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DISAPPEARANCE OF EXTREME POINTS: $d_*(w,1)$ CASE

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Abstract. In this paper, we show that the predual spaces of some Lorentz sequence spaces are strictly convex renormable such that any point of the unit sphere is not an extreme point of the unit ball of the dibual. This will partially answer the Question 2. in the paper of Peter Morris.

1. Introduction

In 1983, Peter Morris showed that a separable space containing an isomorphic copy of c_0 can be renormed to be strictly convex and the unit sphere doesn't intersect with the set of extreme points of the unit ball of the bidual space [6]. At the end of his paper he asked two questions:

Question 1. Which spaces have the property that, under any renorming, the intersection of the set of extreme points of the unit ball and the set of extreme points of the bidual unit ball is nonempty? (The obvious conjecture for Question 1 is: Radon-Nikodym property).

Question 2. Which spaces can be renormed to be strictly convex and $Ext(B_{\widetilde{X}}) \cap Ext(B_{\widetilde{X}^{**}}) = \emptyset$?

In 1985, H. Rosenthal [8] showed that a Banach space X has the Radon-Nikodým property if and only if for every equivalent renorming, $\widetilde{X} := (X, |||\cdot|||)$, the intersection of the set of extreme points of the unit ball and the set of extreme points of the bidual unit ball, $Ext(B_{\widetilde{X}}) \cap Ext(B_{\widetilde{X}^{**}})$ is nonempty, but the second question is still open. See also [2], for an extensive study related to this problems.

The purpose of this paper is to show that the preduals of some Lorentz sequence spaces is an answer to Question 2. To show this we need to find a weak-star closed infinite dimensional subspace of $d^*(w,1)$ which intersects $d_*(w,1)$ at just

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one point and this is done using a well known facts [4, 5] that $d(w,1)$ has the Radon-Nikodým property and hence the Krein-Millman property and it contains a complemented isomorphic copy of ℓ_1.

2. Main Results

We are going to use the notations appearing in the standard texts such as [4, 5]. For every $p \geq 1$ and every nonincreasing sequence of positive numbers $\{w_n\}_{n=1}^{\infty}$, we consider the space $d(w,p)$ of all sequences of scalars $x = (x_1, x_2, \cdots)$ for which $\|x\| = \sup \left(\sum_{n=1}^{\infty} |x_{\pi(n)}|^p w_n\right)^{\frac{1}{p}} < \infty$, the supremum being taken over all permutations π of the integers. If we also assume that $\lim_{n\to\infty} w_n = 0$, $\sum_{n=1}^{\infty} w_n = \infty$ and $w_1 = 1$, $d(w,p)$ is called a Lorentz sequence space [5]. In the following, let's consider $d(w,1)$, for $w := (w_k)_{k=1}^{\infty} \in c_0 \setminus \ell_1$, a decreasing sequence of positive real numbers with $w_1 = 1$. The norm of $x = (x_k)_{k=1}^{\infty} \in d(w,1)$ can also be written as

$$\|x\| = \sup\left(\sum_{n=1}^{\infty} |x_{\pi(n)}| w_n\right) = \sum_{k=1}^{\infty} [x]_k w_k,$$

where $([x]_k)_{k=1}^{\infty}$ is a decreasing rearrangement of $(|x_k|)_{k=1}^{\infty}$, the supremum being taken over all permutations π of the integers. Note that the space $d(w,1)$ with the norm $\|x\| = \sum_{k=1}^{\infty} [x]_k w_k$ is a Banach space [5]. It is known [4, 5] that the dual $d^*(w,1)$ of $d(w,1)$ is the space of bounded real sequences $y = (y_k)_{k=1}^{\infty}$ satisfying

$$\|y\| := \left\| \frac{[Y](n)}{W(n)} \right\|_{\ell_\infty} < \infty,$$

and the predual $d_*(w,1)$ is the space of bounded real sequences $y = (y_k)_{k=1}^{\infty}$ satisfying

$$\|y\| := \left\| \frac{[Y](n)}{W(n)} \right\|_{c_0} < \infty,$$

where $[Y](n) := \sum_{k=1}^{n} [y]_k$ and $W(n) := \sum_{k=1}^{n} w_k$. More on these can be found in the texts [4, 5].

Lemma 1. *There exists an infinite dimensional weak-star closed subspace M of $d^*(w,1)$ such that $M \cap d_*(w,1) = \{0\}$.*

Proof. Since ℓ_1 is complementedly and isomorphically embedded in $d(w,1)$ [5], we can write ℓ_2 as a quotient space of $d(w,1)$,

$$\ell_2 \approx d(w,1)/K,$$

where K is the kernel of the quotient map $Q : d(w,1) \to \ell_2$. Since $d(w,1)$ is a Banach space and the kernel K of the quotient map Q is a closed subspace of $d(w,1)$,

$$\ell_2 \approx (\ell_2)^* \approx K^\perp \subset d^*(w,1).$$

Note that $K^\perp$ is weak-star closed and infinite dimensional. Note also that $d_*(w,1)$ is an M-embedded space, so the dual $d(w,1)$ has the RNP(See, for example, [4] for the details). We claim that its intersection with $d_*(w,1)$ is at most one dimensional. Indeed, if $\dim(Z) \geq 2$ with $Z := d_*(w,1) \cap K^\perp$, it contains a Euclidean plane. But this means that the dual $d(w,1)$ of $d_*(w,1)$ should also contain a Euclidean plane. But this is impossible, because the set of extreme points $Ext(B_{d(w,1)})$ is countable. Indeed, an extreme point $e \in Ext(B_{d(w,1)})$ is of the form

$$e = \frac{1}{W_n} \sum_{i \in I_n} e_i,$$

where $W_n = \sum_{k=1}^n w_k$, $I_n \subset \mathbb{N}$ with $|I_n| = n$ and $\{e_k\}_{k \in \mathbb{N}}$ is the standard basis of $d(w,1)$. The cardinality of the set of extreme points of $B_{d(w,1)}$ can be written as

$$|Ext(B_{d(w,1)})| = \aleph_0 + \aleph_0 \times \aleph_0 + \aleph_0 \times \aleph_0 \times \aleph_0 + \cdots = \aleph_0,$$

where $\aleph_0 = |\mathbb{N}| = |\mathbb{Q}|$. On the other hand, the unit ball of a Euclidean plane has uncountably many extreme points and each of which extends to an extreme point of $B_{d(w,1)}$, as was the case of c_0. We can see this using the fact that $d(w,1)$ has the RNP and hence the KMP(Krein-Milman property), because the two properties are equivalent for dual spaces [3, 4, 7]. Indeed, write

$$X = d(w,1) = V \oplus W$$

with $V = (\mathbb{R}^2, \|\cdot\|_{\ell_2})$ and $W = X/V$. Let $v \in Ext(B_V)$. If $v \oplus 0$ is in $Ext(B_X)$ done. Otherwise, there is a nonzero $w \in W$ such that $v \pm w \in B_X$, moreover $v + w \in S_X$. We can find $f \in S_{X^*}$ which generates the face

$$\Phi = \{v + u \in S_X : \|v + u\| = f(v + u) = 1\}.$$

This is a bounded closed convex subset of B_X and, as we noted above, this set has the KMP, so the set of its extreme points $Ext(\Phi)$ is not empty and $e \in Ext(\Phi)$ is written as

$$e = v + u_0.$$

Since the face Φ is extremal, $e \in Ext(B_X)$. Then

$$M := K^\perp / Z$$

has the required property. ■

Now, we are ready to prove the following theorem by constructing a one-to-one operator T from $d_*(w,1)$ into ℓ_2 such that the kernel of T^{**} equals the infinite dimensional weak-star closed subspace $M \subset d^*(w,1)$ found above.

Theorem 2. *Let E be $d_*(w,1)$, the predual of Lorentz sequence space $d(w,1)$ with $w \in c_0 \setminus \ell_1$ a decreasing sequence of positive numbers with $w_1 = 1$. Then E is isomorphic to a strictly convex space F such that no extreme points of F is preserved. That is,*

$$S_F \cap ExtB_{F^{**}} = \emptyset.$$

We will use the following Lemma:

Lemma 3. (Lemma 1. in [6]). *Let E and Y be Banach spaces and let $T : E \to Y$ be a bounded linear operator. Define*

$$|||x||| = \|x\| + \|Tx\|, x \in E.$$

Then

(i) $|||\cdot|||$ *is an equivalent norm on E;*

(ii) *the norm on $(E, |||\cdot|||)^{**}$ is given by*

$$|||x^{**}||| = \|x^{**}\| + \|T^{**}x^{**}\|;$$

(iii) *if Y is strictly convex and T is one-to-one then $|||\cdot|||$ is strictly convex.*

Proof of the Theorem 2. Let $E := d_*(w,1)$. By the Lemma 1, there is a weak-star closed infinite dimensional subspace $M \subset d^*(w,1)$ which intersects E at just one point. Let $N \subset d(w,1)$ be the prepolar of M, that is $M = N^\perp$. Since N is a closed subspace of a separable Banach space $d(w,1)$, there is a sequence $\{\phi_k\}_{k=1}^\infty \subset N$ such that $N = \overline{span}\{\phi_k : k \in \mathbb{N}\}$, and $\sum_{k=1}^\infty \|\phi_k\|^2 < \infty$. Define $T : d_*(w,1) \to \ell_2$ as

$$T(x) = (\phi_k(x))_{k\in\mathbb{N}}.$$

Then, T is one-to-one and $T^{**} : d^*(w,1) \to \ell_2$ is given as

$$T^{**}x^{**} = (x^{**}(\phi_k))_{k\in\mathbb{N}},$$

and hence the kernel of T^{**} is M. Now let's define a new norm, on $d_*(w,1)$,

$$|||x||| := \|x\| + \|Tx\|.$$

By Lemma 3. this norm is strictly convex, because T is one-to-one. So, using a similar argument as one in [1], for any $x \in d_*(w,1)$ satisfying $|||x||| = 1$, we can find a nonzero vector $y^{**} \in M = \ker(T^{**})$ satisfying

$$\|x \pm y^{**}\| = \|x\|.$$

We describe the argument for completion. Indeed, let

$$\lambda := (\lambda_k)_{k=1}^{\infty} = \left(\frac{[x]_k}{\|x\|}\right)_{k=1}^{\infty}.$$

Then $\lambda \in B_{d_*(w,1)}$ and $\|\lambda\| = 1$. Let $n \in \mathbb{N}$ be the largest integer satisfying

$$\Lambda(n) := \sum_{k=1}^{n} \lambda_k = W(n).$$

For every $m > n$,

$$\Lambda(m) < W(m),$$

because, by definition of the norm of $d_*(w,1)$, $\lim_{m\to\infty} \frac{\Lambda(m)}{W(m)} = 0$. Moreover,

$$a := 1 - \sup\left\{\frac{\Lambda(m)}{W(m)} : m > n\right\} > 0.$$

Since $\Lambda(n) = W(n)$ and $\Lambda(n+1) < W(n+1)$, it is easy to check $\lambda_n > \lambda_{n+1} \geq 0$. So we can choose $\delta > 0$ such that

$$\lambda_n > \lambda_{n+1} + \delta,$$

and define

$$b := \min(a, \delta).$$

Because M is an infinite dimensional subspace of $d^*(w,1)$, there exists an element $z^{**} = (z_k)_{k=1}^{\infty} \in B_{d^*(w,1)} \cap M$ with $z_k = 0$ for $1 \leq k \leq n$ and $\|z^{**}\| = 1$. Now, let's show that $\|\lambda \pm bz^{**}\| = \|\lambda\| = 1$. Indeed, for every $k > n$,

$$\lambda_n > \lambda_{n+1} + \delta \geq \lambda_k + b|z_k| \geq |\lambda_k \pm bz_k|,$$

and for any $m > n$,

$$\begin{aligned}
\Lambda(n) + \sum_{k=n+1}^{m} |\lambda_k \pm bz_k| &\leq \Lambda(n) + \sum_{k=n+1}^{m} \lambda_k + aW(m-n) \\
&< \Lambda(n) + \sum_{k=n+1}^{m} \lambda_k + \left(1 - \frac{\Lambda(m)}{W(m)}\right) W(m) \\
&= W(m) + \left(\sum_{k=n+1}^{m} \lambda_k - \Lambda(m) + \Lambda(n)\right) \\
&= W(m).
\end{aligned}$$

This means that

$$\|\lambda \pm bz^{**}\| = \|\lambda\| = 1.$$

Since λ is a rearrangement of a scalar multiple of $(|x_k|)_{k=1}^{\infty}$, applying the inverse permutation, corresponding to the rearrangement, to z^{**}, we get $\tilde{z}^{**} \in B_{d^*(w,1)} \cap M$, with $\|\tilde{z}^{**}\| = \|z^{**}\| = 1$. Now, define

$$y^{**} := b \cdot \|x\| \cdot \tilde{z}^{**}.$$

Then, $y^{**} \in M$ and

$$\|x \pm y^{**}\| = \|x\|.$$

By choice of $y^{**} \in M = \ker(T^{**})$, we have

$$\|T^{**}(x \pm y^{**})\| = \|T^{**}(x)\| = \|Tx\|.$$

Hence,

$$|||x \pm y^{**}||| = \|x \pm y^{**}\| + \|T^{**}(x \pm y^{**})\| = \|x\| + \|Tx\| = |||x|||.$$

This means that x is not an extreme point of $B_{(d^*(w,1),|||\cdot|||)}$. This proves the theorem. ■

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