \leq 2\mu B_1^2 \end{cases}$$

*The result is sharp.*

*Proof.* Our proof of Theorem 5.1 is essentially similar to the proof of Theorem 3 of Ma and Minda[6]. If  $f(z) \in \Sigma^*(\phi)$ , then there is a Schwarz function  $w(z)$ , analytic in  $\Delta$  with  $w(0) = 0$  and  $|w(z)| < 1$  in  $\Delta$  such that

$$(5.1) \quad -\frac{zf'(z)}{f(z)} = \phi(w(z)).$$

Define the function  $p_1(z)$  by

$$(5.2) \quad p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots$$

Since  $w(z)$  is a Schwarz function, we see that  $\Re p_1(z) > 0$  and  $p_1(0) = 1$ . Define the function  $p(z)$  by

$$(5.3) \quad p(z) := -\frac{zf'(z)}{f(z)} = 1 + b_1z + b_2z^2 + \dots$$

In view of the equations (5.1), (5.2), (5.3), we have

$$(5.4) \quad p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right).$$

and from this equation (5.4), we obtain

$$b_1 = \frac{1}{2}B_1c_1$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2c_1^2.$$

From the equation (5.3), we see that

$$(5.5) \quad b_1 + a_0 = 0$$

$$(5.6) \quad b_2 + b_1a_0 + 2a_1 = 0$$

or equivalently

$$(5.7) \quad a_0 = -b_1 = -\frac{B_1c_1}{2}$$

and

$$\begin{aligned} a_1 &= \frac{1}{2}(b_1^2 - b_2) \\ &= \frac{1}{8} \{ B_1^2c_1^2 - 2B_1c_2 + B_1c_1^2 - B_2c_1^2 \}. \end{aligned}$$

Therefore,

$$(5.8) \quad a_1 - \mu a_0^2 = -\frac{B_1}{4} \{ c_2 - vc_1^2 \}$$

where

$$v := \frac{1}{2} \left[ 1 + B_1 - \frac{B_2}{B_1} - 2\mu B_1 \right].$$

Our result now follows by an application of Lemma 5.2 below. The sharpness is also an immediate consequence of Lemma 5.2. ■

**Lemma 5.2.** [6]. *If  $p_1(z) = 1 + c_1z + c_2z^2 + \dots$  is a function with positive real part in  $\Delta$ , then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0 \\ 2 & \text{if } 0 \leq v \leq 1 \\ 4v - 2 & \text{if } v \geq 1 \end{cases}$$

*When  $v < 0$  or  $v > 1$ , equality holds if and only if  $p_1(z)$  is  $(1+z)/(1-z)$  or one of its rotations. If  $0 < v < 1$ , then equality holds if and only if  $p_1(z)$  is  $(1+z^2)/(1-z^2)$  or one of its rotations. If  $v = 0$ , equality holds if and only if*

$$p_1(z) = \left( \frac{1}{2} + \frac{1}{2}\lambda \right) \frac{1+z}{1-z} + \left( \frac{1}{2} - \frac{1}{2}\lambda \right) \frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

*or one of its rotations. If  $v = 1$ , equality holds if and only if  $p_1$  is the reciprocal of one of the functions such that the equality holds in the case of  $v = 0$ .*

When  $\mu$  is complex, we have the following:

**Theorem 5.2.** *Let  $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots$ . If  $f(z) = 1/z + \sum_{k=0}^{\infty} a_k z^k$  belongs to  $\Sigma^*(\phi)$ , then for  $\mu$  a complex number,*

$$|a_1 - \mu a_0^2| \leq \frac{B_1}{2} \max\left\{1, \left|B_1 - 2\mu B_1 - \frac{B_2}{B_1}\right|\right\}.$$

The result is sharp.

Theorem 5.2 follows from the following result. For a function  $p(z) = 1 + c_1z + c_2z^2 + \dots$  with positive real part, we have

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

## 6. DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR $\Sigma^*(\phi)$

In this section, we discuss differential implications for the subclass  $\Sigma^*(\phi)$ . We shall require the following definition and lemmas:

**Definition 6.1.** [9, Definition 2, p. 817]. *Denote by  $\mathcal{Q}$ , the set of all functions  $f(z)$  that are analytic and injective on  $\overline{\Delta} - E(f)$ , where*

$$E(f) = \left\{ \zeta \in \partial\Delta : \lim_{z \rightarrow \zeta} f(z) = \infty \right\},$$

and are such that  $f'(\zeta) \neq 0$  for  $\zeta \in \partial\Delta - E(f)$ .

**Lemma 6.1.** (cf. Miller and Mocanu [8, Theorem 3.4h, p. 132]). *Let  $q(z)$  be univalent in the unit disk  $\Delta$  and  $\vartheta$  and  $\varphi$  be analytic in a domain  $D$  containing  $q(\Delta)$  with  $\varphi(w) \neq 0$  when  $w \in q(\Delta)$ . Set  $Q(z) := zq'(z)\varphi(q(z))$  and  $h(z) := \vartheta(q(z)) + Q(z)$ . Suppose that either  $h(z)$  is convex, or  $Q(z)$  is starlike univalent in  $\Delta$ . In addition, assume that  $\Re[zh'(z)/Q(z)] > 0$  for  $z \in \Delta$ . If  $p(z)$  is analytic in  $\Delta$  with  $p(0) = q(0)$ ,  $p(\Delta) \subseteq D$  and*

$$(6.1) \quad \vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)),$$

then  $p(z) \prec q(z)$  and  $q(z)$  is the best dominant.



**Lemma 6.2.** [1]. *Let  $q(z)$  be univalent in the unit disk  $\Delta$  and  $\vartheta$  and  $\varphi$  be analytic in a domain  $D$  containing  $q(\Delta)$ . Suppose that  $\Re [\vartheta'(q(z))/\varphi(q(z))] > 0$  for  $z \in \Delta$  and  $zq'(z)\varphi(q(z))$  is starlike univalent in  $\Delta$ . If  $p(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ , with  $p(\Delta) \subseteq D$ , and  $\vartheta(p(z)) + zp'(z)\varphi(p(z))$  is univalent in  $\Delta$ , then*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

*implies  $q(z) \prec p(z)$  and  $q(z)$  is the best subordinant. (Here  $\mathcal{H}[a, n]$  denotes the class of all analytic functions  $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$  ( $z \in \Delta$ )).*

First we prove a differential subordination result for the class  $\Sigma^*(\phi)$ .

**Theorem 6.1.** *Let  $\alpha$  be a nonzero complex number. Let  $q(z)$  be univalent in  $\Delta$ ,  $q(0) = 1$ . Assume that  $q(z)$  or  $(\alpha - 1)q(z) + \alpha q^2(z) - \alpha zq'(z)$  is convex univalent and*

$$(6.2) \quad \Re \left\{ \frac{1 - \alpha}{\alpha} - 2q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0.$$

*If  $f \in \Sigma$  satisfies*

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec (\alpha - 1)q(z) + \alpha q^2(z) - \alpha zq'(z),$$

*then  $-\frac{zf'(z)}{f(z)} \prec q(z)$  and  $q(z)$  is the best dominant.*

*Proof.* Define the function  $p(z)$  by

$$(6.3) \quad p(z) := -\frac{zf'(z)}{f(z)}.$$

Then a computation shows that

$$(6.4) \quad p(z) - \frac{zp'(z)}{p(z)} = -\left( 1 + \frac{zf''(z)}{f'(z)} \right).$$

Using (6.4) and (6.3), we have

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} = (\alpha - 1)p(z) + \alpha p^2(z) - \alpha zp'(z).$$

Define the function  $\vartheta$  and  $\varphi$  by

$$\vartheta(w) = (\alpha - 1)w + \alpha w^2 \quad \text{and} \quad \varphi(w) = -\alpha.$$

Then the functions  $\vartheta$  and  $\varphi$  are analytic in  $\mathbb{C}$  and  $\varphi(w) \neq 0$  in  $\mathbb{C}$ . Also the function  $Q(z) := zq'(z)\varphi(q(z)) = -\alpha zq'(z)$  is starlike in  $\Delta$ . Using (6.2), we see that the function  $h(z) := \vartheta(q(z)) + Q(z) = (\alpha - 1)q(z) + \alpha q(z)^2 + Q(z)$  satisfies  $\Re [zh'(z)/Q(z)] > 0$ . The result now follows by an application of Lemma 6.1. ■

**Theorem 6.2.** Let  $q(z) \neq 0$  be univalent in  $\Delta$  and  $q(0) = 1$ . Let  $zq'(z)/q(z)^2$  be starlike in  $\Delta$ . If  $f \in \Sigma$  and

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 - \frac{zq'(z)}{q(z)^2},$$

then  $-zf'(z)/f(z) \prec q(z)$  and  $q(z)$  is the best dominant.

*Proof.* Let  $p(z)$  be defined by (6.3). From (6.4) and (6.3), we get

$$1 - \frac{zp'(z)}{p(z)^2} = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}$$

and the result follows by an application of Lemma 6.1. ■

The corresponding superordination results are obtained from Lemma 6.2 in a similar manner to Theorems 6.1 and 6.2. The proofs are omitted.

**Theorem 6.3.** Let  $\alpha$  be a nonzero complex number,  $q(z)$  be convex univalent in  $\Delta$ . Assume that  $\Re \left\{ \frac{\alpha-1}{\alpha} + 2q(z) \right\} < 0$ . If  $f \in \Sigma$ ,  $-zf'(z)/f(z) \in \mathcal{H}[1, 1] \cap \mathcal{Q}$  and  $\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$  is univalent in  $\Delta$  and

$$(\alpha - 1)q(z) + \alpha q^2(z) - \alpha zq'(z) \prec \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)},$$

then  $q(z) \prec -\frac{zf'(z)}{f(z)}$  and  $q(z)$  is the best subordinator.

**Theorem 6.4.** Let  $q(z) \neq 0$  be univalent,  $q(0) = 1$  and  $zq'(z)/q(z)^2$  be starlike in  $\Delta$ . If  $f \in \Sigma$ ,  $-zf'(z)/f(z) \in \mathcal{H}[1, 1] \cap \mathcal{Q}$  and  $\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$  is univalent in  $\Delta$

$$1 - \frac{zq'(z)}{q(z)^2} \prec \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)},$$

then  $q(z) \prec -zf'(z)/f(z)$  and  $q(z)$  is the best subordinator.

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