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CLASSIFICATION OF RULED SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

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Dedicated to Professor Bang-yen Chen on the occasion of his 65^{th} birthday.

Abstract. Ruled surfaces with the Gauss map satisfying a partial differential equation which is similar to an eigenvalue problem in a 3-dimensional Euclidean space are studied. Such a Gauss map is said to be of pointwise 1-type, namely, the Gauss map G satisfies $\Delta G = f(G + C)$, where Δ is the Laplacian operator, f is a non-zero function and C is a constant vector. As a result, such ruled surfaces are completely determined by the function f and the vector C when their Gauss map is of pointwise 1-type. New examples of ruled surfaces called cylinders of an infinite type and rotational ruled surfaces are introduced in this regard.

1. Introduction

In the late 1970's B.-Y. Chen introduced the notion of finite type immersion. Essentially submanifolds of finite type immersed into an m-dimensional Euclidean space $\mathbb{E}^m$ are constructed in terms of finitely many $\mathbb{E}^m$-valued eigenfunctions of their Laplacian. Minimal submanifolds of a Euclidean space or minimal submanifolds of a sphere are of the simplest finite type, i.e. 1-type, which are akin to eigenvalue problems with regard to the immersion. Many results on this subject have been collected in the book ([3]) and the motivations and problems were introduced in a survey paper [4]. The notion of finite type immersion is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it.

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In this area, B.-Y. Chen and P. Piccini ([7]) studied submanifolds of Euclidean space with finite type Gauss map and classified compact surfaces with 1-type Gauss map, that is, $\Delta G = \lambda(G + C)$, where Δ is the Laplacian of M, G the Gauss map, C a constant vector and $\lambda \in \mathbb{R}$. Several geometers also studied submanifolds of Euclidean spaces or pseudo-Euclidean spaces with finite type Gauss map ([1, 5, 8, 9-11], etc.).

However, there are some submanifolds satisfying $\Delta G = f(G + C)$ for some smooth function f and a constant vector C. For example, an ordinary helicoid is, up to rigid motion, parameterized by

$$x(t, \theta) = (t \cos\theta, t \sin\theta, h\theta), \quad h \neq 0$$

with respect to a surface patch (t, θ). Then the Gauss map is given by

$$G = \frac{1}{\sqrt{h^2 + t^2}}(h \sin\theta, -h\cos\theta, t)$$

and the Laplacian ΔG of the Gauss map G is obtained as

$$\Delta G = \frac{2h^2}{(h^2 + t^2)^2} G.$$

The right cone C_a which is parameterized by

$$x(u, v) = (v \cos u, v \sin u, av), \quad a \geq 0$$

has the Gauss map G equal to

$$G = \frac{1}{\sqrt{1 + a^2}}(a \cos u, a \sin u, -1).$$

Then, its Laplacian ΔG satisfies

$$\Delta G = \frac{1}{v^2}\Big(G + \Big(0, 0, \frac{1}{\sqrt{1 + a^2}}\Big)\Big).$$

Based on this view, we raise the following question:

Problem. *Classify all submanifolds M in an m-dimensional Euclidean space $\mathbb{E}^m$ satisfying the condition*

$$\Delta G = f(G + C) \tag{1.1}$$

for some non-zero smooth function f and some constant vector C. In this case, we have to determine the submanifold M of $\mathbb{E}^m$, the function f and the constant vector C as well.

A submanifold M in $\mathbb{E}^m$ is said to have *pointwise 1-type Gauss map* if it satisfies (1.1). In particular, if C is zero, it is said to be of *the first kind*. Otherwise, it is said to be of *the second kind* ([5]).

In the present paper, we completely classify ruled surfaces in a 3-dimensional Euclidean space with pointwise 1-type Gauss map of the first kind and the second kind. If f is not constant, it is said to be *proper*. So, a non-proper pointwise 1-type Gauss map is of just an ordinary 1-type.

Throughout this paper, we assume that all the geometric objects are smooth and all surfaces are connected unless otherwise mentioned.

bigskip

2. Preliminaries

Let M be a surface of a 3-dimensional Euclidean space $\mathbb{E}^3$. The map $G : M \to S^2 \subset \mathbb{E}^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of the surface M, where S^2 is the unit sphere in $\mathbb{E}^3$ centered at the origin. For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the metric on M, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. The Laplacian Δ on M is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(\sqrt{\mathcal{G}}\ \tilde{g}^{ij} \frac{\partial}{\partial x^j}\Big). \tag{2.1}$$

Considering the results of [5], [8], [11] concerning mean curvature, we have

Lemma 2.1. *Let M be a surface in a 3-dimensional Euclidean space $\mathbb{E}^3$. Then, the mean curvature H is constant if and only if the Gauss map G is of pointwise 1-type of the first kind.*

In particular, if the surface M is a ruled surface, the first and the second named author ([8]) proved the following theorem :

Theorem 2.2. ([8]). *A ruled surface in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the first kind is an open portion of either a circular cylinder or a helicoid.*

Thus, we have immediately

Corollary 2.3. *The helicoid is the only ruled surface in $\mathbb{E}^3$ with proper pointwise 1-type Gauss map of the first kind.*

3. Main Theorems

In this section, we will classify the ruled surfaces in terms of pointwise 1-type Gauss map. More precisely, we focus on the ruled surfaces in $\mathbb{E}^3$ with proper pointwise 1-type Gauss map of the second kind.

Let M be a cylindrical ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$. Then, M is expressed by

$$x(s,t) = \alpha(s) + t\beta$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ is a plane curve parameterized by arc-length s and β is a constant vector, namely $\beta = (0,0,1)$. In this case, the Gauss map G of M is given by $G = \alpha' \times \beta = (\alpha_2', -\alpha_1', 0)$ and the Laplacian ΔG of G by $\Delta G = (-\alpha_2''', \alpha_1''', 0)$, where the prime denotes the derivative with respect to s.

Suppose that the surface M has pointwise 1-type Gauss map of the second kind. Then, from the equation (1.1) we have the following system of differential equations

$$\begin{aligned} -\alpha_2''' &= f\alpha_2' + fc_1 \\ \alpha_1''' &= -f\alpha_1' + fc_2 \end{aligned} \tag{3.1}$$

where $C = (c_1, c_2, 0)$. On the other hand, the curve $\alpha(s)$ is of unit speed, that is, $(\alpha_1')^2 + (\alpha_2')^2 = 1$. So we may put

$$\alpha_1'(s) = \cos\theta(s), \quad \alpha_2'(s) = \sin\theta(s)$$

for a smooth function $\theta = \theta(s)$. So, it enables equation (3.1) to be rewritten in the form

$$\begin{aligned} (\theta')^2 \sin\theta - \theta'' \cos\theta &= f\sin\theta + fc_1, \\ (\theta')^2 \cos\theta + \theta'' \sin\theta &= f\cos\theta - fc_2, \end{aligned} \tag{3.2}$$

which give

$$(\theta')^2 = f(1 + c_1 \sin\theta - c_2 \cos\theta), \tag{3.3}$$

$$\theta'' = -f(c_1 \cos\theta + c_2 \sin\theta). \tag{3.4}$$

If $\theta' \equiv 0$, the Gauss map G is a constant vector. In this case, M is a part of a plane and $\Delta G = 0$. Therefore, if we choose $C = -G$, we can take an arbitrary non-zero smooth function f making (1.1) hold.

Suppose $\theta' \neq 0$. Taking the derivative of (3.3) and using (3.3) and (3.4) we obtain

$$\theta' = c\sqrt[3]{f} \tag{3.5}$$

for some non-zero constant c. By the composition of trigonometric function, (3.3) and (3.4) we find the differential equation

$$\left(\frac{1}{f}(\theta')^2 - 1\right)^2 + \left(\frac{1}{f}\theta''\right)^2 = c_1^2 + c_2^2, \tag{3.6}$$

which implies with the help of (3.5)

$$\left(c^2 f^{-\frac{1}{3}} - 1\right)^2 + \left(-\frac{1}{2} c \left(f^{-\frac{2}{3}}\right)'\right)^2 = c_1^2 + c_2^2. \tag{3.7}$$

Putting $y = f^{-\frac{1}{3}}$, the equation (3.7) becomes

$$(c^2 y - 1)^2 + \frac{1}{4} c^2 \left((y^2)'\right)^2 = c_1^2 + c_2^2. \tag{3.8}$$

The solution of the differential equation is given by

$$\sin^{-1}\left(\frac{c^2 f^{-\frac{1}{3}} - 1}{\sqrt{c_1^2 + c_2^2}}\right) - \sqrt{c_1^2 + c_2^2 - \left(c^2 f^{-\frac{1}{3}} - 1\right)^2} = \pm c^3 (s + k), \tag{3.9}$$

where k is the constant of integration.

Definition 3.1. A cylindrical ruled surface M over an infinite type base curve α is called a *cylinder of an infinite type.*

Thus, we have

Proposition 3.1. *Let M be a cylindrical ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$. If the Gauss map G is of pointwise 1-type of the second kind, the non-zero smooth function f satisfies the equation* (3.9).

Combining (3.4) and the result of [8], we obtain

Theorem 3.2. *Let M be a cylindrical ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$. If the Gauss map G is of pointwise 1-type of the first kind, then θ' is a constant, that is, the curvature of the base curve is a constant. Furthermore, the surface M is an open part of a circular cylinder.*

On the other hand, it is well-known that the plane curves of finite type are of 1-type, that is, they are part of straight lines or circles (See [6]). Viewing this fact, we have

Theorem 3.3. *Let M be a cylindrical ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$. Then, the Gauss map G is of pointwise 1-type of the second kind if and only if M is an open part of a plane or a cylinder of an infinite type satisfying* (3.9).

Now, we consider a reparametrization of a given non-cylindrical ruled surface for our convenience of consideration.

Proposition 3.4. *Let M be a non-cylindrical ruled surface with parametrization*

$$x_1(s,t) = \alpha_1(s) + t\beta(s)$$

where α_1 is a base curve and β a director vector field satisfying $\langle \alpha'_1, \beta\rangle = 0, \langle \beta, \beta\rangle = 1$ and $\langle \beta', \beta'\rangle = 1$. Then, there exists a reparametrization

$$x(s,t) = \alpha(s) + t\beta(s)$$

for M with the base curve α and the director vector β satisfying $\langle \alpha', \beta\rangle = 0, \langle \alpha', \beta'\rangle = 0, \langle \beta, \beta\rangle = 1$ and $\langle \beta', \beta'\rangle = 1$. *Proof.* For a base curve α_1 which is a regular curve and a director vector field β parameterized by arc-length s, suppose a parametrization $x_1(s,t) = \alpha_1(s) + t\beta(s)$ of M with $\langle \alpha'_1, \beta\rangle = 0, \langle \beta, \beta\rangle = 1$ and $\langle \beta', \beta'\rangle = 1$ is given. For such a base curve α_1 and a director vector β, define a curve α by $\alpha(s) = y_1(s)\alpha_1(s) + y_2(s)\beta(s)$, where y_1 and y_2 are the solutions satisfying a system of ordinary differential equations

$$f_1(s)y'_1(s) + y'_2(s) = 0,$$

$$f_2(s)y_1(s) + f_3(s)y'_1(s) + y_2(s) = 0$$

with a proper initial condition $y_1(0) = (y_1)_0, y_2(0) = (y_2)_0$, where $f_1(s) = \langle \alpha_1(s), \beta(s)\rangle$, $f_2(s) = \langle \alpha'_1(s), \beta'(s)\rangle$ and $f_3(s) = \langle \alpha_1(s), \beta'(s)\rangle$. Then, we easily see α and β satisfy $\langle \alpha', \beta\rangle = 0, \langle \alpha', \beta'\rangle = 0, \langle \beta, \beta\rangle = 1$ and $\langle \beta', \beta'\rangle = 1$. ■

Next, we consider a non-cylindrical ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$ with pointwise 1-type Gauss map.

Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$. As is given by Proposition 3.4, M is parameterized by a base curve α and a director vector field β, up to rigid motion,

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \alpha', \beta\rangle = 0, \langle \alpha', \beta'\rangle = 0, \langle \beta, \beta\rangle = 1$ and $\langle \beta', \beta'\rangle = 1$. Then, we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha' + t\beta'$ and $x_t = \beta$. From this setting, we have an orthonormal frame $\{\beta, \beta', \beta\times\beta'\}$. For later use, we define the smooth functions q, Q and R as follows :

$$q = \langle x_s, x_s\rangle, \quad Q = \langle \alpha', \beta\times\beta'\rangle, \quad R = \langle \beta'', \beta\times\beta'\rangle.$$

In terms of the orthonormal frame $\{\beta, \beta', \beta\times\beta'\}$ we obtain

$$\begin{aligned} &\alpha' = Q\beta\times\beta', \\ &\beta'' = -\beta + R\beta\times\beta', \\ &\alpha'\times\beta = Q\beta' \\ &\beta\times\beta'' = -R\beta', \end{aligned} \tag{3.10}$$

from which, the Gauss map G of M is obtained by

$$G = \left(\frac{1}{||x_s \times x_t||}\right) x_s \times x_t = q^{-1/2}(Q\beta' - t\beta \times \beta')$$

and the smooth function q is given by

$$q = t^2 + Q^2.$$

Denote by H the mean curvature of M. Using (2.1) for the Laplacian operator Δ and the well known equation $\Delta x = -2H$, the mean curvature H of M is obtained as follows:

$$H = \frac{1}{2} q^{-3/2}(-Rt^2 - Q't - Q^2 R). \tag{3.11}$$

Furthermore, the following formula for the Laplacian of the Gauss map of M in $\mathbb{E}^3$ is easily obtained by applying the Gauss formula and the Weingarten formula:

$$\Delta G = 2\mathrm{grad}H + (\mathrm{tr}A^2)G, \tag{3.12}$$

where A denotes the shape operator of the surface M. From (3.11)

$$\begin{aligned} 2\mathrm{grad}H &= 2e_1(H)e_1 + 2e_2(H)e_2 \\ &= \frac{1}{2} q^{-3} A_1 e_1 + q^{-5/2} B_1 e_2 \\ &= \frac{1}{2} q^{-7/2}(2qB_1\beta + tA_1\beta' + QA_1\beta \times \beta'), \end{aligned} \tag{3.13}$$

where $e_1 = \frac{x_s}{||x_s||}, e_2 = \frac{x_t}{||x_t||}$,

$$\begin{aligned} A_1 =& -2R't^4 - 2Q''t^3 + (2QQ'R - 4Q^2R')t^2 \\ &+ (6Q(Q')^2 - 2Q^2Q'')t + (2Q^3Q'R - 2Q^4R') \qquad \text{and} \\ B_1 =& Rt^3 + 2Q't^2 + Q^2Rt - Q^2Q'. \end{aligned}$$

Furthermore, we have

$$\mathrm{tr}A^2 = q^{-3}D_1,$$

where

$$D_1 = R^2t^4 + 2Q'Rt^3 + (2Q^2R^2 + Q'^2 + 2Q^2)t^2 + 2Q^2Q'Rt + Q^4R^2 + 2Q^4.$$

Thus, from (1.1) and (3.12) we have

$$\begin{aligned} &\frac{1}{2} q^{-7/2}(2qB_1\beta + tA_1\beta' + QA_1\beta \times \beta') + q^{-7/2}D_1(Q\beta' - t\beta \times \beta') \\ &= q^{-1/2} f(Q\beta' - t\beta \times \beta') + fC. \end{aligned} \tag{3.14}$$

If we take the inner products on the equation (3.14) with β, β' and $\beta \times \beta'$, respectively, then we have

$$\langle C, \beta \rangle f = q^{-5/2} B_1, \tag{3.15}$$

$$(\langle C, \beta' \rangle + q^{-1/2} Q) f = \frac{1}{2} q^{-7/2} t A_1 + q^{-7/2} Q D_1 \tag{3.16}$$

and

$$(\langle C, \beta \times \beta' \rangle - q^{-1/2} t) f = \frac{1}{2} q^{-7/2} Q A_1 - q^{-7/2} t D_1, \tag{3.17}$$

respectively. From (3.15) and (3.16) we have

$$4qQ^2 B_1^2 = (2qB_1\mu - \lambda t A_1 - 2\lambda D_1 Q)^2 \tag{3.18}$$

and from (3.15) and (3.17)

$$4qt^2 B_1^2 = (\lambda Q A_1 - 2\lambda t D_1 - 2\nu q B_1)^2, \tag{3.19}$$

where we put $\lambda = \langle C, \beta \rangle$, $\mu = \langle C, \beta' \rangle$ and $\nu = \langle C, \beta \times \beta' \rangle$. Also, combining (3.16) and (3.17), we obtain

$$\left(Q^2 A_1 + t^2 A_1\right)^2 = q\left(\nu t A_1 + 2\nu D_1 Q - \mu Q A_1 + 2\mu t D_1\right)^2. \tag{3.20}$$

Differentiating the constant vector $C = \lambda\beta + \mu\beta' + \nu\beta \times \beta'$ with respect to the parameter s, we have

$$(\nu' + \mu R)\beta \times \beta' = 0$$

because of the last equation of (3.10), from which,

$$\nu' + \mu R = 0. \tag{3.21}$$

From the definition of μ, we have

$$\mu' = -\lambda + R\nu. \tag{3.22}$$

If we look at (3.18), it is a polynomial in t with functions of s as the coefficients. Thus, the leading coefficient must be zero, i.e.,

$$\mu R + \lambda R' = 0, \tag{3.23}$$

from which,

$$R\lambda = k \tag{3.24}$$

for some constant k. Consider an open subset $\mathbf{U} = \{p \in M | R(p) \neq 0\}$. Suppose $\mathbf{U} \neq \phi$. From (3.19), we have

$$R^2 = (\lambda R^2 + \nu R)^2. \tag{3.25}$$

On $\mathbf{U}$, (3.25) gives ν is constant on a component $\mathbf{U}_o$ of $\mathbf{U}$. From (3.21), we see that $\mu = 0$ on $\mathbf{U}_o$. Thus, (3.22) and (3.24) imply that λ is constant on $\mathbf{U}_o$ and so is R on $\mathbf{U}_o$. By continuity and connectedness of M, R is a non-zero constant on M. Using (3.25), we see that ν is constant on M. By means of (3.24), λ is also a constant on M. (3.21) with the help of $\nu =$ constant implies $\mu = 0$ and $\lambda = R\nu$.

In this case, if we consider the coefficients of polynomials in t of (3.18) and (3.20), we have

$$Q^2R^2 = \lambda^2(R^2Q - Q'')^2, \tag{3.26}$$

$$Q''^2 = \nu^2(R^2Q - Q'')^2, \tag{3.27}$$

which are respectively derived from the coefficients of t^8 in(3.18) and t^{10} in (3.20). From these two equations, we have

$$Q'' = \pm Q,$$

from which, we get

$$Q(s) = \tilde{A}\cosh s + \tilde{B}\sinh s \quad \text{or} \quad Q(s) = \tilde{C}\cos s + \tilde{D}\sin s$$

for some constants $\tilde{A}$, $\tilde{B}, \tilde{C}$ and $\tilde{D}$. Under an appropriate initial condition and rigid motion, we may assume that

$$Q(s) = \cosh s \quad \text{or} \quad Q(s) = \cos s. \tag{3.28}$$

On the other hand, from the second equation of (3.10), we have

$$\beta'''(s) + a^2\beta'(s) = 0, \tag{3.29}$$

where $a = \sqrt{R^2+1}$. Under an initial condition $\beta(0) = (1,0,0)$, $\beta'(0) = (0,1,0)$, $\beta''(0) = (-1, 0, R)$, we have a unique solution of (3.29) as follows

$$\beta(s) = (1 - \frac{1}{a^2} + \frac{1}{a^2}\cos as, \frac{1}{a}\sin as, \frac{R}{a^2} - \frac{R}{a^2}\cos as). \tag{3.30}$$

Using (3.10) and (3.30), we get, up to rigid motion,

$$\begin{aligned}\alpha(s) = \Bigg(&\frac{R}{a^2}\sinh s - \frac{R}{a^2(a^2+1)}(\sinh s\cos as + a\cosh s\sin as) + d_1,\\ &-\frac{R}{a(a^2+1)}(\sinh s\sin as - a\cosh s\cos as) + d_2,\\ &\frac{1}{a^2}\sinh s + \frac{R^2}{a^2(a^2+1)}(\sinh s\cos as + a\cosh s\sin as) + d_3\Bigg)\end{aligned} \tag{3.31}$$

if $Q(s) = \cosh s$, or,

$$\alpha(s) = \Big(\frac{R}{a^2}\sin s - \frac{R}{2a^2}\{\frac{1}{a+1}\sin(a+1)s + \frac{1}{a-1}\sin(a-1)s\} + k_1,$$

$$\frac{R}{2a}\{\frac{1}{a+1}\cos(a+1)s + \frac{1}{a-1}\cos(a-1)s\} + k_2, \tag{3.32}$$

$$\frac{1}{a^2}\sin s + \frac{R^2}{2a^2}\{\frac{1}{a+1}\sin(a+1)s + \frac{1}{a-1}\sin(a-1)s\} + k_3\Big),$$

if $Q(s) = \cos s$, where d_i and k_i ($i = 1, 2, 3$) are some constants.

Therefore, the ruled surface M is given by

$$x(s,t) = \alpha(s) + t\beta(s),$$

where α and β are obtained by (3.30) and (3.31) or (3.32).

Definition 3.2. A non-cylindrical ruled surface M generated by (3.30) and (3.31) is called a *rotational ruled surface of the first kind* and that generated by (3.30) and (3.32) a *rotational ruled surface of the second kind.*

Next, we consider the case of $R = 0$. If we compute the leading coefficient and the constant term with respect to t in (3.19) with $R = 0$, we have respectively

$$Q'^2 - \nu^2 Q'^2 = 0 \quad \text{and} \quad \nu^2 Q^8 Q'^2 = 0.$$

From these two equations with the property of non-vanishing Q, we see that Q is a nonzero constant. Therefore, the mean curvature H vanishes on M, that is, M is minimal. It contradicts the hypothesis that the Gauss map is of pointwise 1-type of the second kind. Thus, the case of $R = 0$ can never occur. Consequently, we conclude

Proposition 3.5. *Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind. Then, M is a part of a rotational ruled surface of the first kind or the second kind.*

Combining Theorem 3.3, Proposition 3.5 and [8], we have a complete classification :

Theorem 3.6. *Let M be a ruled surface in $\mathbb{E}^3$. Then, M has pointwise 1-type Gauss map if and only if it is a part of a plane, a circular cylinder, a helicoid, a cylinder of an infinite type satisfying* (3.9)*, or a rotational ruled surface of the first kind or the second kind.*

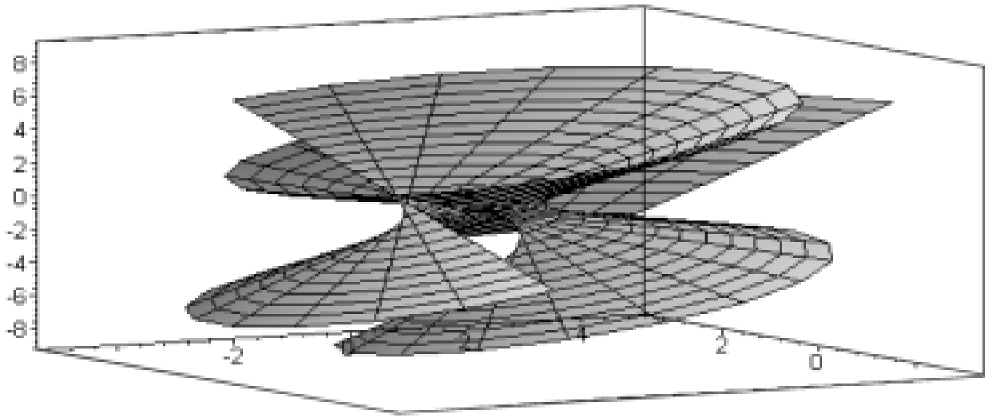

Fig. 3.1. A rotational ruled surface of the first kind with $a = 2$.

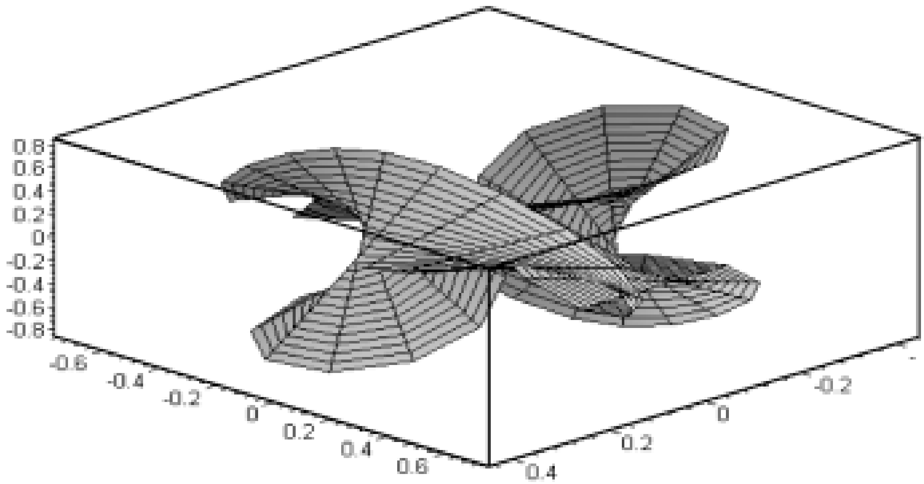

Fig. 3.2. A rotational ruled surface of the second kind with $a = 2$.

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