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AMBARZUMYAN'S THEOREMS FOR VECTORIAL STURM-LIOUVILLE SYSTEMS WITH COUPLED BOUNDARY CONDITIONS

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Abstract. We consider the n-dimensional vectorial Sturm-Liouville problem under some coupled boundary conditions. In some special cases the potential can be reconstructed from one spectrum. We prove that if the spectrum is the same as the spectrum belonging to the zero potential, under periodic boundary conditions or semi-periodic boundary conditions in which case an additional condition on the potential is imposed, then the potential is actually zero.

1. Introduction and Statement of Results

Inverse spectral theory is to reconstruct the potential from spectral data. From a historical viewpoint, the work of Ambarzumyan [1] may be thought of as the first paper in the theory of inverse spectral problems associated with Sturm-Liouville operators. He investigated the Schrödinger operator with Neumann boundary conditions, and found that if its spectrum consists of 0 and infinitely many other square numbers, then the potential is zero. Later it became clear that the case investigated by Ambarzumyan was exceptional; in general two spectra is needed to determine the potential, see [3, 4, 10, 12]. The Ambarzumyan's theorem has been generalized in many directions, we mention only the papers [1, 5, 6, 7, 9, 11, 14, 16], etc. Recently, the paper [6] extended the clasicial Ambarzumyan's theorem for the Sturm-Liouville equation to the vectorial Sturm-Liouville systems with the separated boundary conditions.

For the vectorial Sturm-Liouville systems with some coupled boundary conditions, whether there are other exceptions is a challenging open problem. In the

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present paper, based on the well-known extremal property of the smallest eigenvalue, the theory of matrix differential equations and the vectorial Gelfand-Levitan equation [7], we find two analogs of Ambarzumyan's theorem to Sturm-Liouville systems of n dimension under periodic or semi-periodic boundary conditions, that is, uniqueness of the zero potential can be verified from a part knowledge of one spectrum.

Consider the eigenvalue problems of n-dimensional vectorial Sturm-Liouville systems

$$-\phi''(x) + P(x)\phi(x) = \lambda\phi(x) \quad \text{on } [0, \pi] \tag{1.1}$$

with periodic boundary conditions

$$\phi(0) = \phi(\pi), \quad \phi'(0) = \phi'(\pi), \tag{1.2}$$

and semi-periodic boundary conditions

$$\phi(0) = -\phi(\pi), \quad \phi'(0) = -\phi'(\pi), \tag{1.3}$$

respectively, where $P(x)$ is an $n \times n$ real, symmetric, continuous matrix-valued function, and $\phi(x)$ is a vector-valued function of length n. Note that the problems (1.1), (1.2) and (1.1), (1.3) are self-adjoint eigenvalue problems, and each operator's spectrum, which consists of eigenvalues, is real and can be determined by the variational principle. Counting multiplicities of the eigenvalues, we can arrange those eigenvalues in an ascending order as

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to +\infty.$$

The multiplicity of each eigenvalue λ_k is at most $2n$.

In the case when $P(x) = 0$ on $[0, \pi]$, we can easily calculate the eigenvalues of the problem (1.1) and (1.2) is $\{4k^2 : k = 0, 1, 2, \cdots\}$, and all the eigenvalues are of multiplicity $2n$ except that the eigenvalue 0 is of multiplicity n; the eigenvalues of the problem (1.1) and (1.3) is $\{(2k-1)^2 : k = 1, 2, \cdots\}$, and all the eigenvalues are of multiplicity $2n$.

The main theorems of this paper read as follows.

Theorem 1.1. *Suppose that the elements of $P(x)$ are in $C[0, \pi]$ and $\{4m_j^2 : j = 1, 2, \cdots\} \bigcup \{0\}$ is a subset of the spectrum of the eigenvalue problem (1.1) and (1.2), where 0 is the first eigenvalue of the problem (1.1) and (1.2), m_j is a strictly ascending infinite sequence of positive integers, and each eigenvalue $4m_j^2$ are of multiplicity $2n$, then*

$$P(x) = 0 \text{ on } [0, \pi].$$

Motivated by [6], we can impose an additional condition on the potential function to arrive at theorem 1.2 for the semi-periodic boundary conditions. Theorem 1.2 shows that if the semi-periodic spectrum is trivial, and the second term in the Fourier cosine series of $(P(x))_{ii}$ is nonnegative, then the potential function $P(x)$ has to be trivial.

Theorem 1.2. *Suppose that the elements of* $P(x)$ *are in* $C[0,\pi]$. *Then* $\{1\} \bigcup \{(2m_j-1)^2 : j = 1, 2, \cdots\}$ *is a subset of the spectrum of the eigenvalue problem* (1.1) *and* (1.3), *where* 1 *is the first eigenvalue of the problem* (1.1) *and* (1.3), m_j *is a strictly ascending infinite sequence of positive integers, and each eigenvalue* $(2m_j-1)^2$ *are of multiplicity* $2n$, *and the potential function* $P(x)$ *satisfy*

$$\int_0^\pi (P(x))_{ii} \cos(2x) dx \geq 0$$

if and only if

$$P(x) = 0 \text{ on } [0,\pi],$$

where $(P(x))_{ii}$ *denotes entry of matrix* $P(x)$ *at the* i*-st row and* i*-st column,* $i = 1, 2, \cdots, n$.

2. Analysis of Eigenvalues

This section gives analysis of eigenvalues for the problems (1.1) and (1.2), and (1.1) and (1.3), respectively.

Suppose $P(x)$ is a continuous $n \times n$ symmetric matrix-valued function, I_n is an $n \times n$ identity matrix. Let $\Phi_1(x,\lambda)$ satisfy matrix differential equation

$$\begin{cases} -Y''(x) + P(x)Y(x) = \lambda Y(x) \\ Y(0) = I_n, \quad Y'(0) = 0, \end{cases} \tag{2.1}$$

then, by [7], the solution $\Phi_1(x,\lambda)$ can be expressed as

$$\Phi_1(x,\lambda) = \cos(\sqrt{\lambda}x) I_n + \int_0^x K(x,t)\cos(\sqrt{\lambda}t)dt, \tag{2.2}$$

where $K(x,t)$ is a symmetric matrix-valued function whose entries have continuous partial derivatives up to order two with respect to t and x, and

$$K(0,0) = 0, \ K(x,t) = 0 \text{ for } t > x, \ K(\pi,\pi) = \frac{1}{2}\int_0^\pi P(x)dx. \tag{2.3}$$

Similarly, let $\Phi_2(x,\lambda)$ satisfy matrix differential equation

$$\begin{cases} -Y''(x) + P(x)Y(x) = \lambda Y(x) \\ Y(0) = 0, \quad Y'(0) = I_n, \end{cases} \tag{2.4}$$

then the solution $\Phi_2(x,\lambda)$ can be expressed as

$$\Phi_2(x,\lambda)=\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}I_n+\int_0^x L(x,t)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt, \tag{2.5}$$

where $L(x,t)$ is a symmetric matrix-valued function whose entries have continuous partial derivatives up to order two with respect to t and x, and

$$L(0,0)=0,\ L(x,t)=0 \text{ for } t>x,\ L(\pi,\pi)=\frac{1}{2}\int_0^\pi P(x)dx. \tag{2.6}$$

From (2.1) and (2.4), it is easy to verify that Wronskian determinant

$$\det\begin{pmatrix}\Phi_1(x,\lambda) & \Phi_2(x,\lambda)\\ \Phi_1'(x,\lambda) & \Phi_2'(x,\lambda)\end{pmatrix}$$

is a constant independent of x. As

$$\det\begin{pmatrix}\Phi_1(0,\lambda) & \Phi_2(0,\lambda)\\ \Phi_1'(0,\lambda) & \Phi_2'(0,\lambda)\end{pmatrix}=1,$$

we get that $(\Phi_1(x,\lambda),\Phi_2(x,\lambda))$ is the fundamental matrix of solutions of systems (1.1). Therefore, the general solutions of systems (1.1) have the form

$$\phi(x,\lambda)=(\Phi_1(x,\lambda),\Phi_2(x,\lambda))C,$$

where $C=(c_1,c_2,\cdots,c_{2n})^t, c_k\in\mathbf{C}, k=1,2,\cdots,2n$, and A^t denotes transpose of the matrix A.

If $\phi(x,\lambda)=(\Phi_1(x,\lambda),\Phi_2(x,\lambda))C$ is a nontrivial solution of the problem (1.1) and (1.2) there exists a non-vanishing vector C satisfying the matrix equation

$$\begin{pmatrix}I_n-\Phi_1(\pi,\lambda) & -\Phi_2(\pi,\lambda)\\ -\Phi_1'(\pi,\lambda) & I_n-\Phi_2'(\pi,\lambda)\end{pmatrix}C=0.$$

Therefore, λ is an eigenvalue of the problem (1.1) and (1.2) if and only if the matrix

$$W_1(\lambda)=\begin{pmatrix}I_n-\Phi_1(\pi,\lambda) & -\Phi_2(\pi,\lambda)\\ -\Phi_1'(\pi,\lambda) & I_n-\Phi_2'(\pi,\lambda)\end{pmatrix} \tag{2.7}$$

is singular. Furthermore, the multiplicity of λ is equal to $2n-rankW_1(\lambda)$. Thus, λ is an eigenvalue of multiplicity $2n$ of the problem (1.1) and (1.2) if $W_1(\lambda)$ is a zero-matrix.

If $\phi(x,\lambda) = (\Phi_1(x,\lambda), \Phi_2(x,\lambda))C$ is a nontrival solution of the problem (1.1) and (1.3) there exists a non-vanishing vector C satisfying the matrix equation

$$\begin{pmatrix} I_n + \Phi_1(\pi,\lambda) & \Phi_2(\pi,\lambda) \\ \Phi_1'(\pi,\lambda) & I_n + \Phi_2'(\pi,\lambda) \end{pmatrix} C = 0.$$

Thus, λ is an eigenvalue of the problem (1.1) and (1.3) if and only if the matrix

$$W_2(\lambda) = \begin{pmatrix} I_n + \Phi_1(\pi,\lambda) & \Phi_2(\pi,\lambda) \\ \Phi_1'(\pi,\lambda) & I_n + \Phi_2'(\pi,\lambda) \end{pmatrix} \tag{2.8}$$

is singular. Furthermore, the multiplicity of λ is equal to $2n - rankW_2(\lambda)$. Thus, λ is an eigenvalue of multiplicity $2n$ of the problem (1.1) and (1.3) if $W_2(\lambda)$ is a zero-matrix.

3. Proofs of Theorems

Now we can give the proofs of theorems 1.1 and 1.2.

The proof of theorem 1.1. Since the problem (1.1) and (1.2) has infinitely many eigenvalues of the form $4m_j^2$, m_j are positive integers, $j = 1, 2, \cdots$, and each eigenvalue $4m_j^2$ is of multiplicity $2n$, we have from (2.7)

$$W_1(4m_j^2) = \begin{pmatrix} I_n - \Phi_1(\pi, 4m_j^2) & -\Phi_2(\pi, 4m_j^2) \\ -\Phi_1'(\pi, 4m_j^2) & I_n - \Phi_2'(\pi, 4m_j^2) \end{pmatrix} = 0,$$

which implies, together with (2.2),

$$\begin{aligned} &\Phi_1'(\pi, 4m_j^2) \\ &= (-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)I_n + K(\pi,\pi)\cos(\sqrt{\lambda}\pi) + \int_0^\pi K_x'(\pi,t)\cos(\sqrt{\lambda}t)dt)|_{\lambda=4m_j^2} \\ &= K(\pi,\pi) + \int_0^\pi K_x'(\pi,t)\cos(2m_j t)dt = 0. \end{aligned}$$

The Riemann-Lebesgue lemma tells us

$$K(\pi,\pi) = 0,$$

from (2.3) we obtain

$$\int_0^\pi P(x)dx = 0. \tag{3.1}$$

Next, we verify that $\phi_i(x) = \frac{1}{\sqrt{\pi}} e_i$ is the eigenfunction corresponding to the first eigenvalue 0, where e_i is the unit vector whose i-st component is 1 ($i = 1, 2, \cdots, n$). In fact, by the variational principle and (3.1),

$$\begin{aligned} 0 = \lambda_1 \le & \frac{\int_0^\pi (-\phi_i^{*t}\phi_i'' + \phi_i^{*t} P(x)\phi_i) dx}{\int_0^\pi \phi_i^{*t}\phi_i dx} \\ & = \phi_i^{*t} \left(\int_0^\pi P(x) dx \right) \phi_i = 0, \end{aligned}$$

that is, the equality holds. Therefore, the test function $\phi_i(x) = \frac{1}{\sqrt{\pi}} e_i$ is the eigenfunction corresponding to the first eigenvalue 0. Substituting $\phi_i = \frac{1}{\sqrt{\pi}} e_i$ ($i = 1, 2, \cdots, n$) into equation (1.1), we have

$$P(x)e_i = 0 \text{ on } [0, \pi],\ i = 1, 2, \cdots, n,$$

which are equivalent to

$$P(x)I_n = 0 \text{ on } [0, \pi].$$

Thus $P(x) = 0$ on $[0, \pi]$ arises, as we asserted.

The proof of theorem 1.2. The sufficient part is obvious by a direct calculation as $P(x) = 0$. We shall prove the necessary part. Since the problem (1.1) and (1.3) has infinitely many eigenvalues of the form $(2m_j - 1)^2$, m_j are positive integers, $j = 1, 2, \cdots$, and each eigenvalues $(2m_j - 1)^2$ is of multiplicity $2n$, we have by (2.8)

$$\begin{aligned} & W_2((2m_j - 1)^2) \\ & = \begin{pmatrix} I_n + \Phi_1(\pi, (2m_j - 1)^2) & \Phi_2(\pi, (2m_j - 1)^2) \\ \Phi_1'(\pi, (2m_j - 1)^2) & I_n + \Phi_2'(\pi, (2m_j - 1)^2) \end{pmatrix} = 0, \end{aligned}$$

which implies, together with (2.5),

$$\begin{aligned} 0 & = \Phi_2(\pi, (2m_j - 1)^2) \\ & = \left(\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} I_n + \int_0^\pi L(\pi, t) \frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} dt\right)\Big|_{\lambda = (2m_j - 1)^2} \\ & = \int_0^\pi L(\pi, t) \frac{\sin(2m_j - 1)t}{2m_j - 1} dt. \end{aligned}$$

Using integration by parts, we have

$$L(\pi,\pi)+\int_0^{\pi} L_t'(\pi,t)\cos(2m_j-1)tdt=0.$$

The Riemann-Lebesgue lemma implies, by (2.6),

$$\frac{1}{2}\int_0^{\pi} P(x)dx=L(\pi,\pi)=0,$$

thus

$$\int_0^{\pi} P(x)dx=0. \tag{3.2}$$

Next, we show that $\phi_i(x)=(\sqrt{\frac{2}{\pi}}\sin x)e_i$ is the eigenfunction corresponding to the first eigenvalue 1, $i=1,2,\cdots,n$. In fact, by the variational principle,

$$\begin{aligned}
1=\lambda_1 &\le \frac{\displaystyle\int_0^{\pi}(-\phi_i^{*t}\phi_i''+\phi_i^{*t}P(x)\phi_i)dx}{\displaystyle\int_0^{\pi}\phi_i^{*t}\phi_i dx}\\
&=\frac{2}{\pi}\int_0^{\pi}\sin^2 xdx+\frac{2}{\pi}\int_0^{\pi}(P(x))_{ii}\sin^2 xdx\\
&=1+\frac{1}{\pi}\int_0^{\pi}(P(x))_{ii}dx-\frac{1}{\pi}\int_0^{\pi}(P(x))_{ii}\cos(2x)dx\\
&\le 1,
\end{aligned}$$

by the assumption and (3.2), the equality holds. Therefore, $\phi_i(x)=(\sqrt{\frac{2}{\pi}}\sin x)e_i$ ($i=1,2,\cdots,n$) is the eigenfunction corresponding to the first eigenvalue 1. Substituting $\phi_i=(\sqrt{\frac{2}{\pi}}\sin x)e_i$ into equation (1.1), we get for $i=1,2,\cdots,n$,

$$(\sqrt{\frac{2}{\pi}}\sin x)e_i+(\sqrt{\frac{2}{\pi}}\sin x)P(x)e_i=(\sqrt{\frac{2}{\pi}}\sin x)e_i \text{ on } [0,\pi],$$

that is,

$$P(x)I_n=0 \text{ on } [0,\pi],$$

thus $P(x)=0$ on $[0,\pi]$. This finishes the proof of theorem 1.2.

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REFERENCES

1. V. A. Ambarzumyan, Über eine Frage der Eigenwerttheorie, *Z. Phys.*, **53** (1929), 690-695.

2. F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic, New York, 1964.

3. G. Borg, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Bestimmung der Differentialgleichung durch die Eigenwerte, *Acta Math.*, **78** (1946), 1-96.

4. G. Borg, Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))y = 0$, in: *Proc. 11th Scandinavian Congress of Mathematicians* (*Oslo: Johan Grundt Tanums Forlag*), 1952, pp. 276-287.

5. N. K. Chakravarty and S. K. Acharyya, On an extension of the theorem of V. A. Ambarzumyan, *Proc. R. Soc. Edinb. A*, **110** (1988), 79-84.

6. H. H. Chern, C. K. Law and H. J. Wang, Extension of Ambarzumyan's theorem to general boundary conditions, *J. Math. Anal. Appl.*, **309** (2005), 764-768, (corrigendum).

7. H. H. Chern and C. L. Shen, On the n-dimensional Ambarzumyan's theorem, *Inverse Problems*, **13** (1997), 15-18.

8. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, 1, Interscience, New York, 1953.

9. E. M. Harrell, On the extension of Ambarzunyan's inverse spectral theorem to compact symmetric spaces, *Am. J. Math.*, **109** (1987), 787-795.

10. H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed given data, *SIAM J. Appl. Math.*, **34** (1978), 676-680.

11. N. V. Kuznetsov, Generalization of a theorem of V. A. Ambarzumian, *Dokl. Akad. Nauk SSSR*, **146** (1962), 1259-1262, (in Russian).

12. B. M. Levitan and M. G. Gasymov, Determination of a differential equation by two of its spectra, *Usp. Mat. Nauk*, **19** (1964), 3-63, [*Russian Math. Surveys*, **19** (1964), 1-64].

13. V. A. Marchenko, *Sturm-Liouville Operators and Applications (Operator Theory: Advances and Applications, 22)*, Basel, Birkhäuser, 1986.

14. V. N. Pivovarchik, Ambarzumyan's theorem for a Sturm-Liouville boundary value problem on a star-shaped graph, *Funct. Anal. Appl.*, **39** (2005), 148-151.

15. B. Simon, A new approach to inverse spectral theory: I Fundam. Formalism, *Ann. Math.*, **150** (1999), 1029-1057.

16. C. F. Yang, Z. Y. Huang and X. P. Yang, Ambarzumyan-type Theorems for the Sturm-Liouville Equation on a Graph, *Rocky Mountain J. Math.*, **39** (2009).

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