TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 4, pp. 1385-1409, August 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

SEMIGROUP OF RATIONAL p-ADIC FUNCTIONS FOR COMPOSITION

Abdelaziz Bellagh

Abstract. We are interested in the Julia set of a semigroup of rational functions with coefficients in $\mathbb{C}_p$ where the semigroup operation is composition. We prove that if a semigroup G is generated by a finite number of rational functions of degree at least two with coefficients in a finite extension of $\mathbb{Q}_p$, and has a nonempty Julia set $\mathcal{J}(G)$, then $\mathcal{J}(G)$ is perfect and has an empty interior.

1. Introduction and Results

In recent years, the Julia sets of rational functions with coefficients in a complete local field have been the subject of several investigations.

These sets are studied in [7, 2, 3, 4], etc... .

In this paper, we prove that some of the properties of the Julia set of a rational function with coefficients in $\mathbb{C}_p$ are also true for the Julia set of a semigroup of rational functions with coefficients in $\mathbb{C}_p$ (where the operation of semigroup is composition).

First, we recall some basic notions (exceptional points, spherical distance, equicontinuity, etc...) and some preliminary results we shall need later (see Section 2). Then, we give the definitions of the Julia set $\mathcal{J}(G)$ and the Fatou set $\mathcal{F}(G)$ of a semigroup G of rational functions with coefficients in $\mathbb{C}_p$, and we prove some of their immediate properties (see Proposition 1, Corollary 2 and Corollary 3).

In Section 4 we are interested in the invariance of the Julia set $\mathcal{J}(G)$ under elements of G, and we prove that as in the case of a semigroup G of rational functions with complex coefficients (see [6]) the Julia set $\mathcal{J}(G)$ is backward invariant (see Proposition 2) but not always forward invariant under the elements of G (see Example 1).

It is well known that if R is a rational function of degree at least two with coefficients in $\mathbb{C}_p$, then its Julia set $\mathcal{J}(R)$ is either empty or infinite (see [7]). Section 5 is devoted to the case where the Julia set $\mathcal{J}(G)$ is finite, and we prove:

Received October 31, 2007, accepted September 11, 2008.
Communicated by Julie Tze-Yueh Wang.
2000 *Mathematics Subject Classification*: 11S99.
Key words and phrases: Julia sets, p-Adic iteration, Semigroup.

Theorem 1. *Let $R_1, \ldots, R_n$ be rational functions of degree at least two with coefficients in $\mathbb{C}_p$ and $G = \langle R_1, \ldots, R_n \rangle$, then $\mathcal{J}(G)$ is either empty or infinite.*

Then, we give examples of finite Julia sets (see Corollary 4, Corollary 5, Remark 4 and Example 2).

In Section 6 we prove that $\mathcal{J}(G)$ can be infinite even if all its generators have empty Julia sets (see Corollary 6).

It is well known that if R is a rational function of degree at least two with coefficients in $\mathbb{C}_p$, then its Fatou set $\mathcal{F}(R)$ is not empty (see [3]) and its Julia set $\mathcal{J}(R)$ has an empty interior and $\mathcal{J}(R)$ is either perfect or empty (see [7]). In Section 7 we give partial generalizations of these results. We consider the case where G is a semigroup of rational functions with coefficients in a finite extension of $\mathbb{Q}_p$ and G has a finite number of generators all of which are of degree at least two, and we prove that its Julia set $\mathcal{J}(G)$ is perfect or empty and that $\mathcal{J}(G)$ has an empty interior. More precisely, we have:

Theorem 2. *Let K be a finite extension of the field $\mathbb{Q}_p$ and G the semigroup $\langle R_1, R_2, \ldots, R_n \rangle$, where $R_1, R_2, \ldots, R_n$ are rational functions of degree at least two with coefficients in K, then $\mathcal{J}(G)$ is either perfect or empty.*

Theorem 3. *Let K be a finite extension of $\mathbb{Q}_p$ and $G = \langle R_1, \ldots, R_n \rangle$ where $R_1, \ldots, R_n \in K(z) - K$, then $\mathcal{J}(G)$ has an empty interior.*

In Section 8 we compare the properties of the Julia sets in the p-adic case with the properties of the Julia sets in the complex case. For the properties of the Julia set of a semigroup of rational functions with complex coefficients, the reader can see for example [1, 6, 12].

2. Notations and Preliminary Results

Let p be a fixed prime number, let $\mathbb{C}_p$ be the completion of a fixed algebraic closure of the field $\mathbb{Q}_p$ of p-adic rationals, and let $|.|$ denote the p-adic absolute value on $\mathbb{C}_p$ extending the one on $\mathbb{Q}_p$ such that $|p| = 1/p$. If $r > o$ and $a \in \mathbb{C}_p$, we set $B^+(a, r) = \{x \in \mathbb{C}_p \ / \ |x - a| \leq r\}$ and $B^-(a, r) = \{x \in \mathbb{C}_p \ / \ |x - a| < r\}$. Let $\mathbb{P}^1(\mathbb{C}_p)$ denote the projective line over $\mathbb{C}_p$ (in the sequel $\mathbb{P}^1(\mathbb{C}_p)$ is viewed as $\mathbb{C}_p \cup \{\infty\}$). We define an open (resp., closed) $\mathbb{P}^1(\mathbb{C}_p)$ disk to be any subset of $\mathbb{P}^1(\mathbb{C}_p)$ of the form $\{x \in \mathbb{C}_p / |x - a| < r\}$ or $\{x \in \mathbb{C}_p / |x - a| > r\} \cup \{\infty\}$ (resp., $\{x \in \mathbb{C}_p / |x - a| \leq r\}$ or $\{x \in \mathbb{C}_p / |x - a| \geq r\} \cup \{\infty\}$) where $a \in \mathbb{C}_p, r \in \mathbb{R}^+ - \{0\}$. We define the spherical metric on $\mathbb{P}^1(\mathbb{C}_p)$ as follows (see [2, p. 3]), if (α, β) and (γ, δ) are two points in $\mathbb{P}^1(\mathbb{C}_p)$, we set:

$$\Delta((\alpha, \beta), (\gamma, \delta)) = \frac{|\alpha\delta - \gamma\beta|}{Max(|\alpha|, |\beta|)Max(|\delta|, |\gamma|)}$$

We easily check that the $\mathbb{P}^1(\mathbb{C}_p)$ disks are open sets of $\mathbb{P}^1(\mathbb{C}_p)$ for the spherical metric Δ, and that they form a base for the topological space $\mathbb{P}^1(\mathbb{C}_p)$. We will need the following two properties of the rational functions:

Lemma 1. [8, p. 104]. *If R is a rational function with coefficients in $\mathbb{C}_p$, then there exists a positive constant C such that for any α, β in $\mathbb{P}^1(\mathbb{C}_p)$, we have $\Delta(R(\alpha), R(\beta)) \leq C\Delta(\alpha, \beta)$. In particular R is uniformly continuous on $\mathbb{P}^1(\mathbb{C}_p)$.*

Lemma 2. *If R is a (non-constant) rational function with coefficients in $\mathbb{C}_p$, then R is an open map from the metric space $\mathbb{P}^1(\mathbb{C}_p)$ to itself.*

Proof. Indeed, if U is an open set of $\mathbb{P}^1(\mathbb{C}_p)$, then U is the union of a family $(D_i)_{i \in I}$ of $\mathbb{P}^1(\mathbb{C}_p)$ disks. Given that the image of a $\mathbb{P}^1(\mathbb{C}_p)$ disk by a (non-constant) rational function is either a $\mathbb{P}^1(\mathbb{C}_p)$ disk or $\mathbb{P}^1(\mathbb{C}_p)$ (see [3, p. 5]), we deduce that $R(D_i)$ is an open set of $\mathbb{P}^1(\mathbb{C}_p)$ for any i in I. It follows that $R(U)$ is also an open set of $\mathbb{P}^1(\mathbb{C}_p)$. ■

Now we introduce the notion of equicontinuity. Roughly speaking, the equicontinuity of a family $\mathcal{F}$ of rational functions with coefficients in $\mathbb{C}_p$ in a neighborhood V of a point x, means that if two points of V are close together, then their images under application of an element of $\mathcal{F}$ are also close together. On the other hand, the non-equicontinuity of the family $\mathcal{F}$ means that small errors on x may become arbitrarily large under application of elements of $\mathcal{F}$. More precisely:

Definition 1. Let D be a $\mathbb{P}^1(\mathbb{C}_p)$ disk and $\mathcal{F}$ a family of rational functions with coefficients in $\mathbb{C}_p$, we say that $\mathcal{F}$ is an equicontinuous family on D, if for any real number $\epsilon > 0$ there exists a real number $\eta > 0$ such that for any f in $\mathcal{F}$ and any x, y in D, $\Delta(x, y) < \eta$ implies that $\Delta(f(x), f(y)) < \epsilon$.

The following theorem is an equicontinuity criterion that we shall use afterwards. It is the p-adic analogous of Montel's Theorem in the complex case (see [1, p. 57]).

Hsia's Criterion. [7, p. 691]. Let D be a $\mathbb{P}^1(\mathbb{C}_p)$ disk, let Y be a subset of $\mathbb{P}^1(\mathbb{C}_p)$ whose complement contains at least two points, and let $\mathcal{F}$ be a family of rational functions such that $f(D) \subset Y$ for any f in $\mathcal{F}$. Then $\mathcal{F}$ is an equicontinuous family on D.

In this paper $\mathcal{F}$ will be a semigroup of rational functions. A semigroup of rational functions with coefficients in $\mathbb{C}_p$ is by definition a subset G of $\mathbb{C}_p(z) - \mathbb{C}_p$ containing the polynomial z, such that if $S, R \in G$, then the composition $R \circ S$ of R and S is also in G. If A is a nonempty subset of $\mathbb{C}_p(z) - \mathbb{C}_p$, then the semigroup generated by A is denoted by $\langle A \rangle$. If a semigroup G of rational functions is a subset of $\mathbb{C}_p[z] - \mathbb{C}_p$, we say that G is a polynomial semigroup. If n is a positive integer and R is a rational function with coefficients in $\mathbb{C}_p$, we denote by $R^{[n]}$ the n-th iterate of R, where by convention $R^{[0]}(z) = z$. We also need the notion of degree. If $P(z)$ is a polynomial with coefficients in $\mathbb{C}_p$, the degree of P is denoted by $\deg P$. If $R(z)$ is a rational function with coefficients in $\mathbb{C}_p$ such that $R(z) = \frac{P(z)}{Q(z)}$ where P and Q are relatively prime polynomials (with coefficients in $\mathbb{C}_p$), the ***degree*** of R is by definition $\deg R = \max(\deg P, \deg Q)$.

3. Julia Sets: Definitions and Immediate Properties

In this Section, we will give a definition of the Julia set and the Fatou set of a semigroup of rational functions with coefficients in $\mathbb{C}_p$, which is a natural extension of the definition given in the classical case of the Julia set and the Fatou set of a rational function with coefficients in $\mathbb{C}_p$ (see [7, p. 691]). This definition is analogous to the one given in the complex case (see [6, p. 360] and [1, Theorem 3.3.2]). However, we will see that as in the case of the Julia set and the Fatou set of a semigroup of rational functions with complex coefficients, the properties of these sets are less interesting than in the case the semigroup has only one generator (see [6, p. 361]).

Definition 2. Let G be a semigroup of rational functions with coefficients in $\mathbb{C}_p$ (where the semigroup operation is composition). The *Fatou set* of G is the set of points z of $\mathbb{P}^1(\mathbb{C}_p)$, such that there exists a $\mathbb{P}^1(\mathbb{C}_p)$ disk containing z, where the family of elements of G is equicontinuous for the spherical metric. The Fatou set of G is denoted by $\mathcal{F}(G)$. The *Julia set* of G is the complement of the Fatou set in $\mathbb{P}^1(\mathbb{C}_p)$, and is denoted by $\mathcal{J}(G)$.

Remark 1. It is clear from the above definition that if G is a semigroup of rational functions with coefficients in $\mathbb{C}_p$, then the Fatou set of G is open in the metric space $\mathbb{P}^1(\mathbb{C}_p)$ and the Julia set of G is closed. In addition, if G is generated by a single rational function g, the Julia (resp., Fatou) set of G coincides with the Julia (resp., Fatou) set of g defined in the classical way (for example as in [2, p. 1]) and is denoted by $\mathcal{J}(g)$ (resp., $\mathcal{F}(g)$).

Here we need the following definition:

Definition 3. Let R, S be two rational functions with coefficients in $\mathbb{C}_p$, we say that R and S are conjugate if there exists a rational function φ of degree one, such that $S = \varphi \circ R \circ \varphi^{-1}$.

We know that if R is a rational function of degree at least two, then $\mathcal{J}(R)$ is either empty or infinite (see [7, p. 694]). When R is a degree one rational function, it's well known that R is conjugate either to $z + \alpha$ or to αz (where $\alpha \in \mathbb{C}_p - \{0\}$);

and it is easy to prove that for $\lambda, \mu \in \mathbb{C}_p - \{0\}$ we have $\mathcal{J}(z+\mu) = \emptyset$, and that $\mathcal{J}(\lambda z) = \emptyset$ if $|\lambda| = 1$, $\mathcal{J}(\lambda z) = \{\infty\}$ if $|\lambda| < 1$, $\mathcal{J}(\lambda z) = \{0\}$ if $|\lambda| > 1$. Now we prove some elementary facts about the Fatou set and the Julia set of a semigroup G of rational functions with coefficients in $\mathbb{C}_p$.

Lemma 3. *Let U be an open set of $\mathbb{P}^1(\mathbb{C}_p)$, and Y a subset of $\mathbb{P}^1(\mathbb{C}_p)$ whose complement contains at least two points. Assume that $g(U) \subset Y$ for any g in G, then we have $U \subset \mathcal{F}(G)$.*

Proof. Results from Hsia's Criterion. ∎

We will also need the notions of invariance backward and forward by a rational function.

Definition 4. Let g be a rational function with coefficients in $\mathbb{C}_p$ and A a subset of $\mathbb{P}^1(\mathbb{C}_p)$, we say that:

(1) A is forward invariant by g if $g(A) \subset A$.
(2) A is backward invariant by g if $g^{-1}(A) \subset A$.
(3) A is completely invariant by g if $g(A) \subset A$ and $g^{-1}(A) \subset A$.

Corollary 1. *Let F be a closed set (of $\mathbb{P}^1(\mathbb{C}_p)$) backward invariant by every element g of G. If F contains at least two points, then we have:*

$$\mathcal{J}(G) \subset F.$$

Proof. If we set $U = F^c$, then we have $g(U) \subset U$ for every element g of G. Given that the complement F of U contains at least two points, Lemma 3 implies that $U \subset \mathcal{F}(G)$ and thus $\mathcal{J}(G) \subset F$. ∎

Lemma 4. *Let G_1 and G_2 be two semigroups of rational functions. Assume that $G_1 \subset G_2$, then we have $\mathcal{F}(G_2) \subset \mathcal{F}(G_1)$ and $\mathcal{J}(G_1) \subset \mathcal{J}(G_2)$. In particular if G is a semigroup of rational functions, then for any g in G, we have $\mathcal{F}(G) \subset \mathcal{F}(g)$ and $\mathcal{J}(g) \subset \mathcal{J}(G)$.*

Proof. Let z be a point of $\mathcal{F}(G_2)$. There exists a $\mathbb{P}^1(\mathbb{C}_p)$ disk D containing z where the family of elements of G_2 is equicontinuous. Given that every subfamily of an equicontinuous family is also equicontinuous, we deduce that the family of elements of G_1 is equicontinuous on D. It follows that z is in $\mathcal{F}(G_1)$, and this proves the first inclusion. The second inclusion results by taking the complement in $\mathbb{P}^1(\mathbb{C}_p)$. The final assertion is straightforward. ∎

Remark 2. If R is an element of a semigroup $G = \langle R_i \ / i \in I \rangle$ such that $|\mathcal{J}(R)| \geq 2$ and $R_i^{-1}(\mathcal{J}(R)) \subset \mathcal{J}(R)$ for any i in I, then we deduce from Corollary 1 and Lemma 4 that $\mathcal{J}(G) = \mathcal{J}(R)$. In particular if f, g are two rational functions of degree at least two with the same nonempty Julia set, then we have $\mathcal{J}(\langle f, g \rangle) = \mathcal{J}(f)$.

3.1. Conjugation by a degree one rational function

Now we consider the conjugation of a semigroup G of rational functions by a degree one rational function φ. We set $G_\varphi = \{\varphi \circ g \circ \varphi^{-1} \ / \ g \in G\}$.

Proposition 1. *G_φ is a semigroup of rational functions, and we have the equalities:*

$$\mathcal{F}(G_\varphi) = \varphi(\mathcal{F}(G)) \quad \text{and} \quad \mathcal{J}(G_\varphi) = \varphi(\mathcal{J}(G)).$$

Proof. The fact that G_φ is a semigroup is obvious. For the second assertion, it suffices to prove that $\mathcal{F}(G_\varphi) \subset \varphi(\mathcal{F}(G))$. According to Lemma 1, there exists two positive real numbers b_1, b_2 such that for any x, y in $\mathbb{P}^1(\mathbb{C}_p)$ we have $\Delta(\varphi(x), \varphi(y)) \leq b_1 \Delta(x, y)$, and $\Delta(\varphi^{-1}(x), \varphi^{-1}(y)) \leq b_2 \Delta(x, y)$. If $z \in \mathcal{F}(G_\varphi)$, there exists a $\mathbb{P}^1(\mathbb{C}_p)$ disk D containing z such that for any $\epsilon > 0$ there exists $\eta > 0$ such that for any f in G and for any x, y in D, $\Delta(x, y) < \eta$ implies that $\Delta(f \circ \varphi^{-1}(x), f \circ \varphi^{-1}(y)) \leq b_2 \Delta(\varphi \circ f \circ \varphi^{-1}(x), \varphi \circ f \circ \varphi^{-1}(y)) < \epsilon$. Let $\eta' = \dfrac{\eta}{b_1}$, then for any u, v in $D' = \varphi^{-1}(D)$, $\Delta(u, v) < \eta'$ implies that $\Delta(f(u), f(v)) < \epsilon$. It follows that the family of elements of G is equicontinuous on D', and so z is in $\varphi(\mathcal{F}(G))$. ∎

3.2. Rational functions with good reduction

When $\mathcal{J}(G)$ is empty, Lemma 4 implies that for any g in G, we have $\mathcal{J}(g) = \emptyset$ (but we prove in Section 6 that the converse is false). Here we give a condition on the elements of G sufficient to have $\mathcal{J}(G)$ empty. Let $f(X) = \frac{P(X)}{Q(X)}$ be a rational function of degree d with coefficients in $\mathbb{C}_p$ where $P(X), Q(X)$ are relatively prime polynomials. We can assume that the coefficients of P and Q are in the ring of integers of $\mathbb{C}_p$, and that one of these coefficients has absolute value 1. Then we set $P_1(X, Y) = Y^d P(\frac{X}{Y})$ and $Q_1(X, Y) = Y^d Q(\frac{X}{Y})$. Following [2] we say that the rational function f has *good reduction* if the polynomials P_1 and Q_1 have no common zeros in $\overline{\mathbb{F}_p}^2$ other than $(0, 0)$. If f has not good reduction we say that f has bad reduction. We note that the reduction type (good or bad) is independent of the choice of the polynomials P and Q such that $f(X) = \frac{P(X)}{Q(X)}$. We know that if a semigroup of rational functions has only one generator R and R has good reduction, then for any x, y in $\mathbb{P}^1(\mathbb{C}_p)$, we have $\Delta(R(x), R(y)) \leq \Delta(x, y)$ (see [8, p. 105]), in particular the Julia set of R is empty. It easily follows that:

Corollary 2. *Let A be a nonempty subset of $\mathbb{C}_p(z) - \mathbb{C}_p$ and G the semigroup of rational functions generated by A. If any element of A has good reduction, then the Julia set of G is empty.*

3.3. Fixed points

We need the notions of repelling fixed point, attracting fixed point and neutral fixed point of a rational function. Let R be a rational function with coefficients in $\mathbb{C}_p$ and z_0 a point of $\mathbb{P}^1(\mathbb{C}_p)$. We say that z_0 is a fixed point of R if $R(z_0) = z_0$. When z_0 is in $\mathbb{C}_p$, we say that the fixed point z_0 is *repelling* if $|R'(z_0)| > 1$, *attracting* if $|R'(z_0)| < 1$, *neutral* if $[R'(z_0)| = 1$. When $z_0 \in \mathbb{C}_p$, it is easy to prove that if R is a rational function and z_0 a fixed point of R, the type of the fixed point z_0 (repelling, attracting or neutral) is invariant by conjugation. This allows us to define the type of ∞ when it is a fixed point of R. We know that if R is a rational function with coefficients in $\mathbb{C}_p$, then the repelling fixed points of R are in $\mathcal{J}(R)$ and the non-repelling fixed points of R are in $\mathcal{F}(R)$ (see [3, p. 2]). In this Section we will prove that if the generators of a semigroup G of rational functions have a common non-repelling fixed point ω, then under certain conditions, ω is the centre of a $\mathbb{P}^1(\mathbb{C}_p)$ disk (contained in $\mathcal{F}(G)$) of which we can compute the radius. By conjugating (if necessary) by an appropriate linear fractional map, we can suppose that $\omega = 0$. Now let R be a rational function such that 0 is a non-repelling fixed point of R and let $R(z) = \sum_{i=0}^{\infty} a_i z^{i+1}$ be the expansion of R as a power series in a neighborhood of 0. If the a_i are not all 0 (for i in $\mathbb{N} - \{0\}$) we set $\tau(R)^{-1} = \sup\left\{|a_n|^{\frac{1}{n}} \ / \ n \in \mathbb{N} - \{0\}\right\}$, and if R is a degree one polynomial we set $\tau(R) = \infty$. Then we use the following lemma:

Lemma 5. [9, pp. 38, 47]. *If $0 < t < \tau(R)$, then we have*

$$R(B^+(0,t)) \subset B^+(0,t)$$

Corollary 3. *Let $(R_i)_{i\in I}$ be a family of (non-constant) rational functions with coefficients in $\mathbb{C}_p$ and $G = \langle R_i, / i \in I\rangle$. We assume that 0 is a non-repelling fixed point of R_i for any i in I, and we set $\tau = \inf\{\tau(R_i), / i \in I\}$. If $\tau \neq 0$, then we have $B^-(0,\tau) \subset \mathcal{F}(G)$.*

Proof. If $r < \tau$, we have for any i in I, $R_i(B^+(0,r)) \subset B^+(0,r)$. Hence, the result follows from Hsia's Criterion. ∎

Remark 3. If $G = \langle R_1, R_2, \ldots, R_n\rangle$ where $R_1, R_2, \ldots, R_n$ are (non-constant) rational functions with coefficients in $\mathbb{C}_p$ and ω is a common non-repelling fixed point (in $\mathbb{P}^1(\mathbb{C}_p)$) of $R_1, R_2, \ldots, R_n$, then there exists a ($\mathbb{P}^1(\mathbb{C}_p)$) disk D such that $\omega \in D \subset \mathcal{F}(G)$. Indeed, by conjugating G (if necessary) by an appropriate degree one rational function, we can suppose that $\omega = 0$. Then we apply the preceding corollary. In particular if G is generated by a finite number of polynomials of degree at least two, then $\infty \in \mathcal{F}(G)$ (since ∞ is in this case a common non-repelling fixed point of the generators of G).

4. Invariance of Julia and Fatou Sets

Now we examine the invariance of Julia and Fatou sets. In the case of a single rational function R, we know that the sets $\mathcal{J}(R)$ and $\mathcal{F}(R)$ are completely invariant by R (see, for example, [3, p. 1]). Here we will see that it is not always the case for Fatou and Julia sets of a semigroup of rational functions with coefficients in $\mathbb{C}_p$. We will need the following lemma:

Lemma 6. [5, p. 41]. *Let D be a $\mathbb{P}^1(\mathbb{C}_p)$ disk, let R be a (non-constant) rational function with coefficients in $\mathbb{C}_p$ and $D^* = R(D)$. For any $\alpha > 0$ there exists $\alpha^* > 0$ with the following property: For any u, v in D^* such that $\Delta(u, v) < \alpha^*$, there exists x, y in D such that $R(x) = u$, $R(y) = v$ and $\Delta(x, y) < \alpha$.*

As in the complex case (see [6, p. 360]), we have:

Proposition 2. *Let G be a semigroup of rational functions with coefficients in $\mathbb{C}_p$. For any g in G, $\mathcal{F}(G)$ is forward invariant by g and $\mathcal{J}(G)$ is backward invariant by g (that is to say $g(\mathcal{F}(G)) \subset \mathcal{F}(G)$ and $g^{-1}(\mathcal{J}(G)) \subset \mathcal{J}(G)$).*

Proof. If $g \in G$ and $z \in \mathcal{F}(G)$, there exists a $\mathbb{P}^1(\mathbb{C}_p)$ disk D containing z such that the family of rational functions of G is equicontinuous on D. Then for any $\epsilon > 0$ there exists $\eta > 0$ such that for any f in G and for any x, y in D, $\Delta(x, y) < \eta$ implies that we have $\Delta(f(x), f(y)) < \epsilon$. According to Lemma 6, there exists $\eta' > 0$ with the following property: for any u, v in $g(D)$ such that $\Delta(u, v) < \eta'$, there exists x, y in D such that $u = g(x), v = g(y)$ and $\Delta(x, y) < \eta$. Then for any f in G and any u, v in $g(D)$ such that $\Delta(u, v) < \eta'$, we have

$$\Delta(f(u), f(v)) = \Delta(f(g(x)), f(g(y))) < \epsilon$$

(recall that $f \circ g \in G$ since G is a semigroup). It follows that the family of rational functions of G is equicontinuous on $g(D)$. Hence, we have $g(\mathcal{F}(G)) \subset \mathcal{F}(G)$. The second assertion results from the first because $\mathcal{J}(G)$ is the complement of $\mathcal{F}(G)$. ■

The following example shows that in general we do not have the equality $g^{-1}(\mathcal{J}(G)) = \mathcal{J}(G)$ for any g in G. In fact, we have the same problem in the complex case (see [6, p. 361]).

Example 1. Let p, q be two different prime numbers, such that q is prime to $p-1$, and let G be the semigroup generated by the rational functions with coefficients in $\mathbb{C}_p$, $g_1(z) = \dfrac{z^p - z}{p}$ and $g_2(z) = z^q$. First, we note that if we set $Y = B^+(0,1)^c$, then for any g in G we have $g(Y) \subset Y$ (since $g_1(Y) \subset Y$ and $g_2(Y) \subset Y$). Hence, Lemma 3 implies that $Y \subset \mathcal{F}(G)$. For any k in $\mathbb{N}$ we set

$$A_k = \{x \in \mathbb{C}_p / x = p^k(l + p\omega), \omega \in \mathbb{Z}_p, l \in \mathbb{N}, 1 \leq l \leq p-1\}$$

(then we have $\mathbb{Z}_p = (\cup_{k\geq 0} A_k) \cup \{0\}$). For any m and k in $\mathbb{N}$ we set

$$B_{k,m} = \{x \in \mathbb{C}_p / x = u(l + p\theta), \theta \in \mathbb{Z}_p, l \in \mathbb{N}, 1 \leq l \leq p-1, u^{q^m} = p^k\}.$$

Now we show that the inverse image of A_k by $g_2^{[m]}$ is $B_{m,k}$. It is clear that the image of $B_{m,k}$ by $g_2^{[m]}$ is a subset of A_k. Conversely, if x is a point of $\mathbb{C}_p$ such that $g_2^{[m]}(x) = p^k(l + p\omega)$ and $\omega \in \mathbb{Z}_p, l \in \mathbb{N}, 1 \leq l \leq p-1$, we set $P(y) = y^{q^m} - (l + p\omega)$. Given that q is prime to $p-1$, we deduce that there exists an integer j between 1 and $p-1$ such that $j^{q^m} \equiv l (\bmod\, p)$. Hence, we have $|P(j)| = |j^{q^m} - (l + p\omega)| \leq \frac{1}{p} < 1$ and $|P'(j)| = 1$, and Hensel's Lemma (see, for example, [10]) implies that there exists y_0 in $\mathbb{Z}_p$ such that $|y_0 - j| \leq \frac{1}{p}$ and $P(y_0) = 0$. If we set $y_0 = j + p\theta$, it follows that $\theta \in \mathbb{Z}_p$ and $y_0^{q^m} = l + p\omega$. Then $u = \dfrac{x}{y_0}$ satisfies $u^{q^m} = p^k$, and we have $x = u(j + p\theta)$, where $\theta \in \mathbb{Z}_p$. We deduce that $x \in B_{k,m}$ and that $(g_2^{[m]})^{-1}(A_k) = B_{k,m}$.

According to Proposition 2, we have $g_2^{-1}(\mathcal{J}(G)) \subset \mathcal{J}(G)$. If we set $B = (\bigcup_{m=0}^{\infty} \bigcup_{k=0}^{\infty} B_{k,m}) \bigcup \{0\}$, then $B_{k,0} = A_k$ implies that $\mathbb{Z}_p \subset B$, and taking into account that the Julia set of g_1 is $\mathbb{Z}_p$ (see [11] or [7]), we deduce that $A_k \subset \mathbb{Z}_p = \mathcal{J}(g_1) \subset \mathcal{J}(G)$. It follows that $B_{k,m} \subset \mathcal{J}(G)$, and that $\mathbb{Z}_p \subset B \subset \mathcal{J}(G) \subset Y^c = B^+(0,1)$. However, there exists an element of B which does not belong to $\mathbb{Z}_p$. For example, if w satisfies $g_2(w) = w^q = p$, then its p-adic absolute value is $\dfrac{1}{\sqrt[q]{p}} \notin p^{\mathbb{Z}}$ and thus we have $w \notin \mathbb{Z}_p$.

Furthermore, for a positive integer N sufficiently large we have

$$g_1^{[N]}(w) \in Y.$$

Indeed, we know that $\mathbb{Z}_p$ is the filled Julia set of g_1 (see [4, p. 23]); hence for any x in $B^+(0,1) - \mathbb{Z}_p$ the sequence $g_1^{[n]}(x)$ tends towards ∞. We deduce that in general $\mathcal{J}(G)$ is not completely invariant by the elements of G.

5. The Case Where the Julia Set is Finite

Let R be a rational function with coefficients in $\mathbb{C}_p$, we know that if R is of degree at least two then $\mathcal{J}(R)$ is either empty or infinite (see, for example, [7, p. 694]). In this Section, we prove Theorem 1 which is a generalization of the preceding result to the case of the Julia set of a semigroup of rational functions with coefficients in $\mathbb{C}_p$. Now we suppose that G is an arbitrary semigroup of rational functions (with coefficients in $\mathbb{C}_p$) which can contain degree one rational functions and that G is not necessarily finitely generated. We begin with some preliminary definitions and lemmas. Let a be a point of $\mathbb{P}^1(\mathbb{C}_p)$, the union of the sets $g^{-1}(\{a\})$ where g is an arbitrary element of G is called the *backward orbit* of a by G, and is denoted by $O_G^-(a)$. In the same way, the union of the sets $\{g(a)\}$ where g is an arbitrary element of G is called the *forward orbit* of a by G, and is denoted by $O_G^+(a)$. If $O_G^-(a)$ is finite, we say that a is an *exceptional point* of G. We denote by $E(G)$ the set of exceptional points of G. If G is generated by a non-constant rational function R, the set of exceptional points of G is denoted by E_R. We know that if a rational function R is of degree at least two, then $|E_R| \leq 2$ (see [1, p. 65]). We can easily check that if R is a rational function of degree one with infinite order (that is to say $R^{[n]}(z) \neq z$ for any n in $\mathbb{N} - \{0\}$), then we also have $|E_R| \leq 2$. Given that $E(G) \subset E_g$ for any g in G, it follows that:

Lemma 7. *If G contains a rational function of degree at last two or of degree one with infinite order, then we have $|E(G)| \leq 2$.*

If $\mathcal{J}(G)$ is finite, then the elements of G are either degree one rational functions or rational functions of degree at least two with empty Julia sets (because $\mathcal{J}(g) \subset \mathcal{J}(G)$ for any g in G); and in this case $\mathcal{J}(G)$ is completely invariant by the elements of G.

Lemma 8. *If $\mathcal{J}(G)$ is finite, then $\mathcal{J}(G)$ has at most two elements.*

Proof. If G contains a rational function of degree at least two, then we have $\mathcal{J}(G) \subset E(G)$, hence this lemma results from Lemma 7. If G contains only degree one rational functions, we consider two cases:

(1) Assume G is infinite and $\mathcal{J}(G) = \{x_1, \ldots, x_n\}$ where n is a positive integer. If $n \geq 3$, given that $\mathcal{J}(G)$ is completely invariant by the elements of G and that an element g of G is perfectly determined by the images $g(x_1), g(x_2), g(x_3)$ (because g is of degree one), we deduce that G has at most $n(n-1)(n-2)$ elements, and this contradicts the hypotheses. It follows that $|\mathcal{J}(G)| \leq 2$.

(2) Assume G is finite, then Lemma 1 implies that there exists a positive constant C such that for any g in G and any x, y in $\mathbb{P}^1(\mathbb{C}_p)$, we have $\Delta(g(x), g(y)) \leq C\Delta(x, y)$. It follows that $\mathcal{J}(G) = \emptyset$.

Now we consider the case where $\mathcal{J}(G)$ has one or two elements. ■

Lemma 9. *Let G be a semigroup of rational functions.*

(1) *If $|\mathcal{J}(G)| = 1$, then G is conjugate to a polynomial semigroup.*

(2) *If $|\mathcal{J}(G)| = 2$, then G is conjugate to a semigroup whose elements are all of the form az^m where $a \in \mathbb{C}_p - \{0\}$ and $m \in \mathbb{Z} - \{0\}$.*

Proof.

(1) If $\mathcal{J}(G) = \{a\}$, then by conjugating (if necessary) by the rational function $\varphi(z) = \dfrac{1}{z-a}$ we may assume that $\mathcal{J}(G) = \{\infty\}$. If g is an element of G, we have $g^{-1}(\{\infty\}) = \{\infty\}$, and it follows that g has no pole in $\mathbb{C}_p$ (thus g is a polynomial).

(2) If $\mathcal{J}(G) = \{\alpha, \beta\}$, then by conjugating (if necessary) by the rational function

$$\varphi(z) = \begin{cases} \dfrac{z-\alpha}{z-\beta} & \text{if} \quad \alpha, \beta \in \mathbb{C}_p \\ \dfrac{1}{z-\beta} & \text{if} \quad \alpha = \infty \end{cases}$$

we may assume that $\mathcal{J}(G) = \{0, \infty\}$. It follows that for any g in G, we have $g^{-1}(\{0, \infty\}) = \{0, \infty\} = g(\{0, \infty\})$.

Now we consider two cases:

(a) If $g(0) = 0$ and $g(\infty) = \infty$, then we have $g^{-1}(\{\infty\}) = \{\infty\}$. Hence, g is a polynomial and g is of the form az^m where $a \in \mathbb{C}_p - \{0\}$ and $m \in \mathbb{Z} - \{0\}$ (because $g^{-1}(\{0\}) = \{0\}$).

(b) If $g(0) = \infty$ and $g(\infty) = 0$, we set $h(z) = (g(z))^{-1}$. Then we have $h^{-1}(\{\infty\}) = \{\infty\}$ and $h^{-1}(\{0\}) = \{0\}$. Hence we see as in the first case that h is of the form az^m where $a \in \mathbb{C}_p - \{0\}$ and $m \in \mathbb{Z} - \{0\}$. It follows that g has the form indicated in the lemma.

When $|\mathcal{J}(G)| = 1$, we state: ■

Lemma 10. *Let $(R_i, i \in I)$ be a family of rational functions of degree at least two and $G = \langle R_i, i \in I\rangle$. If $|\mathcal{J}(G)| = 1$, then the set I is infinite.*

Proof. According to Lemma 9, we can assume that G is a polynomial semigroup. Suppose that G is generated by the polynomials (of degree at least two) $P_1, \ldots, P_n$ and that $\mathcal{J}(G) = \{\infty\}$, then ∞ is a common non-repelling fixed point of $P_1, \ldots, P_n$, and Remark 3 implies that $\infty \in \mathcal{F}(G)$ (contradicting the hypotheses). ■

In order to complete the study of the case where $\mathcal{J}(G)$ is finite, we need to examine the Julia set of G when G is a subsemigroup of the semigroup $\mathcal{S}$ of rational functions of the form az^m where $a \in \mathbb{C}_p - \{0\}$ and $m \in \mathbb{Z} - \{0\}$.

Lemma 11. *If G is a subsemigroup of $\mathcal{S}$, then we have:*

$$\mathcal{J}(G) \subset \{0, \infty\}.$$

Proof. For any element g of G, we have $g(\mathbb{C}_p - \{0\}) \subset \mathbb{C}_p - \{0\}$. Given that $\mathbb{C}_p - \{0\}$ is an open set of $\mathbb{P}^1(\mathbb{C}_p)$ whose complement contains two points, Lemma 3 implies that $\mathbb{C}_p - \{0\} \subset \mathcal{F}(G)$ and $\mathcal{J}(G) \subset \{0, \infty\}$. ■

If G is a finitely generated subsemigroup of $\mathcal{S}$, then the following lemma shows that we can't have $|\mathcal{J}(G)| = 2$.

Lemma 12. *Let $G = \langle R_j \ / \ j \in J\rangle$ where $R_j(z) = (\lambda_j z)^{r_j}$, $\lambda_j \in \mathbb{C}_p - \{0\}$ and $r_j \in \mathbb{Z} - \{0, -1, 1\}$ for any j in J. If $\sup\limits_{j\in J} |\lambda_j| \neq \infty$ and $\inf\limits_{j\in J} |\lambda_j| \neq 0$, then we have $\mathcal{J}(G) = \emptyset$.*

Proof. First, we note that every element R of $\mathcal{S}$ can be written in the form $R(z) = (\alpha z)^n$ where $\alpha \in \mathbb{C}_p - \{0\}$ and $n \in \mathbb{Z} - \{0\}$, and that if $T(z) = (\beta z)^m$ where $\beta \in \mathbb{C}_p - \{0\}$ and $m \in \mathbb{Z} - \{0\}$, then we have $R \circ T(z) = (\alpha(m)\beta z)^{nm}$, where $\alpha(m)$ denotes an element of $\mathbb{C}_p - \{0\}$ such that $\alpha(m)^m = \alpha$ (in particular we have $|\alpha(m)| = |\alpha|^{1/m}$). In addition, it is easy to see (by induction on $n \in \mathbb{N} - \{0\}$) that if for $i = 1, \dots, n$ we have $T_i(z) = (\mu_i z)^{m_i}$ where $m_i \in \mathbb{Z} - \{0\}$, $\mu_i \in \mathbb{C}_p - \{0\}$, then we have $T_n \circ T_{n-1} \circ \cdots \circ T_1(z) = (\lambda z)^m$ where $m = m_n m_{n-1} \cdots m_2 m_1$ and $\lambda = \mu_n(m_{n-1}m_{n-2} \cdots m_2 m_1)\mu_{n-1}(m_{n-2} \cdots m_2 m_1) \cdots \mu_2(m_1)\mu_1$. It follows that

$$|\lambda| = |\mu_n|^{\frac{1}{m_{n-1}m_{n-2}\cdots m_2 m_1}} |\mu_{n-1}|^{\frac{1}{m_{n-2}\cdots m_2 m_1}} \cdots |\mu_2|^{\frac{1}{m_1}} |\mu_1| \tag{1}$$

Now we set $\mathcal{E} = \{|\lambda_j|^\epsilon \ / \ j \in J, \ \epsilon = \pm 1\}$ and $\delta = \sup \mathcal{E}$, and we prove that if $R(z) = (\lambda z)^m \in G$, then we have $\delta^{-2} \leq |\lambda| \leq \delta^2$. First, we note that $\delta \in [1, +\infty[$ and that $\mathcal{E} \subset [\delta^{-1}, \delta]$. If R is an element of G, then R can be written in the form $R = T_n \circ T_{n-1} \circ \cdots \circ T_1$, where $T_i(z) = (\mu_i z)^{m_i} \in \{R_j \ / \ j \in J\}$, $\mu_i \in \{\lambda_j \ / \ j \in J\}$ and $m_i \in \{r_j \ / \ j \in J\}$ for $i = 1 \dots, n$. It is easy to see that if $t \in [\delta^{-1}, \delta]$ and $\alpha \in \mathbb{R} - \{0\}$, then we have $t^\alpha \in [\delta^{-|\alpha|}, \delta^{|\alpha|}]$. Hence, it follows from Equality (1) that

$$|\lambda| \leq \delta^{\left|\frac{1}{m_{n-1}m_{n-2}\cdots m_2 m_1}\right|} \delta^{\left|\frac{1}{m_{n-2}\cdots m_2 m_1}\right|} \cdots \delta^{\left|\frac{1}{m_1}\right|} \delta$$

and given that $\dfrac{1}{|m_1|} + \cdots + \dfrac{1}{|m_{n-2} \cdots m_2 m_1|} + \dfrac{1}{|m_{n-1}m_{n-2} \cdots m_2 m_1|} \leq \sum\limits_{i=1}^{n-1} \dfrac{1}{2^i} < 1$, we deduce that $\delta^{-2} \leq |\lambda| \leq \delta^2$.

Now we prove that $0 \in \mathcal{F}(G)$. It suffices to verify that the family of rational functions of G is equicontinuous on the disk $B^-(0,\delta^{-2})$. More precisely, we will prove that for $\epsilon > 0$ and $y_1, y_2 \in B^-(0,\delta^{-2})$ the inequality $\Delta(y_1,y_2) \leq \epsilon/\delta^2$ implies that $\Delta(R(y_1),R(y_2)) \leq \epsilon$ for every element R of G.

If y is an element of $B^-(0,\delta^{-2})$ and $R(z) = (\lambda z)^m$ is an element of G, we have $|\lambda y| \leq \delta^2 |y| < 1$. Then we consider two cases:

(i) If $m > 1$, then $|R(y)| < 1$. Hence, we have

$$\begin{aligned}\Delta(R(y_1),R(y_2)) &= |R(y_1) - R(y_2)| = |(\lambda y_1)^m - (\lambda y_2)^m| \\ &= |(\lambda y_1 - \lambda y_2) \sum_{k+l=n-1} (\lambda y_1)^k (\lambda y_2)^l| \leq |\lambda y_1 - \lambda y_2| \\ &\leq \delta^2 \Delta(y_1,y_2)| \leq \epsilon\end{aligned}$$

(ii) If $m \ < -1$, then for $y \neq 0$, we have $|R(y)| > 1$. Hence, for $y_1 y_2 \neq 0$, we have $\Delta(R(y_1),R(y_2)) = \left|\dfrac{1}{R(y_1)} - \dfrac{1}{R(y_2)}\right| = |(\lambda y_1)^{m'} - (\lambda y_2)^{m'}|$ where $m' = -m$, and as in the first case we have $\Delta(R(y_1),R(y_2)) \leq \epsilon$. For $y_1 \neq 0$ and $y_2 = 0$, we have $\Delta(R(y_1),R(y_2)) = \left|\dfrac{1}{R(y_1)}\right| = |(\lambda y_1)^{m'}| \leq |\lambda y_1| \leq \delta^2 \Delta(y_1, 0)| \leq \epsilon$.

In addition, we have $\infty \in \mathcal{F}(G)$. Indeed, if we set $i(z) = 1/z$ and note H the conjugate of G by i, then H satisfies the hypotheses of the lemma, and the preceding proof implies that $0 \in \mathcal{F}(H) = i(\mathcal{F}(G))$. Hence, we have $\infty = i(0) \in \mathcal{F}(G)$.

Finally, we have $\{0,\infty\} \subset \mathcal{F}(G)$, and according to Lemma 11 we have $\mathcal{J}(G) \subset \{0,\infty\}$. Hence, we deduce that $\mathcal{J}(G) = \emptyset$. ■

Now we are ready to give the proof of Theorem 1.

Proof. [Proof of Theorem 1]. If $\mathcal{J}(G)$ is neither empty nor infinite, Lemma 8 implies that $|\mathcal{J}(G)| \leq 2$, and using Lemmas 9, 10 and 12 we get a contradiction. ■

The following two corollaries give examples of semigroups of rational functions whose elements are of the form az^m (where $a \in \mathbb{C}_p - \{0\}, m \in \mathbb{Z} - \{0\}$) and whose Julia set has one or two elements.

Corollary 4. *If* $G = \langle P_i \ / \ i \in I\rangle$ *where* $P_i(z) = (\lambda_i z)^{m_i}$, $\lambda_i \in \mathbb{C}_p - \{0\}$, $|\lambda_i| \geq 1$ *and* $m_i \in \mathbb{N} - \{0,1\}$ *for any* i *in* I, *then we have* $\mathcal{J}(G) \subset \{0\}$. *In addition* $\mathcal{J}(G) = \{0\}$ *if and only if* $\sup\limits_{i \in I} |\lambda_i| = \infty$.

Proof. For any i in I we have $P_i(B^-(0,1)^c) \subset B^-(0,1)^c$ (since $|\lambda_i| \geq 1$). Hence, Hsia's Criterion implies that $B^-(0,1)^c \subset \mathcal{F}(G)$ and in particular that

$\infty \in \mathcal{F}(G)$. According to Lemma 11 $\mathcal{J}(G)$ is a subset of $\{0,\infty\}$; it follows that $\mathcal{J}(G) \subset \{0\}$.

Now we prove that if we have $m_i \in \mathbb{N}-\{0\}$ for any i in I and $\sup_{i\in I}|\lambda_i| = \infty$, then we have $0 \in \mathcal{J}(G)$. If D is a disk containing 0 and ϵ is a real positive number, then for any real positive number η, there exists an element i of I such that $\Delta(\lambda_i^{-1},0) = |\lambda_i^{-1}| < \eta$. Given that $\Delta(P_i(\lambda_i{}^{-1}), P_i(0)) = \Delta(1,0) = 1 > \epsilon$, we conclude that the family of rational functions of G is not equicontinuous on any disk containing 0. It follows that $0 \in \mathcal{J}(G)$.

Conversely, if we have $m_i \in \mathbb{N}-\{0,1\}$ for any element i of I and $0 \in \mathcal{J}(G)$, then Corollary 3 implies that $\inf_{i\in I}\tau(P_i) = 0$, where $\tau(P_i) = |\lambda_i|^{\frac{-m_i}{m_i-1}}$ (see Section 3.3 page 1391 for the definition of $\tau(P_i)$). Given that $|\lambda_i| \geq 1$ and $\frac{m_i}{m_i-1} \leq 2$, we deduce that $\tau(P_i) \geq |\lambda_i|^{-2}$. It follows that $0 = \inf_{i\in I}\tau(P_i) \geq (\sup_{i\in I}|\lambda_i|)^{-2}$, and this implies that $\sup_{i\in I}|\lambda_i| = \infty$. ■

Corollary 5. *Let $G = \langle Q_i \ / \ i \in I\rangle$ where $Q_i(z) = (\lambda_i z)^{-m_i}$, $\lambda_i \in \mathbb{C}_p - \{0\}$ and $m_i \in \mathbb{N}-\{0,1\}$ for any i in I. Then $\mathcal{J}(G) = \{0,\infty\}$ if and only if $\sup_{i\in I}|\lambda_i| = \infty$ or $\inf_{i\in I}|\lambda_i| = 0$.*

Proof. We verify as in the preceding corollary that if we have $m_i \in \mathbb{N}-\{0\}$ for any i in I and $\sup_{i\in I}|\lambda_i| = \infty$, then $0 \in \mathcal{J}(G)$. Given that $\infty \in Q^{-1}(\{0\}) \subset Q^{-1}(\mathcal{J}(G)) \subset \mathcal{J}(G)$, Lemma 11 implies that $\mathcal{J}(G) = \{0,\infty\}$. In the same way we verify that $\inf_{i\in i}|\lambda_i| = 0$ implies that $\mathcal{J}(G) = \{0,\infty\}$.

Conversely, if we have $\mathcal{J}(G) = \{0,\infty\}$, then Lemma 12 implies that either $\inf_{i\in I}|\lambda_i| = 0$ or $\sup_{i\in I}|\lambda_i| = \infty$. ■

Remark 4. If R is a rational function of the form az^m (where $a \in \mathbb{C}_p-\{0\}, m \in \mathbb{Z}-\{0,1,-1\}$), then R is conjugate (by an appropriate rational function of the form δz where $\delta \in \mathbb{C}_p-\{0\}$) to the rational function z^m. Given that the rational function z^m has good reduction, it follows that R has an empty Julia set. Hence, we deduce from Corollaries 4 and 5 that even if all the elements of a semigroup G have empty Julia sets, the Julia set of G can be nonempty. In particular, we don't have $\mathcal{J}(G) = \overline{\cup_{g\in G}\mathcal{J}(g)}$ as in the complex case (see [6, p. 365]).

Remark 5. If G contains a degree one element, we can have $|\mathcal{J}(G)| = 1$ with G finitely generated. Take for example, $f(z) = pz$, $g(z) = z^2$, $G = \langle f,g\rangle$, then we have $\{\infty\} = \mathcal{J}(f) \subset \mathcal{J}(G)$ and Lemma 11 implies that $\mathcal{J}(G) \subset \{0,\infty\}$. Given that $f(B^+(0,1)) \cup g(B^+(0,1)) \subset B^+(0,1)$, Hsia's Criterion implies that $0 \in \mathcal{F}(G)$. It follows that $\mathcal{J}(G) = \{\infty\}$.

If $|\mathcal{J}(G)| = 1$, are all the elements of G of the form az^m (where $a \in \mathbb{C}_p - \{0\}$ and $m \in \mathbb{Z} - \{0\}$) ? The following example shows that the answer to this question is no.

Example 2. Let G be a semigroup generated by a set $\mathcal{E}$ of polynomials such that every element T of $\mathcal{E}$ satisfies the following condition: All the roots of T have the same p-adic absolute value $\rho(T) > 1$ and $|T(0)| = 1$. In addition, we assume that $\sup_{T\in\mathcal{E}} \rho(T) = \infty$, then it easy to prove that $\mathcal{J}(G) = \{\infty\}$.

6. The Julia Set of the Composition of Two Rational Functions

In this Section we are interested in the following question: " if two rational functions have empty Julia sets, does their composition have an empty Julia set ? " We know that if the two rational functions have both good reduction, then the answer is yes (see [2, p. 4]). Here we give a case where the answer to the previous question is no. In particular, if all the generators of a semigroup G have empty Julia sets, this does not imply that $\mathcal{J}(G)$ is empty (it can even be infinite as we will see). We will need the following lemma whose proof is easy.

Lemma 13. *If $P(x) = u_s x^s + \cdots + u_0$ is a non-constant polynomial with coefficients in $\mathbb{C}_p$ such that $\max(|u_s|, |u_{s-1}|, \ldots, |u_0|) = |u_s| = 1$, then:*

(1) *P has good reduction.*
(2) *$|P(x)| > 1$ if and only if $|x| > 1$.*
(3) *For $|x| > 1$ we have $|P(x)| = |x|^s$.*

Corollary 6. *Let $f, g \in \mathbb{C}_p[x]$. Assume that:*

(1) *$f(x) = a_d x^d + \cdots + a_0$ where*

$$\max(|a_d|, |a_{d-1}|, \ldots, |a_0|) = |a_d| = |a_0| = 1.$$

(2) *The degree d of the polynomial f is 1 or prime to p.*
(3) *All the roots of f' are in the disk $B^-(0, 1)$.*
(4) *The degree m of the polynomial g is prime to p and greater than 2.*
(5) *$g(0) = 0$.*
(6) *There exists a constant c in $\mathbb{C}_p$ such that $0 \neq |c| < 1$ and that for $h(x) = c^{-1}g(cx) = b_m x^m + \cdots + b_1 x$, we have*

$$\max(|b_m|, |b_{m-1}|, \ldots, |b_1|) = |b_m| = 1.$$

Then every fixed point (in $\mathbb{C}_p$) of the polynomial gof is repelling with p-adic absolute value 1, the set $\mathcal{J}(gof)$ is infinite and the sets $\mathcal{J}(f)$ and $\mathcal{J}(g)$ are empty.

Proof. First, we note that according to the hypotheses (of the corollary), the preceding lemma applies to f, h and h'.

Now we show that all the fixed points (in $\mathbb{C}_p$) of the polynomial gof have p-adic absolute value 1.

If $g(f(x)) = x$, we have $h\left(\frac{f(x)}{c}\right) = \frac{x}{c}$ and we prove that $|x| > |c|$. Indeed, if $\left|\frac{x}{c}\right| \leq 1$, the hypotheses on h imply that $|f(x)| \leq |c| < 1$; but $|x| \leq |c| < 1$ and $|f(0)| = 1$ imply that $|f(x)| = |f(0)| = 1$. Thus we have a contradiction.

Then the hypotheses on h imply that $\left|\frac{f(x)}{c}\right| > 1$ and that $h\left(\frac{f(x)}{c}\right)$ has p-adic absolute value $\left|\frac{f(x)}{c}\right|^m$. Hence, we have $|f(x)|^m = |c|^{m-1}|x|$. If $|x| < 1$, again we have the contradiction $|f(0)| = |f(x)| < 1$. If $|x| > 1$, then the hypotheses on f imply that $|f(x)| = |x|^d$. Hence, $|x|^{dm-1} = |c|^{m-1} < 1$, and again we have a contradiction. We conclude that $|x| = 1$.

It remains to prove that every fixed point (in $\mathbb{C}_p$) of the polynomial gof is repelling. First we note that $|(g \circ f)'(x)| = |f'(x)| \left|h'\left(\frac{f(x)}{c}\right)\right|$ and that $|f'(x)| = |da_d x^{d-1}| = 1$ and $\left|\frac{f(x)}{c}\right|^m = \left|\frac{x}{c}\right|$. Hence, $\left|\frac{f(x)}{c}\right| > 1$ and the hypotheses on $h(x)$ and m imply that

$$\left|h'\left(\frac{f(x)}{c}\right)\right| = \left|\frac{f(x)}{c}\right|^{m-1} > 1.$$

It follows that x is a repelling fixed point of $g \circ f$ and that $\mathcal{J}(g \circ f)$ is infinite (because $g \circ f$ is of degree at least two).

In addition, $\mathcal{J}(g) = \mathcal{J}(f) = \emptyset$, because the polynomials $f(x)$ and $h(x)$ both have good reduction (according to Lemma 13) and g is conjugate to h. ∎

Remark 6. If we set $G = \langle f, g \rangle$, then $\mathcal{J}(G)$ is infinite although the generators f and g of G both have an empty Julia set.

7. Two Topological Properties of $\mathcal{J}(G)$

In this Section, we are interested in the following two questions: When is a semigroup G of rational functions with coefficients in $\mathbb{C}_p$ perfect? When does a semigroup G of rational functions with coefficients in $\mathbb{C}_p$ have an empty interior (in the metric space $\mathbb{P}^1(\mathbb{C}_p)$)? We begin by proving some preliminary results.

Lemma 14. *Let G be a semigroup of rational functions with coefficients in $\mathbb{C}_p$ and F a subset of $\mathcal{J}(G)$. If $|F| \geq 2$, then we have $\mathcal{J}(G) = \overline{\bigcup_{g \in G} g^{-1}(F)}$.*

Proof. We set $\mathcal{B} = \overline{\bigcup_{g\in G} g^{-1}(F)}$. Then we have $g^{-1}(F) \subset g^{-1}(\mathcal{J}(G)) \subset \mathcal{J}(G)$ for any g in G. It follows that $\mathcal{B} \subset \mathcal{J}(G)$ (because $\mathcal{J}(G)$ is closed). In addition, we know that a rational function is an open map (according to Lemma 2); hence for any element h of G, we have $h^{-1}(\mathcal{B}) = h^{-1}(\overline{\bigcup_{g\in G} g^{-1}(F)}) \subset \overline{h^{-1}(\bigcup_{g\in G} g^{-1}(F))} \subset \overline{\bigcup_{g\in G} h^{-1}(g^{-1}(F))} \subset \mathcal{B}$. Then, Corollary 1 implies that $\mathcal{J}(G) \subset \mathcal{B}$, because $\mathcal{B}$ is closed and contains at least two elements. ∎

Corollary 7. *If a is an element of $\mathcal{J}(G)$ whose backward orbit $O^-_G(a)$ contains at least two points, then we have $\mathcal{J}(G) = \overline{O^-_G(a)}$.*

Proof. If we apply Lemma 14 with $F = O^-_G(a)$, we obtain that $\mathcal{J}(G) = \overline{\bigcup_{g\in G} g^{-1}(O^-_G(a))} = \overline{O^-_G(a)}$. ∎

7.1. Case where $\mathcal{J}(G)$ is perfect

We have seen that if g is a rational function of degree one (with coefficients in $\mathbb{C}_p$), then $\mathcal{J}(g)$ is either empty or has only one element. For a rational function g (with coefficients in $\mathbb{C}_p$) of degree at least two, we know (see [7, p. 694]) that $\mathcal{J}(g)$ is either empty or perfect (in particular $\mathcal{J}(g)$ is uncountable if it is not empty). The following lemma describes the structure of $\mathcal{J}(G)$ and G when $\mathcal{J}(G)$ is not perfect.

Lemma 15. *Let $G = \langle R_1, \ldots, R_n\rangle$ where $R_1, \ldots, R_n \in \mathbb{C}_p(z) - \mathbb{C}_p$. We suppose that $\mathcal{J}(G)$ has an isolated point a, then we have two cases: 1)*

Either all the points of $\mathcal{J}(G)$ are isolated points and $\mathcal{J}(G) \subset O^+_G(a)$ and $\mathcal{J}(G)$ is at most countable. 2)

Or there exists only one non-isolated point b in $\mathcal{J}(G)$ and $\mathcal{J}(G) \subset O^+_G(a) \cup \{b\}$. In the second case, b is a repelling fixed point of a degree one element of G, the backward orbit of b by G is $\{b\}$, G is conjugate to a polynomial semigroup and $\mathcal{J}(G)$ is countable. In both cases every element of G whose degree is at least two has an empty Julia set.

Proof. Given that a is isolated, there exists a ($\mathbb{P}^1(\mathbb{C}_p)$) disk B containing a such that $B \cap \mathcal{J}(G) = \{a\}$. We set $U = \bigcup_{g\in G} g(B)$, then U is clearly forward invariant by the elements of G. Therefore $|U^c| \le 1$ (if not, Lemma 3 implies that $U \subset \mathcal{F}(G)$ and this contradicts $a \in \mathcal{J}(G)$).

In addition, if $|U^c| = 1$, then there exists an element b of $\mathbb{P}^1(\mathbb{C}_p)$ such that $U^c = \{b\}$; hence for any element g of G we have $g^{-1}(\{b\}) = \{b\}$ (because $g(U) \subset U$). It follows that for any element g of G, the backward orbit of b by g is $\{b\}$, b is an exceptional point of g, b is a fixed point of g.

Now we set

$$\mathcal{E} = \mathcal{J}(G) \text{ if } U^c = \emptyset \text{ and } \mathcal{E} = \mathcal{J}(G) - \{b\} \text{ if } U^c = \{b\}.$$

Then we have $\mathcal{E} \subset U = \left[\bigcup_{g\in G} g(B - \{a\})\right] \bigcup O_G^+(a)$. Given that $B - \{a\} \subset \mathcal{F}(G)$ and $\mathcal{F}(G)$ is forward invariant by the elements of G and that $\mathcal{E} \subset \mathcal{J}(G)$, we deduce that we have $\mathcal{E} \subset O_G^+(a)$.

Now we prove that every element of $\mathcal{E}$ is isolated. Indeed, if a' is a point of $\mathcal{E}$, then there exists an element g of G such that $a' = g(a)$. It follows that the set $B' = g(B)$ is an open set containing a'. If β is an element of $B' \cap \mathcal{E}$, then there exists $\alpha \in B$ such that $g(\alpha) = \beta$. Given that $\mathcal{J}(G)$ is backward invariant by the elements of G, we deduce that $\alpha \in \mathcal{J}(G) \cap B = \{a\}$. It follows that $\beta = g(a) = a'$, and that a' is isolated in $\mathcal{E}$.

Taking into account that G is generated by a finite family of rational functions, the inclusion $\mathcal{E} \subset O_G^+(a)$ implies that $\mathcal{J}(G)$ is at most countable.

Furthermore if R is an element of G of degree at least two, then $\mathcal{J}(R) = \emptyset$ (because we have $\mathcal{J}(R) \subset \mathcal{J}(G)$ and we know that if $\mathcal{J}(R)$ is not empty then $\mathcal{J}(R)$ is infinite and perfect (see [7, p. 694])).

Now we consider two cases:

Case 1. $R_1, \ldots, R_n$, are all of degree at least two. If $|U^c| = 1$, then b is an exceptional point of R_i (for $i = 1, \ldots, n$). Hence, $b \in \mathcal{F}(R_i)$ for $i = 1, \ldots, n$ (see, for example, [5, p. 43]). It follows that b is a common non-repelling fixed point of $R_1, \ldots, R_n$. Hence, Remark 3 implies that $b \in \mathcal{F}(G)$. We deduce that for $|U^c| \leq 1$, we have $\mathcal{J}(G) = \mathcal{E} \subset O_G^+(a)$.

Case 2. for $m \leq n$ the rational functions $R_1, \ldots, R_m$ are all of degree one.

If $|U^c| = 0$ or $|U^c| = 1$ where b is a common non-repelling fixed point of $R_1, \ldots, R_m$, we see as in the first case that $\mathcal{J}(G) = \mathcal{E} \subset O_G^+(a)$.

If $|U^c| = 1$ and b is a repelling fixed point of (for example) R_1, then we have $\mathcal{J}(G) - \{b\} = \mathcal{E} \subset O_G^+(a)$. Hence, $\mathcal{J}(G) \subset O_G^+(a) \cup \{b\}$ and b is a non-isolated point in this case. Indeed, by conjugating G (if necessary) by an appropriate linear fractional map, we can suppose that $b = 0$ and that $R_1(z) = \lambda z$ where $|\lambda| > 1$. It follows that for any n in $\mathbb{N} - \{0\}$, we have $(R_1^{[n]})^{-1}(a) = \dfrac{a}{\lambda^n} \in \mathcal{J}(G)$ (because $\mathcal{J}(G)$ is backward invariant by the elements of G). Given that the sequence $(R_1^{[n]})^{-1}(a)$ tends towards $b = 0$, we deduce that b is not isolated (because the elements of the sequence $(R_1^{[n]})^{-1}(a)$ are all different) and in particular that $\mathcal{J}(G)$ is countable.

Furthermore, by conjugating G (if necessary) by an appropriate linear fractional map, we can suppose that $b = \infty$. Hence, the backward orbit of ∞ is $\{\infty\}$ and we have $R_i^{-1}(\{\infty\}) = \{\infty\}$ for $i = 1, \dots, n$. It follows that the rational functions $R_1, R_2, \dots, R_n$ are polynomials. ■

Remark 7. It follows from Lemma 15 that when G is finitely generated and $\mathcal{J}(G)$ is infinite, then $\mathcal{J}(G)$ is countable if and only if $\mathcal{J}(G)$ is not perfect. However, we have not found a semigroup G where $\mathcal{J}(G)$ is countable.

A straightforward consequence of Lemma 15 is:

Corollary 8. *Let $G = \langle R_1, \dots, R_n \rangle$ where $R_1, \dots, R_n \in \mathbb{C}_p(z) - \mathbb{C}_p$. We suppose that R_1 is of degree at least two and has a nonempty Julia set. Then $\mathcal{J}(G)$ is perfect.*

If G is generated by degree one rational functions, $\mathcal{J}(G)$ can be perfect as in the following example:

Example 3. Let $f(z) = \frac{z}{p}$, $g(z) = z + 1$ and $G = \langle f, g \rangle$. Then we have $0 \in \mathcal{J}(f) \subset \mathcal{J}(G)$ and $-n = g^{[-n]}(0) \in \mathcal{J}(G)$ for every positive integer n. Given that $f^{-1}(-\mathbb{N}) \subset -\mathbb{N}$ and $g^{-1}(-\mathbb{N}) \subset -\mathbb{N}$, we deduce that $f^{-1}(\overline{-\mathbb{N}}) \subset \overline{-\mathbb{N}}$ and $g^{-1}(\overline{-\mathbb{N}}) \subset \overline{-\mathbb{N}}$. Then Corollary 1 and the inclusion $\mathbb{Z}_p = \overline{-\mathbb{N}} \subset \mathcal{J}(G)$ imply that $\mathcal{J}(G) = \mathbb{Z}_p$.

When G is generated by a finite number of rational functions of degree at least two with coefficients in a finite extension of $\mathbb{Q}_p$, Lemma 15 leads to Theorem 2 which is a more precise result.

Proof. [Proof of Theorem 2] If $\mathcal{J}(G) = \mathbb{P}^1(\mathbb{C}_p)$, then $\mathcal{J}(G)$ is perfect.

If $\mathcal{J}(G) \neq \mathbb{P}^1(\mathbb{C}_p)$, then by conjugating G (if necessary) by an appropriate linear fractional map, we can suppose that $\infty \notin \mathcal{J}(G)$. Furthermore, if $\mathcal{J}(G)$ is not empty, Theorem 1 implies that $\mathcal{J}(G)$ is infinite.

Here, we prove that if $\mathcal{J}(G)$ is infinite and has an isolated point a, then a is algebraic over $\mathbb{Q}_p$. Indeed, Lemma 15 implies that $\mathcal{J}(G) \subset O_G^+(a)$. Therefore, if α is another element of $\mathcal{J}(G)$, there exists an element g_1 of G such that $\alpha = g_1(a)$. Given that α is also an isolated point of $\mathcal{J}(G)$, we have $a \in \mathcal{J}(G) \subset O_G^+(\alpha)$. Hence, there exists an element g_2 of G such that $a = g_2(\alpha)$. It follows that if we set $g = g_2 \circ g_1$ and $g(z) = \frac{P(z)}{Q(z)}$ where P and Q are relatively prime polynomials with coefficients in K and $T(z) = zQ(z) - P(z)$, then T is a non-zero polynomial such that $T(a) = 0$.

Now, we show that if $\mathcal{J}(G)$ is infinite and has an isolated point a, then we have a contradiction. Indeed, $\infty \in \mathcal{F}(G)$ implies that there exists a positive real number r such that $\mathcal{J}(G) \subset B^+(a, r)$. In addition according to Lemma 15, all the elements of $\mathcal{J}(G)$ are isolated and we have $\mathcal{J}(G) \subset O_G^+(a) \subset K(a)$. It follows that the compact set $B^+(a,r) \cap K(a)$ contains a convergent sequence of elements of $\mathcal{J}(G)$. Hence, $\mathcal{J}(G)$ has a non-isolated point. This contradiction completes the proof. ■

7.2. Case where $\mathcal{J}(G)$ has an empty interior

We know that if R is a rational function with coefficients in $\mathbb{C}_p$, then $\mathcal{J}(R)$ has an empty interior (see [7, p. 692]). Here we consider the case of the Julia set of a semigroup of rational functions with coefficients in $\mathbb{C}_p$. In particular, we prove that if G is generated by a finite number of rational functions with coefficients in a finite extension of $\mathbb{Q}_p$, then $\mathcal{J}_p(G)$ has an empty interior. We need two preliminary lemmas.

Lemma 16. *Let G be a semigroup of rational functions with coefficients in $\mathbb{C}_p$. We assume that:*

(1) *There exists an element R of G of degree at least two such that $\mathcal{J}(R)$ is not empty.*

(2) *$\mathcal{J}(G)$ is contained in a closed set F (of $\mathbb{P}^1(\mathbb{C}_p)$) which is completely invariant by the elements of G and that $F \neq \mathbb{P}^1(\mathbb{C}_p)$. Then $\mathcal{J}(G)$ has an empty interior.*

Proof. The proof is similar to the proof of [12, Lemma 2]. Given that $\mathcal{J}(R)$ is not empty, Lemma 14 implies that we have

$$\mathcal{J}(G) = \overline{\bigcup_{h\in G} h^{-1}(\mathcal{J}(R))}.$$

Suppose that $\mathcal{J}(G)$ has a nonempty interior, then there exists a ($\mathbb{P}^1(\mathbb{C}_p)$) disk D contained in $\mathcal{J}(G)$. Hence, there exists an element h of G such that $D \cap h^{-1}(\mathcal{J}(R)) \neq \emptyset$. Then we use the following remark: If f is a rational function and B, C are two subsets of $\mathbb{P}^1(\mathbb{C}_p)$, then we have $B \cap f(C) \neq \emptyset$ if and only if $f^{-1}(B) \cap C \neq \emptyset$. In our case we deduce that $h(D) \cap \mathcal{J}(R) \neq \emptyset$.

Now we set $V = h(D)$ and $W = \bigcup_{n\in\mathbb{N}} R^{[n]}(V)$, and we prove that the complement (in $\mathbb{P}^1(\mathbb{C}_p)$) of W has at most one point. Indeed, since W is forward invariant by R, if $|W^c| \geq 2$, Hsia's Criterion would imply that $W \subset \mathcal{F}(R)$, and this contradicts $V \cap \mathcal{J}(R) \neq \emptyset$. It follows that W is either $\mathbb{P}^1(\mathbb{C}_p)$ or the complement (in $\mathbb{P}^1(\mathbb{C}_p)$) of a point. Hence, we have $\overline{W} = \mathbb{P}^1(\mathbb{C}_p)$.

Therefore $F = \mathbb{P}^1(\mathbb{C}_p)$ (because F is a closed set containing W), contradicting the hypothesis. ■

In the sequel, we denote by $\overline{\mathbb{Q}_p}$ the set of elements of $\mathbb{C}_p$ which are algebraic over $\mathbb{Q}_p$.

Lemma 17. *Let K be a finite extension of $\mathbb{Q}_p$ and $G = \langle R_i \ / \ i \in I \rangle$ where $R_i \in K(z) - K$ for any i in I. We assume that:*

(1) *The degrees of the rational functions R_i $(i \in I)$ are bounded by a constant.*

(2) *The set $\mathcal{J}(G) \cap \overline{\mathbb{Q}_p}$ is infinite. Then, there exists a closed set F containing $\mathcal{J}(G)$ and completely invariant by the elements of G, and there exists a positive integer D such that: for any element θ of $\mathbb{C}_p$ with absolute value equal to $p^{-\frac{1}{\rho}}$ where $\rho \in \mathbb{N} - \{0, 1\}$ and ρ prime to D, we have $B^-(\theta, |\theta|) \subset \mathbb{P}^1(\mathbb{C}_p) - F$ (in particular $\mathbb{P}^1(\mathbb{C}_p) - F$ and $\mathcal{F}(G)$ are not empty).*

Proof. First, we prove that if $\mathcal{J}(G) \cap \overline{\mathbb{Q}_p}$ is infinite, then there exists an element of $\mathcal{J}(G) \cap \overline{\mathbb{Q}_p}$ whose backward orbit by G contains at least two elements. We consider two cases:

(i) If G contains a rational function R of degree at least two then $\mathcal{J}(G) \cap \overline{Q_p}$ contains a non-exceptional point of R, since R has at most two exceptional points (see [5, p. 43]).

(ii) If G contains a degree one rational function $h(z) \neq z$, then h^{-1} has at most two fixed points. Hence, $\mathcal{J}(G) \cap \overline{Q_p}$ contains a non-fixed point a of h^{-1}. Given that there exists an element a of $\mathcal{J}(G)$ which is algebraic over Q_p and whose backward orbit by G contains at least two points, Corollary 7 implies that $\mathcal{J}(G) = \overline{O_G^-(a)}$.

We denote by M the field $K(a)$, d the degree of M over $\mathbb{Q}_p$, d_i the degree of R_i (for any i in I) and δ the maximum of the degrees of the rational functions R_i (for i in I) and $D = d(\delta!)$. Let $\mathcal{N}$ be the set of elements of $\mathbb{N} - \{0\}$ whose prime divisors are divisors of D (in particular $1 \in \mathcal{N}$). We denote by L the set containing ∞ and all the elements of $\overline{\mathbb{Q}_p}$ whose degree over M is in $\mathcal{N}$, and we set $F = \overline{L}$ (closure of L in the metric space $\mathbb{P}^1(\mathbb{C}_p)$). Then we note that the product of two elements of $\mathcal{N}$ is an element of $\mathcal{N}$, and that every positive integer dividing an element of $\mathcal{N}$ is also an element of $\mathcal{N}$.

Now we prove that L is completely invariant by the elements of G. For any i in I we set $R_i(z) = \frac{P_i(z)}{Q_i(z)}$ where $P_i(z), Q_i(z) \in K[z]$ and $P_i(z)$ prime to $Q_i(z)$. If $x \in L$ and $y = R_i(x)$ then: For $x \neq \infty$, either x is a pole of R_i and we have $y = \infty \in L$, or x is not a pole of R_i and we have $y = R_i(x) \in L$ (since the degree of $R_i(x)$ over M divides the degree of x over M). For $x = \infty$, we have in the case $\deg P_i < \deg Q_i$ that $y = 0 \in L$, in the case $\deg P_i > \deg Q_i$ that $y = \infty \in L$, and in the case $\deg P_i = \deg Q_i$ that $y = R_i(x) \in K \subset L$. It follows that $R_i(L) \subset L$.

If $x \in L$, $y \neq \infty$ and $R_i(y) = x$, then: For $x \neq \infty$, y is a root of the polynomial $P_i(z) - xQ_i(z)$ (with coefficients in $M(x)$). Hence, the degree of y over $M(x)$ is in $\mathcal{N}$ (since it is less than or equal to d_i). Therefore the degree of y over M is in

$\mathcal{N}$ (since $[M(y):M]=[M(y):M(x)][M(x),M]$, and the degree of x over M is in $\mathcal{N}$). It follow that $y \in L$. For $x=\infty$, y is a pole of R_i and $y \in L$ (since the degree of y over M is in $\mathcal{N}$). It follows that $R_i^{-1}(L) \subset L$. We deduce that $O_G^-(a) \subset L$ and that F is completely invariant by the elements of the semigroup G. Hence, we have $\mathcal{J}(G)=\overline{O_G^-(a)} \subset F$.

Now we prove that if θ is an element of $\mathbb{C}_p$ whose absolute value is $p^{-\frac{1}{\rho}}$, where $\rho \in \mathbb{N}-\{0,1\}$ and ρ prime to D, then we have

$$B^-(\theta,|\theta|) \subset \mathbb{P}^1(\mathbb{C}_p)-F.$$

First, we show that $|\theta| \notin |\overline{L}|$. Indeed, otherwise there would exist a sequence x_n of elements of $L-\{\infty\}$ such that $\lim_{n\to\infty} x_n = x \in \overline{L}$ and $|x|=|\theta|$. Hence, for n sufficiently large, we have $|x_n|=|x|=|\theta|$ and $|\theta|=p^{-\frac{m}{\lambda}}$ where $\lambda \in \mathcal{N}, m \in \mathbb{Z}$ (since the degree of x_n over $\mathbb{Q}_p$ is in $\mathcal{N}$). It follows that $\rho m = \lambda$. Then $\rho \in \mathcal{N}$ contradicting the hypotheses.

In addition, if r_θ denotes the distance between θ and the closed set $\overline{L}$, then we have $r_\theta \geq |\theta|$ (if not, there exists an element x in $\overline{L}$ such that $|x-\theta|<|\theta|$, and thus we have $|x|=|\theta|$ and $|\theta| \in |\overline{L}|$, contradicting what precedes). We deduce that $B^-(\theta,|\theta|) \subset \mathbb{P}^1(\mathbb{C}_p)-F \subset \mathcal{F}(G)$ as promised.

It follows that if q is a prime number greater than D, and θ is a root of the polynomial z^q-p, we have $|\theta|=p^{-\frac{1}{q}}$, and thus

$$\theta \in \mathbb{P}^1(\mathbb{C}_p)-F \subset \mathcal{F}(G). \qquad \blacksquare$$

Now we are ready to give the proof of Theorem 3.

Proof. [Proof of Theorem 3]. First we prove that if $\mathcal{J}(G)\cap\overline{\mathbb{Q}_p}$ is finite then $\mathcal{J}(G)$ has an empty interior. Indeed, if $\mathcal{J}(G)\cap\overline{\mathbb{Q}_p}=\{a_1,a_2,\dots,a_n\}$ we set $\mathcal{E}=\overline{\mathbb{Q}_p}-\{a_1,a_2,\dots,a_n\}$. Hence, we have $\mathcal{E}\subset\mathcal{F}(G)$ and the closure of $\mathcal{E}$ (in the metric space $\mathbb{P}^1(\mathbb{C}_p)$) is $\mathbb{P}^1(\mathbb{C}_p)$. It follows that $\mathcal{J}(G)$ has an empty interior. When $\mathcal{J}(G)\cap\overline{\mathbb{Q}_p}$ is infinite, we first note that we can assume that G contains an element R of degree at least two whose Julia set $\mathcal{J}(R)$ is not empty (if not we replace G by $\langle G\cup\{R_0\}\rangle$ where $R_0(z)=\dfrac{z^p-z}{p}$ (recall that $\mathcal{J}(R_0)=\mathbb{Z}_p$)). Then, Lemma 17 implies that $\mathbb{P}^1(\mathbb{C}_p)-F$ is not empty and we use Lemma 16 to conclude. $\blacksquare$

8. Comparative Table with the Complex Case

In this Section, we compare the properties of the Julia set of a semigroup of rational functions with coefficients in $\mathbb{C}_p$ with the properties of the Julia set of a semigroup of rational functions with complex coefficients. We denote by $\mathcal{J}_p(R)$ the Julia set of a rational function R of degree at least two with coefficients in

$\mathbb{C}_p$. We denote by $\mathcal{J}_p(G)$ the Julia set of a semigroup $G = \langle R_1, \ldots, R_n\rangle$ where $R_1, \ldots, R_n$ are rational functions with coefficients in $\mathbb{C}_p$, $n \neq 1$ and $\deg R_i \geq 2$ for $i = 1, \ldots, n$. We denote by $\mathcal{J}_c(R)$ the Julia set of a rational function R of degree at least two with complex coefficients. We denote by $\mathcal{J}_c(G)$ the Julia set of a semigroup $G = \langle R_1, \ldots, R_n\rangle$ where $R_1, \ldots, R_n$ are rational functions with complex coefficients, $n \neq 1$ and $\deg R_i \geq 2$ for $i = 1, \ldots, n$. In the table, A means always, P means possible, N means never, O.Q means open question.

	Property	$\mathcal{J}_p(R)$	$\mathcal{J}_p(G)$	$\mathcal{J}_c(R)$	$\mathcal{J}_c(G)$
1	backward invariant	A	A	A	A
2	empty	P	P	N	N
3	finite+nonempty	N	N	N	N
4	perfect	P	P	A	A
5	infinite+ not perfect	N	O.Q	N	N
6	infinite+empty interior	P	P	P	P
7	infinite+nonempty interior	N	O.Q	P	P

Comments

(1) $\mathcal{J}_p(R)$ is completely invariant by R (see, for example, [3, p. 1]). $\mathcal{J}_p(G)$ is backward invariant by the elements of G according to Proposition 2, but it is shown in Example 1 that $\mathcal{J}_p(G)$ is not always forward invariant by the elements of G. In the complex case, $\mathcal{J}_c(R)$ is completely invariant by R (see [1, Theorem 3.2.4]). $\mathcal{J}_c(G)$ is backward invariant by the elements of G according to [6, Lemma 2.1], but it is shown in [6, Example 1] that $\mathcal{J}_c(G)$ is not always forward invariant by the elements of G (if $|a| > 1$, and $G = \langle z^2, a^{-1}z^2\rangle$, it is proven that $\mathcal{J}_c(G) = \{z \in \mathbb{C},\ /\ 1 < |z| \leq |a|\}$).

(2) If R has good reduction then $\mathcal{J}_p(R)$ is empty (see [8, p. 105]). If G is a semigroup generated by rational functions (with coefficients in $\mathbb{C}_p$) which all have good reduction, then $\mathcal{J}_p(G) = \emptyset$ according to Corollary 2. In the complex case, $\mathcal{J}_c(R)$ is infinite according to [1, Theorem 4.2.1], hence $\mathcal{J}_c(G)$ is also infinite.

(3) We know that $\mathcal{J}_p(R)$ is empty or infinite (see [7, Theorem 2.9]), and according to Theorem 1 $\mathcal{J}_p(G)$ is empty or infinite. In the complex case, $\mathcal{J}_c(R)$ and $\mathcal{J}_c(G)$ are infinite from what precedes.

(4) and (5) We know that if $\mathcal{J}_p(R)$ is not empty, then $\mathcal{J}_p(R)$ is perfect, and thus $\mathcal{J}_p(R)$ is uncountable (see [7, p. 694]). In the complex case, we know that $\mathcal{J}_c(R)$ is perfect (see [1, Theorem 4.2.4]), and thus $\mathcal{J}_c(R)$ is uncountable. Under the hypotheses of Theorem 2 $\mathcal{J}_p(G)$ is perfect or empty. $\mathcal{J}_c(G)$ is perfect according to [6, Lemma 3.1], and thus $\mathcal{J}_c(G)$ is uncountable.

(6) We know that if $\mathcal{J}_p(R)$ is not empty, then $\mathcal{J}_p(R)$ has an empty interior (see, for example, [7, p. 692] and [5, p. 52]). According to Theorem 3, if G is generated by rational functions with coefficients in a finite extension of $\mathbb{Q}_p$, then $\mathcal{J}_p(G)$ has an empty interior. In the complex case, it is shown in [1, Theorem 1.4.1] that if we define the polynomials T_n (where $n \in \mathbb{N}$) by the relations $T_0(z) = 1,\ T_1(z) = z$, and $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$, then $\mathcal{J}_c(T_n) = \mathcal{J}_c(-T_n) = [-1, 1]$ for $n \geq 2$, and it is shown in [6, Example 2] that if $G = \langle T_n,\ n \in \mathbb{N} - \{0, 1\}\rangle$, then $\mathcal{J}_c(G) = [-1, 1]$.

(7) In the complex case, it is shown in [1, p. 271] that if $R(z) = (1 - 2z^{-1})^2$, then $\mathcal{J}_c(R) = \mathbb{P}^1(\mathbb{C})$. It is shown in [6, Example 3] that if $G = \langle z^2, 2z^2 - 1\rangle$, then $\mathcal{J}_c(G) = \{z \in \mathbb{C} \ / \ |z| \leq 1\}$.

Acknowledgments

The author would like to thank Jean-Paul. Bézivin for his guidance and his numerous suggestions and corrections, and the referee for his/her suggestions on the organization and the presentation of this paper.

References

1. A. Beardon, *Iteration of rational functions*, Springer-Verlag, New York, 1991.
2. R. Benedetto, Reduction, dynamics, and Julia sets of rational functions, *J. Number Theory*, **86** (2001), 175-195.
3. R. Benedetto, Hyperbolic maps in p-adic dynamics, *Ergodic Theory Dynam systems*, **21** (2001), 1-11.
4. R. Benedetto, Components and periodic points in non-archimedean dynamics, *Proc. London. Math. Soc.*, **84** (2002), 231-256.
5. J. P. Bézivin, *Dynamique des fractions rationnelles*, Cours de DEA, Université de Caen 2003, http://www.math.unicaen.fr/ bezivin/.
6. A. Hinkkanen and G. J. Martin, The dynamics of semigroups of rational functions 1, *Proc. London. Math. Soc.*, **73(3)** (1996), 358-384.
7. L. C. Hsia, Closure of periodic points over a non-archimedean field, *J. London. Math. Soc.*, **62(2)** (2000), 685-700.
8. P. Morton and J. Silverman, Periodic points, multiplicities and dynamical units, *J. Reine Angew. Math.*, **461** (1995), 81-122.
9. J. Rivera-Letelier, *Dynamique des fractions rationnelles sur des corps locaux*, Thèse de Doctorat, Université d'Orsay 2003.
10. A. Robert, *A Course in p-adic analysis*, Springer-Verlag, New York, 1986.

11. N. Smart and C. Woodcock, p-adic chaos and random number generation, *Experiment. Math.*, **7** (1998), 333-342.

12. R. Stankewitz, Completely invariant Julia sets of polynomial semigroups, *Proc. Amer. Math. Soc.*, **127(10)** (1999), 2889-2898.

Abdelaziz Bellagh
Faculté de Mathématiques,
Université des Sciences et de la,
Technologie Houari Boumédienne,
Bp. 32, El ALIA, Bab Ezzouar,
16111, Alger, Algérie
E-mail: abellagh@yahoo.fr