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ON THE DISCOUNTED PENALTY AT RUIN IN A JUMP-DIFFUSION MODEL

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Abstract. In this paper, we study the discounted penalty in a perturbed compound Poisson model with two sided jumps. We prove second-order regularity of this function and investigate its asymptotic behavior at infinity. Next, based on Boyarchenko and Levendorskii (2002), we justify an integro-differential equation for the discounted penalty.

1. INTRODUCTION

Consider a family of real-valued processes $X = (X_t, \mathbb{P}_x)$ and let τ be the first exit time of X from the interval $(0, \infty)$(i.e., $\tau = \inf\{t \geq 0, X_t \in \mathbb{R}_-\}$.) For every bounded nonnegative Borel function $g : \mathbb{R}_- \to \mathbb{R}_+$, we consider the discounted penalty function

$$\Phi(x) = \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right]. \tag{1.1}$$

Here $r \geq 0$ and $\mathbb{E}_x[Z] = \int Z(\omega) d\mathbb{P}_x(\omega)$ for a random variable Z. (We follow the convention that $e^{-r\cdot\infty} = 0$ and write $\mathbb{P}$ for $\mathbb{P}_0$.) In the insurance literature, if X_t stands for the surplus process of an insurance company, we see that the ruin probability is a special case of (1.1) by taking $g \equiv 1$ and $r = 0$.

Gerber and Landry (1998) considered the discounted penalty in a general perturbed compound Poisson model with no positive jumps. The basic assumption on the discounted penalty which leads to a renewal equation as a tool to investigate its analytic properties is that it enjoys sufficient second-order regularity. Although the regularity of the discounted penalty was not studied in Gerber and Landry (1998), it was studied in subsequent literatures. For example, Cai and Yang (2005) showed

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that in a jump-diffusion model, the ruin probability under stochastic interest rate indeed satisfies second-order continuous differentiability. They gave a rigorous proof of the integro-differential equation for ruin probability. However, the result of Cai and Yang (2005) did not cover the case of general discounted penalty functions. Hence, the theoretical question remains unanswered whether the regularity, not just continuous differentiability, for discounted penalty assumed in Gerber and Landry (1998) and others does hold. (We refer to Wang and Wu (2000) and Cai (2004) for related works).

In this paper, in a perturbed compound Poisson model with two sided jumps, we study the regularity of the discounted penalty and investigate its asymptotic behavior at infinity. Then we justify an integro-differential equation for the discounted penalty. As a demonstration for possible applications of our results, we calculate the discounted penalty when the jump distribution is exponential (see Theorem C below.) For other applications in financial mathematics, see Mordecki (2002), Hilberink and Rogers (2002), Asmussen et al. (2004), Chen et al. (2007), and many others.

Throughout this paper, we assume that on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there are a standard Brownian motion $W = (W_t; t \in \mathbb{R}_+)$ and a compound Poisson process $Z = (Z_t; t \in \mathbb{R}_+)$ with $Z_t = \sum_{n=1}^{N_t} Y_n$. Here $N = (N_t; t \in \mathbb{R}_+)$ is a Poisson process with parameter $\lambda > 0$ and $Y = (Y_n; n \in \mathbb{N})$ are independent and identically distributed with distribution F. We assume that W, N and Y are independent.

For every $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the law of the process

$$X_t = X_0 + ct + \sigma W_t - Z_t, \qquad t \geq 0, \tag{1.2}$$

where $c \in \mathbb{R}$, $\sigma > 0$ and $X_0 = x$. Then X is a Lévy process. To study the function Φ for X, we may without loss of generality assume that $\int_{\{0\}} dF = 0$ by Lévy-Khintchine formula.

We write $\Phi \in \mathcal{C}_b^k(\mathbb{R}_+)$ if $\Phi^{(i)}(x), 0 \leq i \leq k$, are continuous and bounded on $\mathbb{R}_+$. (Here, for a function h defined on $\mathbb{R}_+$, $h^{(i)}(0)$ is the right-hand i-th derivative of h at 0.) Also we write $\Phi \in \mathcal{C}_0^k(\mathbb{R}_+)$ if $\Phi \in \mathcal{C}_b^k$ and $\Phi^{(i)}(x), 0 \leq i \leq k$, tend to zero as x goes to infinity.

The main results of the paper are stated in the followings.

Theorem A. *We have $\Phi \in \mathcal{C}_b(\mathbb{R}_+)$. Moreover, if we assume that*

(1.3) *the jump distribution F has a density*,

then we have the following additional regularity of Φ:

(*1*) $\Phi \in \mathcal{C}_b^2(\mathbb{R}_+)$,

(*2*) $\Phi \in \mathcal{C}_0^2(\mathbb{R}_+)$ *whenever* $\lim_{x\to\infty} \Phi(x) = 0$.

Proposition 2.1 and Corollary 2.1 below give conditions under which $\lim_{x\to\infty} \Phi(x) = 0$.

Theorem B. *Assume that the characteristic exponent Ψ of X admits an analytic continuation into a tube domain $\mathbb{R} + i\mathcal{O}$, where $\mathcal{O}$ is an open set in $\mathbb{R}$ containing 0, and that (1.3) is satisfied. Then the discounted penalty Φ satisfies the integro-differential equation*

$$\begin{aligned}(1.4)\qquad &\frac{\sigma^2}{2}\Phi''(x) + c\Phi'(x) + \lambda \int_{-\infty}^{x} \Phi(x-y)dF(y) \\ &+\lambda \int_{x}^{\infty} g(x-y)dF(y) - (\lambda + r)\Phi(x) = 0, \quad x > 0.\end{aligned}$$

Theorem C. *Assume the jump distribution F is exponential with parameter η. Then for every $r > 0$ and every nonnegative function g such that $\int_{-\infty}^{0} g(y)\eta e^{\eta y}dy < \infty$, the discounted penalty Φ is given by the formula*

$$\begin{aligned}(1.5)\qquad \Phi(x) = &\frac{\left[(\lambda+r)g(0) - \lambda\int_{-\infty}^{0} g(y)\eta e^{\eta y}dy\right] - \left(\frac{\sigma^2}{2}\rho_2^2 + c\rho_2\right)g(0)}{\frac{\sigma^2}{2}(\rho_1^2 - \rho_2^2) - c(\rho_2 - \rho_1)} e^{\rho_1 x} \\ &+ \frac{\left[(\lambda+r)g(0) - \lambda\int_{-\infty}^{0} g(y)\eta e^{\eta y}dy\right] - \left(\frac{\sigma^2}{2}\rho_1^2 + c\rho_1\right)g(0)}{\frac{\sigma^2}{2}(\rho_2^2 - \rho_1^2) - c(\rho_1 - \rho_2)} e^{\rho_2 x} \\ &x \in \mathbb{R}_+.\end{aligned}$$

Here $\rho_1 < \rho_2 < 0$ are the two strictly negative zeros of the characteristic polynomial of the operator

$$(1.6)\qquad \mathcal{A} = \frac{\sigma^2}{2}\frac{d^3}{dx^3} + \left(c + \frac{\eta\sigma^2}{2}\right)\frac{d^2}{dx^2} + (c\eta - \lambda - r)\frac{d}{dx} - \eta r.$$

The paper is organized as follows. In Section 2 we use the distribution of Brownian motion stopped at an independent exponential time to give an integral equation satisfied by Φ of simple structure. In Section 3, based on this integral equation, we prove Theorem A. To prove Theorem B, we recall in Section 4 a result of Boyarchenko and Levendorskii (2002) which shows that the discounted penalty for a large class of Lévy processes is a weak solution to an integro-differential equation. Then we prove Theorem B by showing that if the discounted penalty is twice continuously differentiable, it is a strong solution. When the jump distribution is exponential, by Theorem B, we derive an ODE for the discounted penalty. By solving the ODE and using Theorem A, we prove Theorem C in Section 4.

2. Preliminaries

In this and the next sections, to prove the desired regularity of Φ under the general assumptions that F is an arbitrary distribution on $\mathbb{R}$ with $\int_{\{0\}} dF = 0$ and that $c \in \mathbb{R}$, it is plain that we may without loss of generality assume that $\sigma = 1$. A simple change-of-variable argument will lead to the integral equations satisfied by functions Φ with general $\sigma > 0$. Set $X^c_t = X_0 + ct + W_t$ for all $t \in \mathbb{R}_+$, and write J as the first jump time of the process X.

Proposition 2.1. $\lim_{x\to\infty} \Phi(x) = 0$ *whenever any of the following conditions holds:*

(*1*) $r = 0$ *and* $\lim_{t\to\infty} X_t = \infty$ *almost surely,*

(*2*) $r > 0$.

Proof. Write $\Phi(x) = \mathbb{E}\left[e^{-r\tau_x} g(x + X_{\tau_x})\right]$, where $\tau_x = \inf\{t \geq 0; ct + W_t - Z_t \leq -x\}$. First consider the case that $r = 0$ and $\lim_{t\to\infty} X_t = \infty$ almost surely. The assumption that $\lim_{t\to\infty} X_t = \infty$ almost surely implies that almost surely values of X are bounded below and hence $\tau_x = \infty$ for all large x. By Dominated Convergence Theorem, we get that $\lim_{x\to\infty} \Phi(x) = 0$. Next, consider the case that $r > 0$. Note that τ_x is nondecreasing and unbounded. This implies that $\tau_x \uparrow \infty$ as $x \to \infty$ $\mathbb{P}-$a.s. By Dominated Convergence Theorem again, the result follows. ■

Corollary 2.1. *If* $r = 0$ *and* $\mathbb{E}X_1 > 0$*, then* $\lim_{x\to\infty} \Phi(x) = 0$.

Proof. This follows from the fact that by the Law of Large Numbers for Lévy processes(see Sato (1999) pages 246-247) $\lim_{t\to\infty} X_t = \infty$ almost surely. ■

Next we show that Φ satisfies an integral equation. First, we compute some functionals of τ and X_τ.

Lemma 2.1. *For* $x > 0$*, we have* $\mathbb{E}_x\left[e^{-r\tau}; \tau < J\right] = e^{-(\Gamma+c)x}$ *and*

$$\mathbb{E}_x\left[e^{-r\tau} g(X_\tau); \tau \geq J\right] = \int dF(y) \int_0^\infty dw \Phi(w - y)$$
$$\left[\frac{\lambda}{\Gamma} e^{c(w-x)-|w-x|\Gamma} - \frac{\lambda}{\Gamma} e^{c(w-x)-(w+x)\Gamma}\right].$$

Here

$$\Gamma = \sqrt{c^2 + 2(\lambda + r)}, \tag{2.1}$$

Proof. Fix $x > 0$. Write $T_{\alpha,\beta} = \inf\{t \geq 0; \alpha t + W_t = \beta\}$ and let $h(s;\alpha,\beta)$ be the density of $T_{\alpha,\beta}$ under $\mathbb{P}$. (See Borodin and Salminen (2002) page 295 formula 2.0.2.) Clearly we have $\mathbb{E}_x\left[e^{-r\tau};\tau < J\right] = \mathbb{E}\left[e^{-rT_{c,-x}};T_{c,-x} < J\right]$. Hence, by independence of W and J and the fact that J is an exponential random variable with mean $\frac{1}{\lambda}$, we get

$$\begin{aligned}\mathbb{E}_x\left[e^{-r\tau};\tau < J\right] =& \int_0^\infty dt\lambda e^{-\lambda t}\int_0^t dse^{-rs}h(s;c,-x) = \int_0^\infty dse^{-(r+\lambda)s}h(s;c,-x)\\ =& \mathbb{E}_0\left[e^{-(r+\lambda)T_{c,-x}}\right] = \exp\left\{-cx - x\sqrt{c^2+2(\lambda+r)}\right\},\end{aligned}$$

where the last equality follows from Borodin and Salminen (2002) page 295 formula 2.0.1. This proves the first equation.

To prove the second equation, we observe that $\{\tau \geq J\} \supseteq \{\min_{s\leq J} X_s^c > 0\}$. Also,

$$\begin{aligned}\mathbb{P}_x\left[\tau \geq J, \min_{0\leq s\leq J} X_s^c \leq 0\right] =& \mathbb{P}_x\left[\tau \geq J, \min_{0\leq s\leq J} X_s^c = 0\right]\\ =& \mathbb{P}_x\left[\tau \geq J, X_J^c = 0\right] \leq \mathbb{P}_x\left[X_J^c = 0\right] = 0,\end{aligned}$$

by the independence of J and W. These imply $\{\tau \geq J\} = \{\min_{s\leq J} X_s^c > 0\}$, $\mathbb{P}_x-$a.s. From this and Strong Markov Property, we have

$$\begin{aligned}\mathbb{E}_x\left[e^{-r\tau}g(X_\tau);\tau \geq J\right] =& \mathbb{E}_x\left[e^{-rJ}\Phi(X_J);\tau \geq J\right]\\ =& \mathbb{E}_x\left[e^{-rJ}\Phi(X_J^c - Y_1);\min_{s\leq J} X_s^c > 0\right].\end{aligned}$$

By the independence of W, J and Y_1, this gives

$$\begin{aligned}&\mathbb{E}_x\left[e^{-r\tau}g(X_\tau);\tau \geq J\right]\\ &= \frac{\lambda}{\lambda+r}\int dF(y)\int_0^\infty dt(\lambda+r)e^{-(\lambda+r)t}\mathbb{E}_x\left[\Phi(X_t^c - y);\min_{s\leq t} X_s^c > 0\right] \qquad (2.2)\\ &= \frac{\lambda}{\lambda+r}\int dF(y)\mathbb{E}_x\left[\Phi(X_{J'}^c - y);\min_{s\leq J'} X_s^c > 0\right],\end{aligned}$$

where J' is an exponential random variable with mean $\frac{1}{\lambda+r}$ and is independent of X^c.

To complete the proof, we calculate the density of $\frac{\lambda}{\lambda+r}\mathbb{P}_x\left[\min_{s\leq J'} X_s^c > 0, X_{J'}^c \in dw\right]$. First, note that $\mathbb{P}_x\left[\min_{s\leq J'} X_s^c > 0, X_{J'}^c \leq z\right] = 0$ for all $z \leq 0$. Second, observe that $\mathbb{P}_x\left[\min_{s\le J'} X^c_s \le 0, X^c_{J'} \le 0\right] = \mathbb{P}_x\left[X^c_{J'} \le 0\right]$. Hence for all $z > 0$,

$$
\begin{aligned}
&\frac{\lambda}{\lambda+r}\mathbb{P}_x\left[\min_{s\le J'} X^c_s > 0, X^c_{J'} \le z\right]\\
&= \frac{\lambda}{\lambda+r}\left(\mathbb{P}_x\left[X^c_{J'} \le z\right] - \mathbb{P}_x\left[\min_{s\le J'} X^c_s \le 0, X^c_{J'_1} \le z\right]\right)\\
&= \frac{\lambda}{\lambda+r}\left(\mathbb{P}_x\left[0 < X^c_{J'} \le z\right] - \mathbb{P}_x\left[\min_{s\le J'} X^c_s \le 0, 0 < X^c_{J'} \le z\right]\right)\\
&= \int_0^z \left[\frac{\lambda}{\Gamma}e^{c(w-x)-|w-x|\Gamma} - \frac{\lambda}{\Gamma}e^{c(w-x)-(w+x)\Gamma}\right]dw,
\end{aligned}
\tag{2.3}
$$

where the last equation follows from Borodin and Salminen (2002) page 250 formula 1.0.5 and page 252 formula 1.2.5 and Γ is given by (2.1). The last equation then gives us the desired density on $\mathbb{R}_+$ and hence on $\mathbb{R}$. Plugging this density into (2.2) gives the desired equation. ■

Note that $\Phi(x) = g(0)\mathbb{E}_x[e^{-r\tau}\mathbf{1}(\tau < J)] + \mathbb{E}_x\left[e^{-r\tau}g(X_\tau)\mathbf{1}(\tau \ge J)\right]$. By Lemma 2.1, we obtain an integral equation for Φ.

Theorem 2.1. *For $x \ge 0$, Φ satisfies the following integral equation*

$$
\Phi(x) = e^{-(\Gamma+c)x}g(0) + \frac{\lambda}{\Gamma}H(x) - \frac{\lambda}{\Gamma}e^{-(c+\Gamma)x}\int dF(y)\int_0^\infty dw\Phi(w-y)e^{cw-\Gamma w}
\tag{2.4}
$$

where Γ is given by (2.1) and

$$
H(x) = \int dF(y)\int_0^\infty dw\Phi(w-y)e^{c(w-x)-|w-x|\Gamma}.
\tag{2.5}
$$

3. Proof of Theorem A

To prove Theorem A, we need the following lemmas.

Lemma 3.1. *Suppose that* (1.3) *holds. Then* $x \mapsto \int dF(y)\Phi(x-y)$ *is continuous on* $\mathbb{R}$.

Proof. The proof is merely an application of the fact that the convolution of an integrable function and a bounded function is continuous. The proof of the latter fact follows from a slight modification of the proof of Stein and Shakarchi(2005) Proposition 2.2.5, and we omit the details. ■

Lemma 3.2. *Suppose* (1.3) *holds. Then the function* $H(x)$ *defined in* (2.5) *is twice continuously differentiable on* $\mathbb{R}_+ = [0,\infty)$. *Moreover, its first order and second order derivatives satisfy the following integral equations:*

$$\begin{aligned} H'(x) &= -(c+\Gamma)e^{-(c+\Gamma)x}\int_0^x dw \int dF(y)\Phi(w-y)e^{cw+\Gamma w} \\ &\quad -(c-\Gamma)e^{-(c-\Gamma)x}\int_x^\infty dw \int dF(y)\Phi(w-y)e^{cw-\Gamma w} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} H''(x) &= (c+\Gamma)^2 e^{-(c+\Gamma)x}\int_0^x dw \int dF(y)\Phi(w-y)e^{cw+\Gamma w} \\ &\quad +(c-\Gamma)^2 e^{-(c-\Gamma)x}\int_x^\infty dw \int dF(y)\Phi(w-y)e^{cw-\Gamma w} \\ &\quad -2\Gamma \int dF(y)\Phi(x-y). \end{aligned} \tag{3.7}$$

Proof. Using the definition of H and Fubini's Theorem, we write

$$\begin{aligned} H(x) &= e^{-(c+\Gamma)x}\int_0^x dw \int dF(y)\Phi(w-y)e^{cw+\Gamma w} \\ &\quad +e^{-(c-\Gamma)x}\int_x^\infty dw \int dF(y)\Phi(w-y)e^{cw-\Gamma w}. \end{aligned}$$

By Lemma 3.1, $w \mapsto \int dF(y)\Phi(w-y)$ is continuous on $\mathbb{R}$. Hence, by Fundamental Theorem of Calculus, H is differentiable on $(0,\infty)$ and its first order derivative is given by (3.6). Similarly, by (3.6), H' is continuously differentiable on $(0,\infty)$ and its derivative is given by (3.7). Note that $H'(0+)$ and $H''(0+)$ can be calculated directly by definition and we omit the details. ∎

Lemma 3.3. *If* $\lim_{x\to\infty}\Phi(x) = 0$ *and* (1.3) *holds, then*

$$\lim_{x\to\infty} e^{-(c-\Gamma)x}\int_x^\infty dw \int dF(y)\Phi(w-y)e^{cw-\Gamma w} = 0$$

and

$$\lim_{x\to\infty} e^{-(c+\Gamma)x}\int_0^x dw \int dF(y)\Phi(w-y)e^{cw+\Gamma w} = 0.$$

Proof. By the integral equation (2.4) and Lemma 3.2, we deduce that Φ is a continuously differentiable function on $\mathbb{R}_+$. In addition, note that $c\pm\Gamma \gtrless 0$. Hence, by L'Hôspital's rule and Fundamental Theorem of Calculus,

$$\begin{aligned}&\lim_{x\to\infty} e^{-(c-\Gamma)x}\int_x^\infty dw\int dF(y)\Phi(w-y)e^{cw-\Gamma w}\\ &=\lim_{x\to\infty}\frac{-e^{(c-\Gamma)x}\int dF(y)\Phi(x-y)}{(c-\Gamma)e^{(c-\Gamma)x}}=0,\end{aligned}$$

where the last equality follows from the assumption that $\lim_{x\to\infty}\Phi(x)=0$ and Dominated Convergence Theorem. The proof of the second equality follows similarly. ■

Proof of Theorem A. By (2.4) and the boundedness of Φ, we obtain that $\Phi\in\mathcal{C}_b(\mathbb{R}_+)$. Next, assume that (1.3) holds. Clearly we have

$$e^{-(\Gamma+c)x}-\frac{\lambda}{\Gamma}e^{-(c+\Gamma)x}\int dF(y)\int_0^\infty dw\Phi(w-y)e^{cw-\Gamma w}\in\mathcal{C}_0^2(\mathbb{R}_+).$$

By (2.4), to prove (1), it suffices to show that H defined in (2.5) is in $\mathcal{C}_b^2(\mathbb{R}_+)$. Note that $\Gamma-c>0$. Hence, by Lemma 3.2, for all $x\in\mathbb{R}_+$,

$$\begin{aligned}&|H'(x)|\\ &\le(c+\Gamma)\|\Phi\|_\infty e^{-(c+\Gamma)x}\int_0^x dwe^{(c+\Gamma)w}\\ &\quad+(\Gamma-c)\|\Phi\|_\infty e^{-(c-\Gamma)x}\int_x^\infty dwe^{(c-\Gamma)w}\\ &=\|\Phi\|_\infty\left[e^{-(c+\Gamma)x}\left(e^{(c+\Gamma)x}-1\right)+e^{-(c-\Gamma)x}e^{(c-\Gamma)x}\right]\le 3\|\Phi\|_\infty<\infty.\end{aligned}\tag{3.8}$$

This shows that $H(x)\in\mathcal{C}_b^1(\mathbb{R}_+)$. Since $\left|\int dF(y)\Phi(x-y)\right|\le\|\Phi\|_\infty$ for all $x\in\mathbb{R}_+$, we deduce from the above estimate and (3.7) that $H\in\mathcal{C}_b^2(\mathbb{R}_+)$ as well. This gives (1).

Assume further that $\lim_{x\to\infty}\Phi(x)=0$. By Lemma 3.3 and Eq. (3.6), we get that $H\in\mathcal{C}_0^1(\mathbb{R}_+)$. Similarly, by Lemma 3.3 and Equations (3.6) and (3.7), we get that $H\in\mathcal{C}_0^2(\mathbb{R}_+)$ and (2) is proved. The proof is complete. ■

4. Proofs of Theorem B and Theorem C

First we recall a result of Boyarchenko and Levendorskii (2002) in the setting of general Lévy processes. Consider a family $(X_t,\mathbb{P}_x)$ of Lévy processes on $\mathbb{R}^d$. Here under $\mathbb{P}_x$, $X_0=x$ a.s.. Then for every $t\in\mathbb{R}_+$, $\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle z,X_t\rangle}\right]=\mathrm{e}^{-t\Psi(z)}, z\in\mathbb{R}^d$, where the characteristic exponent Ψ of X is of the form:

$$\Psi(z)=\frac{1}{2}\langle z,Az\rangle-\mathrm{i}\langle\gamma,z\rangle+\int_{\mathbb{R}^d}\left(1-\mathrm{e}^{\mathrm{i}\langle z,y\rangle}+\mathrm{i}\langle z,y\rangle\mathbf{1}_{\{y;|y|\le1\}}(y)\right)\nu(\mathrm{d}y).\tag{4.1}$$

Here $\gamma \in \mathbb{R}^d$, A is a symmetric nonnegative definite $d \times d$ matrix, and ν is a Borel measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty$. We assume further that the resolvent kernel are absolutely continuous. (For details, see Bertoin (1996).)

The infinitesimal generator L of X has a domain containing $\mathcal{C}_0^2(\mathbb{R}^d)$ and for any $f \in \mathcal{C}_0^2(\mathbb{R}^d)$,

$$\begin{aligned} Lf(x) &= \frac{1}{2}\sum_{j,k=1}^{d} A_{jk}\frac{\partial^2 f}{\partial x_j \partial x_k}(x) + \sum_{j=1}^{d}\gamma_j \frac{\partial f}{\partial x_j}(x) \\ &\quad + \int_{\mathbb{R}^d}\left(f(x+y) - f(x) - \sum_{j=1}^{d} y_j \frac{\partial f}{\partial x_j}(x)\mathbf{1}_{\{y;|y|\le 1\}}\right)\nu(\mathrm{d}y). \end{aligned} \tag{4.2}$$

Moreover, we write

$$\begin{aligned} \widetilde{L}f(x) &= \frac{1}{2}\sum_{j,k=1}^{d} A_{jk}\frac{\partial^2 f}{\partial x_j \partial x_k}(x) - \sum_{j=1}^{d}\gamma_j \frac{\partial f}{\partial x_j}(x) \\ &\quad + \int_{\mathbb{R}^d}\left(f(x-y) - f(x) + \sum_{j=1}^{d} y_j \frac{\partial f}{\partial x_j}(x)\mathbf{1}_{\{y;|y|\le 1\}}(y)\right)\nu(\mathrm{d}y). \end{aligned} \tag{4.3}$$

(In fact $\widetilde{L}$ is the infinitesimal generator for the dual process $\tilde{X} = -X$).

Let B be a closed set in $\mathbb{R}^d$. Set $T_B = \inf\{t \ge 0; X_t \in B\}$, the first entrance time of B. Define for any $g \in L_\infty(B)$ and $r \ge 0$,

$$P_B^r g(x) = \mathbb{E}_x\left[\mathrm{e}^{-rT_B} g(X_{T_B})\right]. \tag{4.4}$$

(Clearly, if $d = 1$ and $B = (-\infty, 0]$, then $P_B^r g(x) = \Phi(x)$.) To give the following definition, we write $\langle f_1, f_2\rangle = \int f_1(x) f_2(x) dx$. Also put $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ if ϕ is infinitely differentiable and has a compact support. We say that a bounded Borel measurable function $h : \mathbb{R}^d \to \mathbb{R}$ is a *weak solution* of the boundary value problem:

$$\begin{cases} (r-L)h = 0, & \text{in } B^c, \\ h = g, & \text{on } B, \end{cases} \tag{4.5}$$

if $h(x) = g(x)$ for any $x \in B$ and, for any $\phi \in \mathcal{C}_c^\infty(B^c)$, $\langle h, (r - \widetilde{L})\phi\rangle = 0$. Also, we say that h is a *strong solution* of the boundary problem (4.5) if $h(x) = g(x)$ for all $x \in B$, $h \in \mathcal{C}^2(B^c)$ and $(r - L)h = 0$ on B^c.

The following theorem is taken from Boyarchenko and Levendorski (2002).

Theorem 4.1. *Assume that the characteristic exponent Ψ of X admits an analytic continuation into a tube domain $\mathbb{R}^d + \mathrm{i}\mathcal{O}$, where $\mathcal{O}$ is an open set in $\mathbb{R}^d$ containing 0. Let B be a closed set in $\mathbb{R}^d$. Then for any $r \geq 0$ and $g \in L_\infty(B)$, $P_B^r g$ is a weak solution of the boundary value problem (4.5).*

Moreover, we have the following:

Theorem 4.2. *Given $r \geq 0$ and B a closed set in $\mathbb{R}^d$. Suppose $P_B^r g$ is a weak solution of (4.5) and $P_B^r g \in \mathcal{C}^2(B^c) \cap \mathcal{C}(\overline{B^c})$. Then $P_B^r g$ is a strong solution to (4.5) in the following two cases:*

1. *g is bounded continuous on B and ν is a finite measure.*
2. *g is a bounded Borel measurable function on B and ν is an absolutely continuous finite measure.*

Proof. Write $H(x) = P_B^r g(x)$. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$, set $|\alpha| = \sum_{j=1}^d \alpha_j$ and write $\mathrm{D}^\alpha f(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} f(x)$. By partial integration, if $\phi \in \mathcal{C}_c^\infty(B^c)$, $(-1)^{|\alpha|} \int \mathrm{D}^\alpha \phi(x) f(x) dx = \int \phi(x) \mathrm{D}^\alpha f(x) dx$ for all $f \in \mathcal{C}^{|\alpha|}(B^c)$. From this, one immediately gets

$$\begin{aligned} &\int \left[\frac{1}{2}\sum_{j,k=1}^d A_{jk}\frac{\partial^2 \phi}{\partial x_j \partial x_k} - \sum_{j=1}^d \gamma_j \frac{\partial \phi}{\partial x_j}\right](x) H(x) \mathrm{d}x \\ &= \int \left[\frac{1}{2}\sum_{j,k=1}^d A_{jk}\frac{\partial^2 H}{\partial x_j \partial x_k} + \sum_{j=1}^d \gamma_j \frac{\partial H}{\partial x_j}\right](x) \phi(x) \mathrm{d}x. \end{aligned} \tag{4.6}$$

On the other hand, we have

$$\begin{aligned} &\int \mathrm{d}x \int \nu(\mathrm{d}y) H(x) \left[\phi(x-y) - \phi(x) + \mathbf{1}_{\{y;|y|\leq 1\}} \sum_{j=1}^d y_j \frac{\partial \phi}{\partial x_j}(x)\right] \\ &= \int \nu(\mathrm{d}y) \int \mathrm{d}x H(x) \left[\phi(x-y) - \phi(x) + \mathbf{1}_{\{y;|y|\leq 1\}} \sum_{j=1}^d y_j \frac{\partial \phi}{\partial x_j}(x)\right] \\ &= \int \nu(\mathrm{d}y) \int \mathrm{d}x \phi(x) \left[H(x+y) - H(x) - \mathbf{1}_{\{y;|y|\leq 1\}} \sum_{j=1}^d y_j \frac{\partial H}{\partial x_j}(x)\right]. \end{aligned}$$

Then applying the Fubini's Theorem again gives

$$\begin{aligned}
&\int \mathrm{d}x \int \nu(\mathrm{d}y) H(x) \left[\phi(x-y) - \phi(x) + \mathbf{1}_{\{y;|y|\le 1\}} \sum_{j=1}^{d} y_j \frac{\partial \phi}{\partial x_j}(x)\right] \\
&= \int \mathrm{d}x \int \nu(\mathrm{d}y) \phi(x) \left[H(x+y) - H(x) - \mathbf{1}_{\{y;|y|\le 1\}} \sum_{j=1}^{d} y_j \frac{\partial H}{\partial x_j}(x)\right].
\end{aligned} \tag{4.7}$$

Now, combining (4.6) and (4.7) gives that $\langle H, \widetilde{L}\phi\rangle = \langle LH, \phi\rangle$. Hence $\langle (r-L)H, \phi\rangle = \langle H, (r-\widetilde{L})\phi\rangle = 0$ for all $\phi \in \mathcal{C}_c^\infty(B^c)$. This implies that $(r-L)H = 0$ a.s. on B^c.

To show that $(r-L)H(x) = 0$ for every $x \in B^c$, it suffices to show that $(r-L)H$ is continuous on B^c. Since H is in $\mathcal{C}^2(B^c)$, by (4.2), we need only show that

$$\int \left(H(x+y) - H(x) - \mathbf{1}_{\{y;|y|\le 1\}} \sum_{j=1}^{d} y_j \frac{\partial H}{\partial x_j}(x)\right) \nu(\mathrm{d}y) \tag{4.8}$$

is continuous in B^c. First, assume that g is continuous and ν is a finite measure. Since $H \in \mathcal{C}^2(B^c) \cap \mathcal{C}(\overline{B^c})$, $H = g$ on ∂B by the definition of H, and $g \in \mathcal{C}(B)$, we have H is continuous on $\mathbb{R}^d$. Hence, using the assumption ν is a finite measure, we conclude that the function in (4.8) is continuous on B^c. This proves (1). To prove (2), we assume that g is bounded and ν is an absolutely continuous finite measure. Then a modification of Lemma 3.1 also gives that the function $\int H(x+y)\nu(dy)$ in (4.8) is continuous. This proves (2). ■

Proof of Theorem B. Note that the infinitesimal generator L of X in (1.2) is

$$L\Phi(x) = \frac{\sigma^2}{2}\Phi''(x) + c\Phi'(x) + \lambda \int \Phi(x-y)dF(y) - \lambda\Phi(x).$$

Hence Theorem B follows from Theorem 4.1 and Theorem 4.2. ■

Proof of Theorem C. Assume that the jump distribution has the density $dF(y) = \eta e^{-\eta y}\mathbf{1}(y>0)dy$ for some $\eta > 0$. Assume first that g is bounded. Then (1.4) becomes

$$\begin{aligned}
&\frac{\sigma^2}{2}\Phi''(x) + c\Phi'(x) + \lambda \int_0^x \Phi(y)\eta e^{-\eta(x-y)} dy \\
&+\lambda \int_{-\infty}^0 g(y)\eta e^{-\eta(x-y)} dy - (\lambda + r)\Phi(x) = 0.
\end{aligned} \tag{4.9}$$

Differentiating both sides of (4.9) yields

$$
\begin{aligned}
0 = &\frac{\sigma^2}{2}\Phi'''(x) + c\Phi''(x) + \lambda\eta\Phi(x) - \lambda\eta^2 e^{-\eta x}\int_0^x \Phi(y)e^{\eta y}dy \\
&-\lambda\eta^2 e^{-\eta x}\int_{-\infty}^0 g(y)e^{\eta y}dy - (\lambda+r)\Phi'(x).
\end{aligned}
$$

By (4.9), we can rewrite the last equation as

$$
\begin{aligned}
0 = &\frac{\sigma^2}{2}\Phi'''(x) + c\Phi''(x) + \lambda\eta\Phi(x) \\
&+\eta\left(\frac{\sigma^2}{2}\Phi''(x) + c\Phi'(x) - (\lambda+r)\Phi(x)\right) - (\lambda+r)\Phi'(x).
\end{aligned}
$$

Then Φ satisfies the ordinary differential equation $\mathcal{A}\Phi = 0$ on $(0,\infty)$, where $\mathcal{A}$ is given in (1.6). Next, we consider the boundary conditions for Φ. By the definition of Φ, we have $\Phi(0) = g(0)$. Also by Proposition 2.1, we have $\lim_{x\to\infty}\Phi(x) = 0$. Clearly, (4.9) gives another boundary condition: $\frac{\sigma^2}{2}\Phi''(0+) + c\Phi'(0+) = (\lambda + r)g(0) - \lambda\int_{-\infty}^0 g(y)\eta e^{\eta y}dy$. Then as shown in Appendix A, if $r > 0$, $\Phi(x)$ is given by the formula (1.5). Now, if g is an nonnegative function such that $\int_{-\infty}^0 g(y)\eta e^{\eta y}dy < \infty$, by approximation of g by bounded functions, we see that (1.5) still holds for g. The proof is now complete. ∎

Appendix A. Solutions of Ordinary Differential Equations

We consider the ordinary differential equation(ODE):

$$\mathcal{A}f = 0, \qquad \text{on } (0,\infty) \tag{A.1}$$

subject to the boundary conditions

$$f(0) = K_1, \tag{A.2}$$

$$\lim_{x\to\infty} f(x) = 0, \tag{A.3}$$

$$\frac{\sigma^2}{2}f''(0) + cf'(0) = K_2, \tag{A.4}$$

for some constants $K_1, K_2 \in \mathbb{R}$. Here, $\mathcal{A}$ is given by (1.6).

Lemma A.1. *For* $r > 0$, the characteristic polynomial $p(x)$ of (1.6) has three distinct real zeros ρ_2, ρ_1, ρ with $\rho_2 < -\eta < \rho_1 < 0 < \rho$.

Proof. Set $p_1(x) = \frac{\sigma^2}{2}x^2 + cx + \lambda\int_0^\infty e^{-xy}\eta e^{-\eta y}dy - (\lambda + r)$. First, note that $p(x) = p_1(x)(x+\eta)$. Hence, on $(-\eta,\infty)$, $p(x)$ and $p_1(x)$ have the same zero set. Using differential calculus, one sees that p_1 is strictly convex on $(-\eta, \infty)$ with $p_1(0) = -r < 0$. Since $\lim_{x \to (-\eta)+} p_1(x) = \infty$ and $\lim_{x \to \infty} p_1(x) = \infty$, there are two distinct solutions for $p_1(x)$ in $(-\eta, 0)$ and $(0, \infty)$ respectively. Similarly, there is one solution for $p(x)$ in $(-\infty, -\eta)$. ∎

The general solution for (A.1) is given by $f(x) = Ae^{\rho_1 x} + Be^{\rho_2 x} + Ce^{\rho x}$ for some constants A, B and C. Since $\lim_{x \to \infty} f(x) = 0$ and $\rho_2 < \rho_1 < 0 < \rho$, we get $C = 0$. This gives $f(x) = Ae^{\rho_1 x} + Be^{\rho_2 x}$. By the boundary conditions (A.2) and (A.4) respectively, we get

$$A + B = K_1$$

and

$$A\left(\frac{\sigma^2}{2}\rho_1^2 + c\rho_1\right) + B\left(\frac{\sigma^2}{2}\rho_2^2 + c\rho_2\right) = K_2.$$

Simple algebra leads to

$$A = \frac{K_2 - \left(\frac{\sigma^2}{2}\rho_2^2 + c\rho_2\right)K_1}{\frac{\sigma^2}{2}(\rho_1^2 - \rho_2^2) - c(\rho_2 - \rho_1)}$$

and

$$B = \frac{K_2 - \left(\frac{\sigma^2}{2}\rho_1^2 + c\rho_1\right)K_1}{\frac{\sigma^2}{2}(\rho_2^2 - \rho_1^2) - c(\rho_1 - \rho_2)}.$$

Hence, the solution of (A.1) with the boundary condition (A.2), (A.3) and (A.4) is given by

$$f(x) = \frac{K_2 - \left(\frac{\sigma^2}{2}\rho_2^2 + c\rho_2\right)K_1}{\frac{\sigma^2}{2}(\rho_1^2 - \rho_2^2) - c(\rho_2 - \rho_1)}e^{\rho_1 x} + \frac{K_2 - \left(\frac{\sigma^2}{2}\rho_1^2 + c\rho_1\right)K_1}{\frac{\sigma^2}{2}(\rho_2^2 - \rho_1^2) - c(\rho_1 - \rho_2)}e^{\rho_2 x}.$$

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