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MULTIPLE COMBINATORIAL STOKES' THEOREM WITH BALANCED STRUCTURE

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. Combinatorics of complexes plays an important role in topology, nonlinear analysis, game theory, and mathematical economics. In 1967, Ky Fan used door-to-door principle to prove a combinatorial Stokes' theorem on pseudomanifolds. In 1993, Shih and Lee developed the geometric context of general position maps, π-balanced and π-subbalanced sets and used them to prove a combinatorial formula for multiple set-valued labellings on simplexes. On the other hand, in 1998, Lee and Shih proved a multiple combinatorial Stokes' theorem, generalizing the Ky Fan combinatorial formula to multiple labellings. That raises a question : Does there exist a unified theorem underlying Ky Fan's theorem and Shih and Lee's results? In this paper, we prove a multiple combinatorial Stokes' theorem with balanced structure. Our method of proof is based on an incidence function. As a consequence, we obtain a multiple combinatorial Sperner's lemma with balanced structure.

1. INTRODUCTION

The theory of combinatorics of complexes may be traced back to 1928 [14] when Sperner discovered a combinatorial lemma, that is globally called *Sperner's lemma*, which gave a drastic simplification of proofs of two topological theorems, namely theorems of invariance of domain and invariance of dimension. In 1929, Knaster, Kuratowski and Mazurkiewicz [5] used Sperner's lemma to give a combinatorial proof of Brouwer's fixed-point theorem. In 1967, Scarf [10] used Sperner's lemma to give a constructive proof of Brouwer's fixed-point theorem and in 1974, Kuhn [6] gave a constructive proof of the fundamental theorem of algebra based on the

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combinatorial Stokes' theorem. In 1973, Shapley [11] generalized Sperner's lemma with balancd structure, and gave a simple proof of Scarf's theorem concerning the nonemptyness of cores of NTU games. On the other hand, in 1945, Tucker [15] proved a combinatorial lemma in the cube which gave a combinatorial proof of Lusternik-Schnirelmann's topological theorem. In 1967, Ky Fan [2] proved a combinatorial theorem which is called combinatorial Stokes' theorem, giving a common generalization of Sperner's lemma and Tucker's combinatorial lemma. In 1992, Shih and Lee [12] proved a combinatorial Lefschetz fixed-point formula, put Sperner's lemma into the form of "alternating sum", and showed that Sperner's lemma is the case of the Lefchetz number one for any simplicial map on a triangulation of a simplex. In 1989, Bapat [1] proved a multiple Sperner's lemma which gave a combinatorial proof of Gale's theorem [3]. In 1993, Shih and Lee [13] obtained a multiple balanced Sperner's lemma which is a common generalization of Shapley's theorem [11] and Bapat's theorem[1]. In 1998, Lee and Shih [7] also used their lemma to prove a multiple Stokes' theorem. In 2006, Lee and Shih [8] proved a structure theorem for coupled balanced games without side payments. In 2007, Hwang and Shih [4] using Shih-Lee's theorem to prove an equilibrium market game. In 2008, Meunier [9] gave a different approach of Lee and Shih's result.

The purpose of this paper is to give further generalizations of Lee and Shih's results concerning Stokes' theorem on pseudomanifolds [13]. We prove a multiple combinatorial Stokes' theorem with balanced structure. Instead of a search algorithm modifying Ky Fan's door-to-door principle [2], we prove our result by an incidence function. As a consequence, we obtain a multiple combinatorial Sperner's lemma with balanced structure. The paper is organized as follows. In Section 2 we introduce some basic definitions and notations. In Section 3 we study some properties of general position maps, π-balanced and π-subbalanced collections. In Sections 4 and 5 we prove our main results, multiple combinatorial Stokes' theorem with balanced structure and multiple combinatorial Sperner's lemma with balanced structure.

2. Definitions and Notations

For convenience sake, we recall some definitions and notations in this section. The notion of pseudomanifolds is an abstraction of surfaces and curves in a discrete sense which may be defined as follows, see also [2].

An (*abstract*) *complex* is a finite collection $\mathcal{K}$ of nonempty finite sets such that

(K1) if $\sigma \in \mathcal{K}$ and $\tau \subset \sigma$, $\tau \neq \emptyset$, then $\tau \in \mathcal{K}$.

Elements of $\mathcal{K}$ are called *simplexes* of $\mathcal{K}$. A simplex σ of $\mathcal{K}$ is called a *k-simplex* of $\mathcal{K}$ if the cardinality $|\sigma|$ of σ is $k + 1$. For a k-simplex σ of $\mathcal{K}$, the subsets τ of

σ such that $|\tau| = r + 1$ are called the *r-faces* of σ ($0 \le r \le k$). The *vertex set* $V(\mathcal{K})$ of $\mathcal{K}$ is the union of all simplexes of $\mathcal{K}$.

Let n be a positive integer. An $(n-1)$*-pseudomanifold* is a complex $\mathcal{K}$ with the following two properties:

(M1) Every simplex of $\mathcal{K}$ is a face of at least one $(n-1)$-simplex of $\mathcal{K}$.

(M2) Every $(n-2)$-simplex of $\mathcal{K}$ is a common face of at most two distinct $(n-1)$-simplexes of $\mathcal{K}$.

An $(n-2)$-simplex τ of an $(n-1)$-pseudomanifold $\mathcal{K}$ is called a *boundary* $(n-2)$*-simplex* of $\mathcal{K}$ if τ is a face of exactly one $(n-1)$-simplex of $\mathcal{K}$. The set of all boundary $(n-2)$-simplexes and their faces is denoted by $\partial\mathcal{K}$.

Sometimes oriented simplexes, that is, simplexes with orientations, are considered. The notion of the orientations of a nonempty finite set is defined below.

Let $\sigma = \{v_1, \ldots, v_n\}$ be a finite set of cardinality $n \ge 2$. We call an n-tuple with distinct components of the elements of σ an *ordering* of σ. Two orderings $(v_{i_1}, \ldots, v_{i_n})$ and $(v_{j_1}, \ldots, v_{j_n})$ of σ are said to *have the same orientation* if the permutation $\binom{v_{i_1} \cdots v_{i_n}}{v_{j_1} \cdots v_{j_n}}$ is even. Having the same orientation is an equivalence relation and it partitions the set of $n!$ orderings of σ into two equivalence classes. Each of the equivalence classes is called an *orientation* on σ, and if we fix one of them arbitrarily, the other one is called the *opposite orientation*. The orientation on σ determined by the ordering $(v_{i_1}, \ldots, v_{i_n})$ is denoted by $(+1)[v_{i_1}, \ldots, v_{i_n}]$ and the opposite orientation of $(+1)[v_{i_1}, \ldots, v_{i_n}]$ is denoted by $(-1)[v_{i_1}, \ldots, v_{i_n}]$. For the case $n = 1$, we call the two symbols $(+1)[v_1]$ and $(-1)[v_1]$ orientations on the one-point set $\{v_1\}$ and they are defined to be opposite orientations on $\{v_1\}$.

Given an orientation $\omega = \varepsilon[v_1, \ldots, v_n]$ on the set $\sigma = \{v_1, \ldots, v_n\}$ where $\varepsilon = \pm 1$ and $n \ge 2$. For each $i = 1, \ldots, n$, the *induced orientation* on $\sigma \setminus \{v_i\}$ from ω is the well defined orientation $(-1)^{i-1}\varepsilon[v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$ on $\sigma \setminus \{v_i\}$.

We now consider the notion of orientable pseudomanifolds which is an abstraction of orientable surfaces in a discrete sense.

Let $\mathcal{K}$ be an $(n-1)$-pseudomanifold. If there is a map ω defined on the set of all $(n-1)$-simplexes of $\mathcal{K}$ such that following two conditions are satisfied:

(C1) For each $(n-1)$-simplex σ of $\mathcal{K}$, $\omega(\sigma)$ is an orientation on σ.

(C2) If τ is an $(n-2)$-simplex of $\mathcal{K}$ which is a common face of two distinct $(n-1)$-simplexes σ_1 and σ_2 of $\mathcal{K}$, then $\omega(\sigma_1)$ and $\omega(\sigma_2)$ induce opposite orientations on τ.

Then $\mathcal{K}$ is said to be *orientable* and the ordered pair $(\mathcal{K}, \omega)$ is called a *coherently oriented* $(n-1)$*-pseudomanifold.*

The following notion of triangulations of geometric simplexes are also needed in this paper.

Given a finite set $\sigma = \{v_1, \ldots, v_n\}$ in a Euclidean space, the *affine hull* $aff(\sigma)$ of σ is the set of all affine combinations of $v_1, \ldots, v_n$, that is, $aff(\sigma) =$

$\{\sum_{i=1}^{n} \lambda_i v_i; \sum_{i=1}^{n} \lambda_i = 1\}$, and the *convex hull* $conv(\sigma)$ of σ is the set of all convex combinations of $v_1, \ldots, v_n$, that is, $conv(\sigma) = \{\sum_{i=1}^{n} \lambda_i v_i; \sum_{i=1}^{n} \lambda_i = 1, \text{ each } \lambda_i \geq 0\}$. The set σ is said to be *affinely independent* if the following is true:

(A1) If $\sum_{i=1}^{n} \lambda_i v_i = 0$ and if $\sum_{i=1}^{n} \lambda_i = 0$ then $\lambda_1 = \ldots = \lambda_n = 0$.

If $\sigma = \{v_1, \ldots, v_n\}$ $n \geq 1$ is affinely independent, the convex hull of σ is called the (*geometric*) $(n-1)$-*simplex* spanned by σ, the points $v_1, \ldots, v_n$ are called the *vertices* of the simplex, and we denote this simplex by $\overline{v_1 \ldots v_n}$. For $1 \leq i_1 < \ldots < i_k \leq n$, the $(k-1)$-simplex $\overline{v_{i_1} \ldots v_{i_k}}$ is called a $(k-1)$-*face* of the$(n-1)$-simplex $\overline{v_1 \ldots v_n}$. The *open simplex* spanned by the set σ is the set $Int(\overline{v_1 \ldots v_n}) = \{\sum_{i=1}^{n} \lambda_i v_i; \sum_{i=1}^{n} \lambda_i = 1, \text{ each } \lambda_i > 0\}$.

A finite collection T of (geometric) simplexes is called a *triangulation* of an $(n-1)$-simplex $\overline{a_1 \ldots a_n}$ if the following three conditions are satisfied:

(T1) $\overline{a_1 \ldots a_n} = \bigcup_{s \in T} s$.

(T2) If $s \in T$ and if t is a face of s then $t \in T$.

(T3) If $s, t \in T$ and $s \cap t \neq \emptyset$, then $s \cap t$ is a common face of s and t.

A point $v \in \overline{a_1 \ldots a_n}$ is called a *vertex* of T if v is a vertex of some simplex of T. The set of all vertices of T is denoted by $V(T)$. Let $\widetilde{T}$ be the collection of all affinely independent subsets σ of $V(T)$ that spans a simplex of T. Then $\widetilde{T}$ is an (abstract) complex, known as the *vertex scheme* of T. Let us now fix an orientation $\omega = \varepsilon[a_1, \ldots, a_n]$ $(\varepsilon = \pm 1)$ on the set $\{a_1, \ldots, a_n\}$. For each $(n-1)$-simplex $\sigma = \{v_1, \ldots, v_n\}$ of $\widetilde{T}$, define an orientation $\omega(\sigma)$ on σ by $\omega(\sigma) = \varepsilon[v_1, \ldots, v_n]$ or $\omega(\sigma) = (-1)\varepsilon[v_1, \ldots, v_n]$ according as $det(\alpha_{ij}) > 0$ or $det(\alpha_{ij}) < 0$ respectively, where (α_{ij}) is the $n \times n$ real matrix satisfying

$$v_i = \sum_{j=1}^{n} \alpha_{ij} a_j \quad (\sum_{j=1}^{n} \alpha_{ij} = 1) \quad \text{for } i = 1, \ldots, n. \tag{2.1}$$

Thus we may view an orientation ω on $\{a_1, \ldots, a_n\}$ a map whose domain is the set of all $(n-1)$-simplexes of $\widetilde{T}$ which assigns an orientation $\omega(\sigma)$ to σ for each $(n-1)$-simplex σ of $\widetilde{T}$. It is well known that $(\widetilde{T}, \omega)$ forms a coherently oriented $(n-1)$-pseudomanifold.

The following geometric context of general position maps, π-balanced and π-subbalanced collections compare with [11], are given in [13] and they are the key definitions of our theory.

For a given nonempty finite set N, the collection of all nonempty subsets of N is denoted by 2^N. Thus $|2^N| = 2^{|N|} - 1$. Let $A = \{a_i\}_{i \in N}$ be an affinely independent set. If $S \in 2^N$, the simplex spanned by $\{a_i\}_{i \in S}$ is denoted by A^S, and the *barycenter* of A^S is the point $m_S = \frac{1}{|S|} \sum\limits_{i \in S} a_i$. Let $\pi : 2^N \to A^N$ and $p \in P \in 2^N$. We say that π is a *Shapley map* if

(S1) for each $S \in 2^N$, $\pi(S) \in A^S$,

and that π is a *general position map* if the following two conditions are satisfied:

(G1) For each $S \in 2^N$, $\pi(S) \in Int(A^S)$.

(G2) For each $S \in 2^N$, if $\mathcal{B} \subset 2^S$ and $|\mathcal{B}| < |S|$, then $m_S \notin aff(\pi(\mathcal{B}))$.

A collection $\mathcal{B}$ of subsets of N is said to be *π-balanced with respect to P* if it satisfies the following two conditions:

(B1) $\mathcal{B} \subset 2^P$.

(B2) $m_P \in conv(\pi(\mathcal{B}))$.

A collection $\mathcal{B}$ of subsets of N is said to be *π-subbalanced with respect to (P, p)* if it satisfies the following two conditions:

(SB1) $\mathcal{B} \subset 2^P$.

(SB2) $conv(\pi(\mathcal{B})) \cap (m_P, m_{P \setminus \{p\}}] \neq \emptyset$

where $(m_P, m_{P\setminus\{p\}}] = \{(1-\lambda)m_P + \lambda m_{P\setminus\{p\}}; 0 < \lambda \leq 1\}$.

To formulate the multiple combinatorial Stokes' theorem and Sperner's lemma with balanced structure, we need to introduce the following further notations.

Let m and n be positive integers, let $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$, let $\pi : 2^N \to A^N$ where $A^N = \overline{a_1 \ldots a_n}$, and let $\mathcal{K}$ be an $(n-1)$-pseudomanifold. An *m-labelling* in $\mathcal{K}$ is a multiple set-valued map $\varphi : V(\mathcal{K}) \to (2^N)^m$ where $(2^N)^m$ is the Cartesian product $2^N \times \ldots \times 2^N$ of m factors. For each vertex v of $\mathcal{K}$, $\varphi(v)$ is an m-tuple $(\varphi_1(v), \ldots, \varphi_m(v))$ where each $\varphi_i(v) \in 2^N$. Given $\sigma \in \mathcal{K}$ and $f : \sigma \to M$, we shall use the notation $\varphi_f(\sigma)$ to denote the subcollection $\{\varphi_{f(v)}(v); v \in \sigma\}$ of 2^N. Let $p \in P \in 2^N$. If the collection $\varphi_f(\sigma)$ is π-balanced with respect to P or π-subbalanced with respect to (P, p), then we call the pair (σ, f) a *π-balanced pair with respect to P* or a *π-subbalanced pair with respect to (P, p)*. The pair (σ, f) is called a *boundary pair* if $\sigma \in \partial\mathcal{K}$. The set of all π-balanced pairs with respect to N is denoted by $\mathcal{K}_\pi(\varphi)$ and the set of all π-subbalanced boundary pairs with respect to (N, n) is denoted by $\partial\mathcal{K}_\pi(\varphi)$. Suppose further that $\mathcal{K}$ is orientable. Let $(\mathcal{K}, \omega)$ be a coherently oriented $(n-1)$-pseudomanifold and let $\omega' = \varepsilon'[a_1, \ldots, a_n]$ ($\varepsilon' = \pm 1$) be an orientation on $A = \{a_1, \ldots, a_n\}$. We define the sets $\mathcal{K}_\pi^+(\varphi)$, $\mathcal{K}_\pi^-(\varphi)$, $\partial\mathcal{K}_\pi^+(\varphi)$, and $\partial\mathcal{K}_\pi^-(\varphi)$ as follows. For a given $(n-1)$-simplex $\sigma = \{v_1, \ldots, v_n\}$ and a map $f : \sigma \to M$. Let $\omega(\sigma) = \varepsilon[v_1, \ldots, v_n]$ $(\varepsilon = \pm 1)$ and let

$$\pi(\varphi_{f(v_i)}(v_i)) = \sum_{j=1}^{n} \beta_{ij} a_j \quad (\sum_{j=1}^{n} \beta_{ij} = 1) \quad \text{for} \quad i = 1, \ldots, n. \tag{2.2}$$

We call the pair (σ, f) a *positive pair* or a *negative pair* if $\varepsilon\varepsilon' detB > 0$ or $\varepsilon\varepsilon' detB < 0$ respectively, where B is the $n \times n$ matrix (β_{ij}). The sets of all positive and negative pairs of $\mathcal{K}_\pi(\varphi)$ are denoted by $\mathcal{K}_\pi^+(\varphi)$ and $\mathcal{K}_\pi^-(\varphi)$ respectively. For a given boundary $(n-2)$-simplex $\tau = \{v_1, \ldots, v_{n-1}\}$ and a map $g : \tau \to M$. Let σ be the unique $(n-1)$-simplex of $\mathcal{K}$ containing τ, let $\omega(\sigma)$ induce the orientation $\varepsilon[v_1, \ldots, v_{n-1}]$ $(\varepsilon = \pm 1)$ on τ and let

$$\pi(\varphi_{g(v_i)}(v_i)) = \sum_{j=1}^{n} \gamma_{ij} a_j \quad (\sum_{j=1}^{n} \gamma_{ij} = 1) \quad \text{for} \quad i = 1, \ldots, n-1. \tag{2.3}$$

We call the pair (τ, g) a *positively boundary pair* or a *negatively boundary pair* if $\varepsilon\varepsilon' detC > 0$ or $\varepsilon\varepsilon' detC < 0$ respectively, where C is the $(n-1) \times (n-1)$ matrix of the first $n-1$ columns of the $(n-1) \times n$ matrix (γ_{ij}). The sets of all positively and negatively boundary pairs of $\partial\mathcal{K}_\pi(\varphi)$ are denoted by $\partial\mathcal{K}_\pi^+(\varphi)$ and $\partial\mathcal{K}_\pi^-(\varphi)$ respectively. Given any set Ω of pairs (σ, f) where $\sigma \in \mathcal{K}$ and $f : \sigma \to M$, the set of all pairs (σ, f) of Ω such that f is one-to-one is denoted by Ω_*. For example, we have $\partial\mathcal{K}_\pi^-(\varphi)_* = \{(\sigma, f) \in \partial\mathcal{K}_\pi^-(\varphi); f \text{ is one-to-one}\}$.

Let T be a triangulation of an $(n-1)$-simplex $A^N = \overline{a_1 \ldots a_n}$, let $\varphi : V(T) \to (2^N)^m$ and let $\pi : 2^N \to A^N$, where m and n are positive integers, $N = \{1, \ldots, n\}$, and where $A = \{a_1, \ldots, a_n\}$ is an affinely independent set. Let $\widetilde{T}$ be the vertex scheme of T. Then φ is an m-labelling in the $(n-1)$-pseudomanifold $\widetilde{T}$. Let $M = \{1, \ldots, m\}$. Given a $(k-1)$-simplex s of T and a map $f : \sigma \to M$ where σ is the vertex set of s. The pair (s, f) is said to be *k-labelled* under (φ, π) if

(L1) there exists a $P \in 2^N$ such that $|P| = k$, $s \subset A^P$, and $\varphi_f(\sigma)$ is π-balanced with respect to P.

The pair (s, f) is said to be *fixed* under (φ, π) if

(F1) $\pi(\varphi_f(\sigma)) \subset aff(\sigma)$.

Let (s, f) be a fixed pair under (φ, π), let $s = \overline{v_1 \ldots v_k}$, and let

$$\pi(\varphi_{f(v_i)}(v_i)) = \sum_{j=1}^{k} \lambda_{ij} v_j \quad (\sum_{j=1}^{k} \lambda_{ij} = 1) \quad \text{for} \quad i = 1, \ldots, k. \tag{2.4}$$

We call the pair (s, f) a *positively fixed pair* or a *negatively fixed pair* under (φ, π) if $det(\lambda_{ij}) > 0$ or $det(\lambda_{ij}) < 0$ respectively. For each $P \in 2^N$, the set of all pairs

(s, f) such that $s \subset A^P$ and $f : \sigma \to M$, where $s \in T$ and σ is the vertex set of s, is denoted by H^P. We define H_*^P to be the set of all pairs $(s, f) \in H^P$ such that f is one-to-one. Then the number of positively fixed k-labelled pairs minus the number of negatively fixed k-labelled pairs, in H^P or H_*^P, under (φ, π) is denoted by φ_k^P or φ_{k*}^P respectively.

3. Balancedness and General Position Maps

We shall list some basic properties of general position maps, π-balanced and π-subbalanced collections here, for their detail proofs please see [13].

It follows from (**G1**) that

(Π1) general position maps are one-to-one.

(**B1**), (**B2**), (**SB1**) and (**SB2**) give that

(Π2) if $\mathcal{B} \subset 2^N$ is π-subbalanced with respect to (P, p) then $\mathcal{B} \cup \{\{p\}\}$ is π-balanced with respect to P.

Let $\pi_0 : 2^N \to A^N$ be a Shapley map. From (**S1**), (**G1**) and (**G2**), it follows that

(Π3) for each $\varepsilon > 0$ there exists a general position map $\pi : 2^N \to A^N$ such that $\max\limits_{S \in 2^N} \|\pi(S) - \pi_0(S)\| < \varepsilon$ where $\|\cdot\|$ is the Euclidean norm;

when $\varepsilon > 0$ is taken small enough, we have the following additional property:

(Π4) if $P \in 2^N$ and if $\mathcal{B} \subset 2^N$ is π-balanced with respect to P, then $\mathcal{B}$ is π_0-balanced with respect to P.

It follows from (**G1**), (**G2**), (**B1**), (**B2**), (**SB1**), (**SB2**) and Carathéodory theorem that *the following* $(\Pi 5) \sim (\Pi 8)$ *are always true under the condition that* $\pi : 2^N \to A^N$ *is a general position map.*

(Π5) A minimal π-balanced collection $\mathcal{B} \subset 2^N$ with respect to $P \in 2^N$ is of cardinality $|P|$ and the set $\pi(\mathcal{B})$ spans a $(|P| - 1)$-simplex, moreover, the barycenter m_P of A^P is contained in the open simplex $Int(conv(\pi(\mathcal{B})))$.

(Π6) A minimal π-subbalanced collection $\mathcal{B} \subset 2^N$ with respect to (P, p) ($p \in P \in 2^N$ with $|P| \geq 2$) is of cardinality $|P| - 1$ and the set $\pi(\mathcal{B})$ spans a $(|P| - 2)$-simplex, moreover, the set $(m_p, m_{P \setminus \{p\}}] \cap conv(\pi(\mathcal{B}))$ is a singleton which is contained in the open simplex $Int(conv(\pi(\mathcal{B})))$.

(Π7) If $p \in P \in 2^N$ ($|P| \geq 2$), if $\mathcal{B} \subset 2^N$ is π-balanced with respect to P, and if $|\mathcal{B}| = |P|$, then there is a unique subcollection $\mathcal{B}_1$ of $\mathcal{B}$ such that $\mathcal{B}_1$ is π-subbalanced with respect to (P, p), and $|\mathcal{B}_1| = |P| - 1$.

(Π8) If $p \in P \in 2^N$ ($|P| \geq 2$), if $\mathcal{B} \subset 2^N$ is π-subbalanced with respect to (P, p) but is not π-balanced with respect to P, and if $|\mathcal{B}| = |P|$, then there are exactly two subcollections $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}$ such that they are π-subbalanced with respect to (P, p), and $|\mathcal{B}_1| = |\mathcal{B}_2| = |P| - 1$.

We conclude this section by proving the following $(\Pi 9)$ and $(\Pi 10)$.

Let $N = \{1, \ldots, n\}$ $(n \geq 2)$, $A = \{a_i\}_{i \in N}$ an affinely independent set, $\pi : 2^N \to A^N$ a general position map, $\mathcal{B} = \{S_1, \ldots, S_n\} \subset 2^N$, and

$$\pi(S_i) = \sum_{j=1}^{n} \beta_{ij} a_j \quad (\sum_{j=1}^{n} \beta_{ij} = 1) \quad \text{for } i = 1, \ldots, n. \tag{3.1}$$

(Π9) If $\mathcal{B}$ is π-balanced with respect to N and $\mathcal{B}_1 = \{S_1, \ldots, S_{n-1}\}$ is π-subbalanced with respect to (N, n) then

$$(detB)(detB_1) > 0 \tag{3.2}$$

where B is the $n \times n$ matrix (β_{ij}) and where B_1 is the $(n-1) \times (n-1)$ matrix obtained by deleting the nth row and the nth column of (β_{ij}).

(Π10) If $\mathcal{B}_1 = \mathcal{B} \setminus \{S_n\}$ and $\mathcal{B}_2 = \mathcal{B} \setminus \{S_{n-1}\}$ are π-subbalanced with respect to (N, n) then

$$(detB_1)(detB_2) > 0 \tag{3.3}$$

where B_1 is the same in (3.2) and B_2 is the $(n-1) \times (n-1)$ matrix obtained by deleting the $(n-1)$th row and the nth column of (β_{ij}).

We now begin to prove (Π9) and (Π10). Let $E = \{e_1, \ldots, e_n\}$ be the standard basis of the Euclidean n-space R^n. The *affine map* $F : aff(A) \to aff(E)$ defined by $F(\sum_{i=1}^{n} \lambda_i a_i) = \sum_{i=1}^{n} \lambda_i e_i$ $(\sum_{i=1}^{n} \lambda_i = 1)$ preserves affine combinations and affinely independent sets, moreover, since the origin of R^n is not contained in $aff(E)$, we have

(I1) a subset of $aff(E)$ is affinely independent if and only if it is linearly independent.

Thus, by taking an affine map if necessary, we may assume that $A = E$ and $a_1 = e_1, \ldots, a_n = e_n$. It follows that

$$detB = det(\pi(S_1), \ldots, \pi(S_n)), \tag{3.4}$$

$$detB_1 = det(\pi(S_1), \ldots, \pi(S_{n-1}), a_n), \text{ and} \tag{3.5}$$

$$detB_2 = det(\pi(S_1), \ldots, \pi(S_{n-2}), \pi(S_n), a_n). \tag{3.6}$$

Here, $det(\pi(S_1),\ldots,\pi(S_n))$ means that if we express $\pi(S_1),\ldots,\pi(S_n)$ in terms of $e_1,\ldots,e_n$, that is, by (3.1), $\pi(S_i)=\sum\limits_{j=1}^{n}\beta_{ij}e_j=(\beta_{i_1},\ldots,\beta_{i_n})$ $(i=1,\ldots,n)$, then $det(\pi(S_1),\ldots,\pi(S_n))=det(\beta_{ij})$, and so on.

To prove (3.2), let us assume that $\mathcal{B}$ is π-balanced with respect to N and $\mathcal{B}_1$ is π-subbalanced with respect to (N,n). By (Π5) and **(I1)**, $\pi(\mathcal{B})$ is linearly independent, so that

$$det(\pi(S_1),\ldots,\pi(S_n))\neq 0. \tag{3.7}$$

(Π5) also shows that $m_N\in Int(conv(\pi(\beta)))$, so that

$$m_N=\sum_{i=1}^{n}\lambda_i\pi(S_i) \text{ where } \sum_{i=1}^{n}\lambda_i=1 \text{ and each } \lambda_i>0. \tag{3.8}$$

Similarly, by (Π6), we have

$$\alpha_1 m_{N\setminus\{n\}}+(1-\alpha_1)m_N=\sum_{i=1}^{n-1}\mu_i\pi(S_i) \text{ where } 0<\alpha_1\leq 1 \text{ and each } \mu_i>0. \tag{3.9}$$

From (3.8) and (3.9) , it follows that

$$a_n=\sum_{i=1}^{n-1}(\lambda_i(1+\frac{n-1}{\alpha_1})-\mu_i(\frac{n-1}{\alpha_1}))\pi(S_i)+\lambda_n(1+\frac{n-1}{\alpha_1})\pi(S_n). \tag{3.10}$$

By (3.4), (3.5) and (3.10) we have

$$detB_1=\lambda_n(1+\frac{n-1}{\alpha_1})detB. \tag{3.11}$$

From (3.4), (3.7) and (3.11), (3.2) follows.

To see (3.3), let $\mathcal{B}_1$ and $\mathcal{B}_2$ be π-subbalanced with respect to (N,n). By (Π2) , the collection $\mathcal{B}_1\cup\{\{n\}\}$ is π-balanced with respect to N, if we replace $\mathcal{B}$ in (Π9) by $\mathcal{B}_1\cup\{\{n\}\}$, then (3.2) implies that

$$detB_1\neq 0. \tag{3.12}$$

From (Π6), it follows that (3.9) and the following (3.13) hold:

$$\begin{aligned}&\alpha_2 m_{N\setminus\{n\}}+(1-\alpha_2)m_N\\ &=\sum_{i=1}^{n-2}\mu_i'\pi(S_i)+\mu_n'\pi(S_n) \text{ where } 0<\alpha_2\leq 1 \text{ and } \mu_n'>0\end{aligned} \tag{3.13}$$

By substituting

$$m_N = \frac{n-1}{n} m_{N\setminus\{n\}} + \frac{1}{n} a_n$$

into the above (3.9) and (3.13), we obtain two equations without the term of m_N. If we use these two equations to eliminating $m_{N\setminus\{n\}}$, then we have

$$\begin{aligned} \delta_2 \mu'_n \pi(S_n) &= \sum_{i=1}^{n-2} (\delta_1 \mu_i - \delta_2 \mu'_i) \pi(S_i) + \delta_1 \mu_{n-1} \pi(S_{n-1}) \\ &\quad + \left\{ \frac{\delta_2(1-\alpha_2)}{n} - \frac{\delta_1(1-\alpha_1)}{n} \right\} a_n \end{aligned} \tag{3.14}$$

where

$$\delta_i = \{\alpha_i + (1-\alpha_i)\frac{n-1}{n}\}^{-1} \quad (i = 1, 2).$$

(3.5), (3.6) and (3.14) imply that

$$det B_2 = \frac{\delta_1 \mu_{n-1}}{\delta_2 \mu'_n} det B_1. \tag{3.15}$$

As δ_1, δ_2, μ_{n-1} and μ'_n are positive, (3.3) follows from (3.12) and (3.15).

4. Multiple Combinatorial Stokes' Theorem with Balanced Structure

In 1974, Kuhn [6] gave a constructive proof of the fundamental theorem of algebra based on the combinatorial Stokes' Theorem which is a generalization of the celebrated Sperner's lemma [14]. Under the considerations of multiple labellings and balanced structures, Shih and Lee [13] established a combinatorial formula as a generalized Sperner's lemma that is a unification of the results of Shapley [11](balanced version) and Bapat [1](multiple version). The following multiple combinatorial Stokes' theorem with balanced structure generalizes the above formula.

Theorem 1. *Let φ be an m-labelling in an $(n-1)$-pseudomanifold $\mathcal{K}$ where m and n are positive integers with $n \geq 2$, and let $\pi : 2^N \to A^N$ be a general position map where $N = \{1, \ldots, n\}$ and $A = \{a_1, \ldots, a_n\}$ is an affinely independent set. Then*

$$|\mathcal{K}_\pi(\varphi)| \equiv m |\partial \mathcal{K}_\pi(\varphi)| \quad \textit{(mod 2)} \tag{4.1}$$

and

$$|\mathcal{K}_\pi(\varphi)_*| \equiv (m-n+1) |\partial \mathcal{K}_\pi(\varphi)_*| \quad \textit{(mod 2).} \tag{4.2}$$

Suppose further, $(\mathcal{K}, \omega)$ *is a coherently oriented* $(n-1)$*-pseudomanifold, then*

$$(-1)^{n-1}\{|\mathcal{K}_\pi^+(\varphi)| - |\mathcal{K}_\pi^-(\varphi)|\} = m\{|\partial\mathcal{K}_\pi^+(\varphi)| - |\partial\mathcal{K}_\pi^-(\varphi)|\}, \tag{4.3}$$

and

$$(-1)^{n-1}\{|\mathcal{K}_\pi^+(\varphi)_*| - |\mathcal{K}_\pi^-(\varphi)_*|\} = (m-n+1)\{|\partial\mathcal{K}_\pi^+(\varphi)_*| - |\partial\mathcal{K}_\pi^-(\varphi)_*|\}, \tag{4.4}$$

that is, (4.3) *and* (4.4) *are independent of the choices of the orientations* $\varepsilon'[a_1, \ldots, a_n]$ $(\varepsilon' = \pm 1)$ *on* A*.*

The idea of the following proof of Theorem 1 is by counting the sums of incidences between two sets D and R in two different ways.

Let $M = \{1, \ldots, m\}$ as before. Let

$$D = \{(\tau, g) \mid \tau \in \mathcal{K},\ |\tau| = n-1,\ g : \tau \to M\} \tag{4.5}$$

and

$$R = \{(\sigma, f) \mid \sigma \in \mathcal{K},\ |\sigma| = n,\ f : \sigma \to M\}. \tag{4.6}$$

By our definition as above, we have

$$D_* = \{(\tau, g) \in D \mid g \text{ is one-to-one }\} \tag{4.7}$$

and

$$R_* = \{(\sigma, f) \in R \mid f \text{ is one-to-one }\}. \tag{4.8}$$

Define an incidence relation $\prec$ from D into R by $(\tau, g) \prec (\sigma, f)$ if and only if the following three conditions are satisfied:

(R1) $(\tau, g) \in D$ and $(\sigma, f) \in R$,

(R2) $\varphi_g(\tau)$ is π-subbalanced with respect to (N, n),

(R3) $\tau \subset \sigma$ and $g = f\,|_\tau$ (the restriction of f to τ).

Put

$D_1 = \{(\tau, g) \in D \mid \tau \in \partial\mathcal{K}$ and $\varphi_g(\tau)$ is π-subbalanced with respect to $(N, n)\}$,

$D_2 = \{(\tau, g) \in D \mid \tau \notin \partial\mathcal{K}$ and $\varphi_g(\tau)$ is π-subbalanced with respect to $(N, n)\}$,

$D_3 = \{(\tau, g) \in D \mid \varphi_g(\tau)$ is not π-subbalanced with respect to $(N, n)\}$,

$R_1 = \{(\sigma, f) \in R \mid \varphi_f(\sigma)$ is π-balanced with respect to $N\}$,

$R_2 = \{(\sigma, f) \in R \mid \varphi_f(\sigma)$ is π-subbalanced with respect to (N, n) but not π-balanced with respect to N $\}$,

$R_3 = \{(\sigma, f) \in R \mid \varphi_f(\sigma)$ is not π-subbalanced with respect to $(N, n)\}$.

It is clear that

$$\{D_1, D_2, D_3\} \quad \text{partitions} \quad D$$

and

$$\{D_{1*}, D_{2*}, D_{3*}\} \quad \text{partitions} \quad D_*.$$

From (Π1), (Π7) and (4.6), it follows that if $\varphi_f(\sigma)$ is π-balanced with respect to N then it is π-subbalanced with respect to (N, n), so that

$$\{R_1, R_2, R_3\} \quad \text{partitions} \quad R$$

and

$$\{R_{1*}, R_{2*}, R_{3*}\} \quad \text{partitions} \quad R_*.$$

Let

$$D_r \quad = \quad \{d \in D \mid d \prec r\} \quad (r \in R) \tag{4.9}$$

and

$$R_d \quad = \quad \{r \in R \mid d \prec r\} \quad (d \in D). \tag{4.10}$$

We claim that

$$|D_r| = 1 \ \ (r \in R_1), \tag{4.11}$$

$$|D_r| = 2 \ \ (r \in R_2), \tag{4.12}$$

$$|D_r| = 0 \ \ (r \in R_3), \tag{4.13}$$

and

$$|D_{r*}| = 1 \ \ (r \in R_{1*}), \tag{4.14}$$

$$|D_{r*}| = 2 \ \ (r \in R_{2*}), \tag{4.15}$$

$$|D_{r*}| = 0 \ \ (r \in R_{3*}), \tag{4.16}$$

To see (4.11) $\sim$ (4.13), let us fix $r = (\sigma, f) \in R$, By (4.6), we have

$$|\varphi_f(\sigma)| = |\{\varphi_{f(v)}(v); v \in \sigma\}| \leq |\sigma| = n. \tag{4.17}$$

Let

$$\begin{aligned} \sigma &= \{v_1, \ldots, v_n\},\ \tau_1 = \sigma \setminus \{v_n\},\ \tau_2 = \sigma \setminus \{v_{n-1}\}, \\ g_1 &= f \mid \tau_1 \text{ and } g_2 = f \mid \tau_2. \end{aligned} \tag{4.18}$$

Then, by (**K1**), $\tau_1 \in \mathcal{K}$ and $\tau_2 \in \mathcal{K}$, so that, by (4.5), (4.6) and (4.18),

$$(\tau_1, g_1) \in D \quad \text{and} \quad (\tau_2, g_2) \in D. \tag{4.19}$$

Case 1. $r = (\sigma, f) \in R_1$. By $(\Pi 5)$ and (4.17), we have

$$|\varphi_f(\sigma)| = n,$$

so that, by $(\Pi 7)$ and interchanging the indices of the elements of σ if necessary, we may assume that

$$\begin{aligned}&\varphi_{g_1}(\tau_1) \text{ is the unique } \pi- \text{ subbalanced subcollection of } \varphi_f(\sigma)\\ &\text{with respect to } (N, n).\end{aligned} \tag{4.20}$$

Hence, by (4.18), (4.19), (4.20), (**R1**), (**R2**) and (**R3**), we have

$$D_r = \{(\tau_1, g_1)\} \quad (r \in R_1),$$

and (4.11) follows.

Case 2. $r = (\sigma, f) \in R_2$. By $(\Pi 6)$ and (4.17), we have

$$|\varphi_f(\sigma)| = n - 1 \text{ or } n.$$

We discuss two subcases separately.

Case 2.1. $|\varphi_f(\sigma)| = n - 1$. Then by $(\Pi 6)$,

$$\varphi_f(\sigma) \text{ is a minimal } \pi\text{-subbalanced with respect to } (N, n). \tag{4.21}$$

Since

$$\varphi_f(\sigma) = \{\varphi_{f(v_1)}(v_1), \ldots, \varphi_{f(v_n)}(v_n)\}$$

and $|\varphi_f(\sigma)| = n - 1$, we may assume, by interchanging the indices of the elements of σ if necessary, that

$$\varphi_{f(v_{n-1})}(v_{n-1}) = \varphi_{f(v_n)}(v_n),$$

so that

$$\varphi_{g_1}(\tau_1) = \varphi_{g_2}(\tau_2) = \varphi_f(\sigma). \tag{4.22}$$

Hence, by (4.9), (4.18), (4.19), (4.21), (4.22), (**R1**), (**R2**) and (**R3**), we have

$$D_r = \{(\tau_1, g_1), (\tau_2, g_2)\}. \tag{4.23}$$

Case 2.2. $|\varphi_f(\sigma)| = n$. Then by $(\Pi 1)$ and $(\Pi 8)$,

(4.24) $\varphi_f(\sigma)$ contains exactly two minimal π-subbalanced subcollections

$\{\varphi_{f(v)}(v); v \in \tau_1\}$ and $\{\varphi_{f(v)}(v); v \in \tau_2\}$ with respect to (N, n) for some $\tau_1 \subset \sigma$ and $\tau_2 \subset \sigma$, where

(4.25) $$|\tau_1| = |\tau_2| = n - 1 \text{ and } \tau_1 \neq \tau_2.$$

We have

$$n = |\sigma| \geq |\tau_1 \cup \tau_2| = |\tau_1| + |\tau_2| - |\tau_1 \cap \tau_2| = 2(n-1) - |\tau_1 \cap \tau_2|,$$

so that

(4.26) $$|\tau_1 \cap \tau_2| \geq n - 2.$$

(4.25) and (4.26) implies that

(4.27) $$|\tau_1 \cap \tau_2| = n - 2.$$

By (4.27), We may assume, without loss of generality, that

(4.28) $$\tau_1 = \sigma \setminus \{v_n\} \text{ and } \tau_2 = \sigma \setminus \{v_{n-1}\}.$$

Hence by (4.9), (4.18), (4.19), (4.24), (**R1**), (**R2**) and (**R3**), (4.23) also holds for this subcase, and (4.12) is true.

Case 3. $r = (\sigma, f) \in R_3$. $\varphi_f(\sigma)$ contains no π-subbalanced subcollection with respect to (N, n), so that, by (4.9) and (**R2**)

(4.29) $$D_r = \emptyset \quad (r \in R_3),$$

and (4.13) follows. This proves that (4.11), (4.12) and (4.13) are true, and by the same argument, so are (4.14), (4.15) and (4.16). We next claim that

(4.30) $$|R_d| = m(d \in D_1),$$

(4.31) $$|R_d| = 2m(d \in D_2),$$

(4.32) $$|R_d| = 0(d \in D_3).$$

and in case $m \geq n$,

(4.33) $$|R_{d*}| = m - n + 1(d \in D_{1*}),$$

(4.34) $$|R_{d*}| = 2(m - n + 1)(d \in D_{2*}),$$

(4.35) $$|R_{d*}| = 0(d \in D_{3*}).$$

To see (4.30) $\sim$ (4.32) , let us fix $d = (\tau, g) \in D$. By (4.5), $|\tau| = n-1$, we may write

$$\tau = \{v_1, \ldots, v_{n-1}\}. \tag{4.36}$$

Case 1'. $d = (\tau, g) \in D_1$. Then $\tau \in \partial\mathcal{K}$, so that τ is a face of exactly one $(n-1)$-simplex σ of $\mathcal{K}$, say

$$\sigma = \{v_1, \ldots, v_n\}. \tag{4.37}$$

Since $M = \{1, \ldots, m\}$, there are exactly m extensions $f_1, \ldots, f_m$ of g to the set σ into M, where

$$f_k(v_j) = \begin{cases} g(v_j), & \text{if } j = 1, \ldots, n-1 \\ \quad k, & \text{if } j = n \end{cases} \tag{4.38}$$

for $k = 1, \ldots, m$. By (4.10), (4.36), (4.37), (4.38), (**R1**), (**R2**) and (**R3**),

$$R_d = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \tag{4.39}$$

and (4.30) follows.

Case 2'. $d = (\tau, g) \in D_2$. Then $\tau \notin \partial\mathcal{K}$, so that, by (**M1**) and (**M2**), τ is a face of exactly two distinct $(n-1)$-simplexes σ and σ' of $\mathcal{K}$, say,

$$\sigma = \tau \cup \{v_n\} \text{ and } \sigma' = \tau \cup \{v_n'\}. \tag{4.40}$$

For each $k = 1, \ldots, m$, let $f_k : \sigma \to M$ and $f_k' : \sigma' \to M$ be such that

$$f_k|_\tau = f_k'|_\tau = g \text{ and } f_k(v_n) = f_k'(v_n') = k. \tag{4.41}$$

From (4.10), (4.36), (4.40), (4.41), (**R1**), (**R2**) and (**R3**), it follows that

$$R_d = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \cup \{(\sigma', f_1'), \ldots, (\sigma', f_m')\}, \tag{4.42}$$

and (4.31) follows.

Case 3'. $d = (\tau, g) \in D_3$. Then $\varphi_g(\tau)$ is not π-subbalanced with respect to (N, n), so that by (4.10) and (**R2**),

$$R_d = \emptyset, \tag{4.43}$$

and (4.32) follows. This prove (4.30), (4.31) and (4.32). To see (4.33)$\sim$ (4.35) let us assume that $m \geq n$ and fix $d = (\tau, g) \in D_*$. Since g is one-to-one, so that

$$|g(\tau)| = |\tau| = n-1.$$

If $d = (\tau, g) \in D_{1*}$, then there are exactly $m - n + 1$ injective extensions of g to σ into M, namely,

$$R_{d*} = \{(\sigma, f_k) \mid k \in M \setminus g(\tau)\} \quad (d \in D_{1*})$$

where τ, σ, and f_k are the same as in (4.36), (4.37) and (4.38) respectively. Similarly, if we define f_k and f'_k as in (4.41), then we have

$$R_{d*} = \{(\sigma, f_k) \mid k \in M \setminus g(\tau)\} \cup \{(\sigma', f'_k); k \in M \setminus g(\tau)\} \quad (d \in D_{2*}).$$

It is clear that

$$R_{d*} = \emptyset \quad (d \in D_{3*}).$$

This prove (4.33), (4.34) and (4.35). We now claim that

$$R_1 = \mathcal{K}_\pi(\varphi), \tag{4.44}$$

$$D_1 = \partial\mathcal{K}_\pi(\varphi), \tag{4.45}$$

and

$$R_{1*} = \mathcal{K}_\pi(\varphi)_*, \tag{4.46}$$

$$D_{1*} = \partial\mathcal{K}_\pi(\varphi)_*. \tag{4.47}$$

For any pair (σ, f), it follows from π is a general position map that the following $(a) \sim (f)$ are equivalent.

(*a*) $(\sigma, f) \in \mathcal{K}_\pi(\varphi)$.

(*b*) (σ, f) is a π-balanced pair with respect to N.

(*c*) $\varphi_f(\sigma)$ is a π-balanced collection with respect to N.

(*d*) $\sigma \in \mathcal{K}$, $|\sigma| = n$, $f : \sigma \to M$, $\varphi_f(\sigma)$ is a π-balanced with respect to N.

(*e*) $(\sigma, f) \in R$, $\varphi_f(\sigma)$ is a π-balanced with respect to N.

(*f*) $(\sigma, f) \in R_1$

Thus (4.44) holds. Similarly, for any pair (τ, g), the following $(a)' \sim (f)'$ are equivalent.

(*a′*) $(\tau, g) \in \partial\mathcal{K}_\pi(\varphi)$.

(*b′*) (τ, g) is a π-subbalanced boundary pair with respect to (N, n).

(*c′*) $\varphi_g(\tau)$ is a π-subbalanced collection with respect to (N, n), $\tau \in \partial\mathcal{K}$.

(*d′*) $\tau \in \mathcal{K}$, $|\tau| = n - 1$, $g : \tau \to M$, $\varphi_g(\tau)$ is a π-subbalanced with respect to (N, n), $\tau \in \partial\mathcal{K}$.

(*e′*) $(\tau, g) \in D$, $\varphi_g(\tau)$ is a π-subbalanced with respect to (N, n), $\tau \in \partial\mathcal{K}$.

(*f′*) $(\tau, g) \in D_1$

This shows that (4.45) is true. By the same reason, so are (4.46) and (4.47). Define λ:$D \times R \to \{0, 1\}$ by

$$\lambda(d,r) = \begin{cases} 1 & \text{if } d \prec r, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{r\in R}\sum_{d\in D}\lambda(d,r) &= \sum_{r\in R_1}\sum_{d\in D}\lambda(d,r) + \sum_{r\in R_2}\sum_{d\in D}\lambda(d,r) + \sum_{r\in R_3}\sum_{d\in D}\lambda(d,r) \\ &= \sum_{r\in R_1}|D_r| + \sum_{r\in R_2}|D_r| + \sum_{r\in R_3}|D_r| \\ &= \sum_{r\in R_1}1 + \sum_{r\in R_2}2 + \sum_{r\in R_3}0 \\ &= |R_1| + 2|R_2| + 0 \\ &= |\mathcal{K}_\pi(\varphi)| + 2|R_2| \end{aligned}$$

so that

$$\sum_{r\in R}\sum_{d\in D}\lambda(d,r) = |\mathcal{K}_\pi(\varphi)| + 2|R_2| \tag{4.48}$$

and

$$\begin{aligned} \sum_{d\in D}\sum_{r\in R}\lambda(d,r) &= \sum_{d\in D_1}\sum_{r\in R}\lambda(d,r) + \sum_{d\in D_2}\sum_{r\in R}\lambda(d,r) + \sum_{d\in D_3}\sum_{r\in R}\lambda(d,r) \\ &= \sum_{d\in D_1}|R_d| + \sum_{d\in D_2}|R_d| + \sum_{d\in D_3}|R_d| \\ &= \sum_{d\in D_1}m + \sum_{d\in D_2}2m + \sum_{d\in D_3}0 \\ &= m|D_1| + 2m|D_2| + 0 \\ &= m|\partial\mathcal{K}_\pi(\varphi)| + 2m|D_2| \end{aligned}$$

so that

$$\sum_{d\in D}\sum_{r\in R}\lambda(d,r) = m|\partial\mathcal{K}_\pi(\varphi)| + 2m|D_2| \tag{4.49}$$

(4.48) and (4.49) imply that

$$|\mathcal{K}_\pi(\varphi)| + 2|R_2| = m|\partial\mathcal{K}_\pi(\varphi)| + 2m|D_2|$$

this proves (4.1). Similarly, if $m \geq n$ then we have

(4.50) $$|\mathcal{K}_\pi(\varphi)_*| + 2|R_{2*}| = (m-n+1)|\partial\mathcal{K}_\pi(\varphi)_*| + 2(m-n+1)|D_{2*}|$$

Because of the injectivity, we see that

(4.51) $$\mathcal{K}_\pi(\varphi)_* = \emptyset \quad \text{if} \quad m < n \text{ and}$$

(4.52) $$\partial\mathcal{K}_\pi(\varphi)_* = \emptyset \quad \text{if} \quad m < n-1$$

so that both sides of (4.2) are zeros if $m < n$. Thus, (4.2) follows from (4.50), (4.51) and (4.52). This completes the proof of (4.2).

Suppose further, $(\mathcal{K}, \omega)$ is a coherently oriented $(n-1)$-pseudomanifold and $\omega' = \varepsilon'[a_1, \ldots, a_n]$ $(\varepsilon' = \pm 1)$. We claim that

(4.53) $$\{\mathcal{K}^+_\pi(\varphi), \mathcal{K}^-_\pi(\varphi)\} \text{ partitions } \mathcal{K}_\pi(\varphi),$$

(4.54) $$\{\partial\mathcal{K}^+_\pi(\varphi), \partial\mathcal{K}^-_\pi(\varphi)\} \text{ partitions } \partial\mathcal{K}_\pi(\varphi),$$

and

(4.55) $$\{\mathcal{K}^+_\pi(\varphi)_*, \mathcal{K}^-_\pi(\varphi)_*\} \text{ partitions } \mathcal{K}_\pi(\varphi)_*,$$

(4.56) $$\{\partial\mathcal{K}^+_\pi(\varphi)_*, \partial\mathcal{K}^-_\pi(\varphi)_*\} \text{ partitions } \partial\mathcal{K}_\pi(\varphi)_*.$$

Given $(\sigma, f) \in \mathcal{K}_\pi(\varphi)$ with $\omega(\sigma) = \varepsilon[v_1, \ldots, v_n]$ $(\varepsilon = \pm 1)$. Let

(4.57) $$S_i = \varphi_{f(v_i)}(v_i) \text{ for } i = 1, \ldots, n.$$

From (*a*), (*c*), (2.2), (3.1) and (4.57), it follows, by applying $(\Pi 9)$, that (3.2) holds, so that $detB \neq 0$, thus

$$(\sigma, f) \in \mathcal{K}^+_\pi(\varphi) \text{ if and only if } \varepsilon\varepsilon' detB > 0, \text{ and}$$

$$(\sigma, f) \in \mathcal{K}^-_\pi(\varphi) \text{ if and only if } \varepsilon\varepsilon' detB < 0.$$

This proves (4.53). To prove (4.54), let $(\tau, g) \in \partial\mathcal{K}_\pi(\varphi)$ and let σ be the unique $(n-1)$-simplex of $\mathcal{K}$ containing τ with $\omega(\sigma) = \varepsilon[v_1, \ldots, v_n]$ $(\varepsilon = \pm 1)$ and $\tau = \sigma \setminus \{v_n\}$. Then the induced orientation on τ from $\omega(\sigma)$ is

$$(-1)^{n-1}\varepsilon[v_1, \ldots, v_{n-1}].$$

Let

(4.58) $$S_i = \varphi_{g(v_i)}(v_i) \text{ for } i = 1, \ldots, n-1, \text{ and } S_n = \{n\}.$$

By (*a*)′, (*c*)′,$(\Pi 2)$ with $(P, p) = (N, n)$, $\{S_1, \ldots, S_n\}$ is π-balanced with respect to N, so that, by (2.3) and $(\Pi 9)$ with $B_1 = C$, $detB_1 \neq 0$, thus

(4.59) $$(\tau, g) \in \partial\mathcal{K}^+_\pi(\varphi) \text{ if and only if } (-1)^{n-1}\varepsilon\varepsilon' detB_1 > 0, \text{ and}$$

(4.60) $$(\tau, g) \in \partial\mathcal{K}^-_\pi(\varphi) \text{ if and only if } (-1)^{n-1}\varepsilon\varepsilon' detB_1 < 0.$$

This proves (4.54). By the same reason, (4.55) and (4.56) are also true. Let $(\tau, g) \prec (\sigma, f)$, $\omega(\sigma) = \varepsilon[v_1, \ldots, v_n]$ $(\varepsilon = \pm 1)$, and $\tau = \sigma \setminus \{v_n\}$ and (2.2) holds. we call (τ, g) ***positive*** or ***negative*** in (σ, f) if

$$(-1)^{n-1}\varepsilon\varepsilon' det B_1 > 0 \tag{4.61}$$

or

$$(-1)^{n-1}\varepsilon\varepsilon' det B_1 < 0 \tag{4.62}$$

respectively, where B_1 is the $(n-1) \times (n-1)$ matrix obtained by deleting the nth row and nth column of B in (2.2).
Put

$$D_r^+ = \{d \in D \mid d \prec r,\ \ d \text{ is positive in } r\} \quad (r \in R), \tag{4.63}$$

$$D_r^- = \{d \in D \mid d \prec r,\ \ d \text{ is negative in } r\} \quad (r \in R), \tag{4.64}$$

$$R_d^+ = \{r \in R \mid d \prec r,\ \ d \text{ is positive in } r\} \quad (d \in D), \tag{4.65}$$

$$R_d^- = \{r \in R \mid d \prec r,\ \ d \text{ is negative in } r\} \quad (d \in D). \tag{4.66}$$

We claim that

$$|R_d^+| = m \ \text{ and } \ |R_d^-| = 0 \quad (d \in \partial\mathcal{K}_\pi^+(\varphi)), \tag{4.67}$$

$$|R_d^+| = 0 \ \text{ and } \ |R_d^-| = m \quad (d \in \partial\mathcal{K}_\pi^-(\varphi)), \tag{4.68}$$

$$|R_d^+| = |R_d^-| = m \quad (d \in D_2), \tag{4.69}$$

$$|R_d^+| = |R_d^-| = 0 \quad (d \in D_3), \tag{4.70}$$

and if $m \geq n$ then

$$|R_{d*}^+| = m - n + 1 \ \text{ and } \ |R_{d*}^-| = 0 \quad (d \in \partial\mathcal{K}_\pi^+(\varphi)_*), \tag{4.71}$$

$$|R_{d*}^+| = 0 \ \text{ and } \ |R_{d*}^-| = m - n + 1 \quad (d \in \partial\mathcal{K}_\pi^-(\varphi)_*), \tag{4.72}$$

$$|R_{d*}^+| = |R_{d*}^-| = m - n + 1 \quad (d \in D_{2*}), \tag{4.73}$$

$$|R_{d*}^+| = |R_{d*}^-| = 0 \quad (d \in D_{3*}). \tag{4.74}$$

If $d = (\tau, g) \in \partial\mathcal{K}_\pi^+(\varphi)$ then, from (4.38), (4.39), (4.45), (4.54), (4.59), (4.61), (4.62), (4.65) and (4.66), it follows that

$$R_d^+ = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \text{ and } R_d^- = \emptyset \quad (d \in \partial\mathcal{K}_\pi^+(\varphi)).$$

This proves (4.67). If $d = (\tau, g) \in \partial\mathcal{K}_\pi^-(\varphi)$ then from (4.38), (4.39), (4.45), (4.54), (4.60), (4.61), (4.62), (4.65) and (4.66), it follows that

$$R_d^+ = \emptyset \text{ and } R_d^- = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \quad (d \in \partial\mathcal{K}_\pi^-(\varphi)).$$

This proves (4.68). If $d = (\tau, g) \in D_2$ then by (C2) and (4.40), $\omega(\sigma)$ and $\omega(\sigma')$ induce opposite orientations on τ, that is, we may assume

$$\omega(\sigma) = \varepsilon[v_1, \ldots, v_{n-1}, v_n] \text{ and } \omega(\sigma') = (-1)\varepsilon[v_1, \ldots, v_{n-1}, v_n'] \quad (\varepsilon = \pm 1),$$

so, by (4.41), (4.42), (4.61), (4.62), (4.65) and the fact that

$$(-1)^{n-1}\varepsilon\varepsilon' detB_1(-1)^{n-1}(-1)\varepsilon\varepsilon' detB_1 < 0,$$

we have the following statement:

R_d^+ is one of the two sets $\{(\sigma, f_1), \ldots, (\sigma, f_m)\}$ and $\{(\sigma', f_1'), \ldots, (\sigma', f_m')\}$ and R_d^- is the other one.

This proves (4.69). If $d = (\tau, g) \in D_3$ then by, (4.43), (4.65) and (4.66),

$$R_d^+ = R_d^- = \emptyset \quad (d \in D_3).$$

This proves (4.70). By the same reason, (4.71) $\sim$ (4.74) are also true. We finally claim that

(4.75) $\quad |D_r^+| = \dfrac{1+(-1)^{n-1}}{2}$ and $|D_r^-| = \dfrac{1-(-1)^{n-1}}{2} \quad (r \in \mathcal{K}_\pi^+(\varphi)),$

(4.76) $\quad |D_r^+| = \dfrac{1-(-1)^{n-1}}{2}$ and $|D_r^-| = \dfrac{1+(-1)^{n-1}}{2} \quad (r \in \mathcal{K}_\pi^-(\varphi)),$

(4.77) $\quad |D_r^+| = |D_r^-| = 1 \quad (r \in R_2),$

(4.78) $\quad |D_r^+| = |D_r^-| = 0 \quad (r \in R_3),$

and

(4.79) $\quad |D_{r*}^+| = \dfrac{1+(-1)^{n-1}}{2}$ and $|D_{r*}^-| = \dfrac{1-(-1)^{n-1}}{2} \quad (r \in \mathcal{K}_\pi^+(\varphi)_*),$

(4.80) $\quad |D_{r*}^+| = \dfrac{1-(-1)^{n-1}}{2}$ and $|D_{r*}^-| = \dfrac{1+(-1)^{n-1}}{2} \quad (r \in \mathcal{K}_\pi^-(\varphi)_*),$

(4.81) $\quad |D_{r*}^+| = |D_{r*}^-| = 1 \quad (r \in R_{2*}),$

(4.82) $\quad |D_{r*}^+| = |D_{r*}^-| = 0 \quad (r \in R_{3*}).$

Let $r = (\sigma, f)$ and $\omega(\sigma) = \varepsilon[v_1, \ldots, v_n]$ $(\varepsilon = \pm 1)$. If $r = (\sigma, f) \in \mathcal{K}_\pi^+(\varphi)$, then $\varepsilon\varepsilon' detB > 0$ and by $(\Pi 9)$ we have $\varepsilon\varepsilon' detB_1 > 0$, thus

$$(-1)^{n-1}\varepsilon\varepsilon' detB_1 > 0 \quad \text{if } n \text{ is odd},$$
$$(-1)^{n-1}\varepsilon\varepsilon' detB_1 < 0 \quad \text{if } n \text{ is even},$$

so that

$$D_r^+ = \{(\tau_1, g_1)\} \text{ and } D_r^- = \emptyset \quad (r \in \mathcal{K}_\pi^+(\varphi)) \text{ if } n \text{ is odd,}$$
$$D_r^+ = \emptyset \text{ and } D_r^- = \{(\tau_1, g_1)\} \quad (r \in \mathcal{K}_\pi^+(\varphi)) \text{ if } n \text{ is even.}$$

This proves (4.75). Similarly, we have

$$D_r^+ = \emptyset \text{ and } D_r^- = \{(\tau_1, g_1)\} \quad (r \in \mathcal{K}_\pi^-(\varphi)) \text{ if } n \text{ is odd,}$$
$$D_r^+ = \{(\tau_1, g_1)\} \text{ and } D_r^- = \emptyset \quad (r \in \mathcal{K}_\pi^-(\varphi)) \text{ if } n \text{ is even.}$$

This proves (4.76). If $r = (\sigma, f) \in R_2$ then by $(\Pi 10)$, (3.3), (4.18), (4.23) and (4.57), we have

$$\omega(\sigma) \text{ induces } (-1)^{n-1}\varepsilon[v_1, \ldots, v_{n-1}] \text{ on } \tau_1,$$
$$\omega(\sigma) \text{ induces } (-1)^{n-2}\varepsilon[v_1, \ldots, v_{n-2}, v_n] \text{ on } \tau_2,$$
$$(-1)^{n-1}\varepsilon\varepsilon' detB_1(-1)^{n-2}\varepsilon\varepsilon' detB_2 < 0,$$

so that, one of the two pairs (τ_1, g_1) and (τ_2, g_2) is positive in r and the other one is negative in r, thus (4.77) is true. If $r = (\sigma, f) \in R_3$ then $D_r^+ = D_r^- = \emptyset$, this proves (4.78). By the same reason, (4.79) $\sim$ (4.82) are also true.
Define $\Lambda : D \times R \to \{-1, 0, 1\}$ by

$$\Lambda(d, r) = \begin{cases} 1, & \text{if } d \prec r \text{ and } d \text{ is positive in } r, \\ -1, & \text{if } d \prec r \text{ and } d \text{ is negative in } r, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\sum_{r \in R} \sum_{d \in D} \Lambda(d, r) \\ &= \sum_{r \in R} (|D_r^+| - |D_r^-|) \\ &= \Big(\sum_{r \in \mathcal{K}_\pi^+(\varphi)} + \sum_{r \in \mathcal{K}_\pi^-(\varphi)} + \sum_{r \in R_2} + \sum_{r \in R_3})(|D_r^+| - |D_r^-|\Big) \\ &= |\mathcal{K}_\pi^+(\varphi)| \Big\{ \frac{1 + (-1)^{n-1}}{2} - \frac{1 - (-1)^{n-1}}{2} \Big\} \\ &\quad + |\mathcal{K}_\pi^-(\varphi)| \Big\{ \frac{1 - (-1)^{n-1}}{2} - \frac{1 + (-1)^{n-1}}{2} \Big\} + |R_2|(1 - 1) + |R_3|(0 - 0) \\ &= (-1)^{n-1} \{ |\mathcal{K}_\pi^+(\varphi)| - |\mathcal{K}_\pi^-(\varphi)| \}, \end{aligned}$$

so that

$$\sum_{r \in R} \sum_{d \in D} \Lambda(d, r) = (-1)^{n-1} \{ |\mathcal{K}_\pi^+(\varphi)| - |\mathcal{K}_\pi^-(\varphi)| \} \tag{4.83}$$

and

$$
\begin{aligned}
&\sum_{d\in D}\sum_{r\in R}\Lambda(d,r)\\
&= \sum_{d\in D}(|R_d^+|-|R_d^-|)\\
&= \Big(\sum_{d\in\partial\mathcal{K}_\pi^+(\varphi)}+\sum_{d\in\partial\mathcal{K}_\pi^-(\varphi)}+\sum_{d\in D_2}+\sum_{d\in D_3}\Big)(|R_d^+|-|R_d^-|\Big)\\
&= |\partial\mathcal{K}_\pi^+(\varphi)|(m-0)+|\partial\mathcal{K}_\pi^-(\varphi)|(0-m)+|D_2|(m-m)+|D_3|(0-0)\\
&= m\{|\partial\mathcal{K}_\pi^+(\varphi)|-|\partial\mathcal{K}_\pi^-(\varphi)|\}
\end{aligned}
$$

so that

$$\sum_{d\in D}\sum_{r\in R}\Lambda(d,r)=m\{|\partial\mathcal{K}_\pi^+(\varphi)|-|\partial\mathcal{K}_\pi^-(\varphi)|\} \tag{4.84}$$

(4.83) and (4.84) imply (4.3). Similarly, if $m\geq n$ then (4.4) holds. It is clear that if $m<n$ then both sides of (4.4) are zeros. Thus (4.4) is true. This completes the proof of Theorem 1.

5. Multiple Combinatorial Sperner's Lemma with Balanced Structure

The following multiple Sperner's lemma with balanced structure is a consequence of Theorem 1.

Theorem 2. *Let T be a triangulation of an $(n-1)$-simplex $A^N=\overline{a_1\ldots a_n}$, let $\varphi: V(T)\to(2^N)^m$ and $\pi: 2^N\to A^N$, where m and n are positive integers, $N=\{1,\ldots,n\}$ and π is a general position map, such that*

(F2) $\varphi(V(T)\cap A^S)\subset(2^S)^m$ *for all* $S\in 2^N$.

Then, for each $P\in 2^N$, we have

$$\varphi^P_{|P|}=m^{|P|}, \tag{5.1}$$

and, if $m\geq|P|$, we have

$$\varphi^P_{|P|*}=\frac{m!}{(m-|P|)!}. \tag{5.2}$$

We shall apply Theorem 1 and inductive method to prove Theorem 2. The details are as follows.

Let $P \in 2^N$. Then $1 \leq |P| \leq n$. If $P = \{p\}$, a singleton, then $A^P = \overline{a_p} = \{a_p\}$, so that

$$H^P = \{(\{a_p\}, f_1), \ldots, (\{a_p\}, f_m)\}$$

where $f_i : \{a_p\} \to M$ $(M = \{1, \ldots, m\})$ is the function such that $f_i(a_p) = i$ for $i = 1, \ldots, m$. By **(F2)**, we have

$$\varphi(a_p) = (\{p\}, \ldots, \{p\}).$$

And since π is general position map, $\pi(\{p\}) = a_p$ it follows that

$$\pi(\varphi_{f_i(a_p)}(a_p)) = \pi(\varphi_i(a_p)) = \pi(\{p\}) = a_p,$$

which shows that $(\{a_p\}, f_i)$ is a positively fixed pair under (φ, π) for $i = 1, \ldots, m$, thus, we have

$$\varphi^P_{|P|} = |H^P| - 0 = m = m^{|P|} \quad (|P| = 1).$$

And since each f_i is one-to-one, we have $H^P_* = H^P$ then

$$\varphi^P_{|P|*} = |H^P_*| - 0 = m = \frac{m!}{(m - |P|)!} \quad (|P| = 1).$$

Thus the theorem holds for $|P| = 1$. Suppose now, $1 < |P| \leq n$, assume

(5.3) $$\varphi^S_{|S|} = m^{|S|} \text{ for all } S \in 2^N \text{ with } 1 \leq |S| < |P|$$

and

(5.4) $$\varphi^S_{|S|*} = \frac{m!}{(m - |S|)!} \text{ for all } S \in 2^N \text{ with } 1 \leq |S| < |P| \text{ if } m \geq |P|.$$

Fix $p \in P$. Let $P = \{n_1, \ldots, n_{|P|}\}$ where $n_{|P|} = p$. Then we have

(5.5) $$A^P = \overline{a_{n_1} \ldots a_{n_{|P|}}}.$$

Let

(5.6) $$K = \{s \in T \mid s \subset A^P\}.$$

It is well known, by **(T1)**, **(T2)**, **(T3)** and (5.6), that K is a triangulation of A^P with the vertex scheme

(5.7) $$\widetilde{K} = \{\sigma \mid \sigma \text{ spans } s \text{ for some } s \in K\}$$

that is,

$$s \in K \text{ if and only if } \sigma \in \widetilde{K} \quad (\sigma \text{ spans } s).$$

Let σ be a $(|P|-1)$–simplex of $\widetilde{K}$, σ spans s, we may write

$$\sigma = \{v_1, \ldots, v_{|P|}\} \in \widetilde{K}, \quad s = \overline{v_1 \ldots v_{|P|}} \in K. \tag{5.8}$$

Because of the dimension, by (5.5), (5.6) and (5.8),we have

$$aff(\sigma) \subset aff(s) \subset aff(A^P) \subset aff(\{a_{n_1}, \ldots, a_{n_{|P|}}\}) \subset aff(\sigma),$$

so that

$$aff(\sigma) = aff(A^P) \tag{5.9}$$

and

$$aff(\{v_1, \ldots, v_{|P|}\}) = aff(\{a_{n_1}, \ldots, a_{n_{|P|}}\}). \tag{5.10}$$

By (5.10), we may write

$$v_j = \sum_{k=1}^{|P|} \alpha_{jk} a_{n_k} \quad (\sum_{k=1}^{|P|} \alpha_{jk} = 1) \quad \text{for} \quad j = 1, \ldots, |P|, \tag{5.11}$$

and

$$a_{n_k} = \sum_{j=1}^{|P|} \alpha'_{kj} v_j \quad (\sum_{j=1}^{|P|} \alpha'_{kj} = 1) \quad \text{for} \quad k = 1, \ldots, |P|. \tag{5.12}$$

By (5.11), (5.12) and the affine independence of σ,

$$A' = A^{-1}$$

where A and A' are the $|P| \times |P|$ matrices (α_{jk}) and (α'_{kj}) in (5.11) and (5.12), respectively. Fix an orientation $\omega = \varepsilon'[a_{n_1}, \ldots, a_{n_{|P|}}]$ $(\varepsilon' = \pm 1)$. Then, as we mentioned before, $(\widetilde{K}, \omega)$ is an coherently oriented $(|P|-1)$-pseudomanifold, and by (5.8), (5.11) and compare with (2.1), the orientation $\omega(\sigma)$ of σ is given by $\omega(\sigma) = \varepsilon'[v_1, \ldots, v_{|P|}]$ or $\omega(\sigma) = (-1)\varepsilon'[v_1, \ldots, v_{|P|}]$ if $detA > 0$ or $detA < 0$ respectively, that is,

$$\omega(\sigma) = \varepsilon[v_1, \ldots, v_{|P|}] \quad (\varepsilon = \varepsilon' detA/|detA|). \tag{5.13}$$

For the given σ in (5.8), let $f : \sigma \to M$, By **(F2)** with $S = P$, we have

$$\varphi_{f(v_i)}(v_i) \subset P \text{ for } i = 1, \ldots, |P|, \tag{5.14}$$

so that, by (**G1**) and (5.14),

(5.15) $$\pi(\varphi_{f(v_i)}(v_i)) \in Int(A^{\varphi_{f(v_i)}(v_i)}) \subset A^P \text{ for } i = 1, \ldots, |P|,$$

thus, by (5.5) and (5.15), we may write

(5.16) $$\pi(\varphi_{f(v_i)}(v_i)) = \sum_{k=1}^{|P|} \beta_{ik} a_{n_k} \quad (\sum_{k=1}^{|P|} \beta_{ik} = 1) \text{ for } \quad i = 1, \ldots, |P|.$$

(5.12) and (5.16) implies that

(5.17) $$\pi(\varphi_{f(v_i)}(v_i)) = \sum_{j=1}^{|P|} \lambda_{ij} v_j \quad (\sum_{j=1}^{|P|} \lambda_{ij} = 1) \text{ for } \quad i = 1, \ldots, |P|,$$

where

$$\lambda_{ij} = \sum_{k=1}^{|P|} \beta_{ik} \alpha'_{kj} \quad \text{for} \quad i = 1, \ldots, |P| \text{ and } \quad j = 1, \ldots, |P|.$$

Thus $\Lambda = BA^{-1}$
where B and Λ are the $|P| \times |P|$ matrices (β_{ik}) and (λ_{ij}) in (5.16) and (5.17) respectively. Compare (2.2) with (5.16) and compare (2.4) with (5.17), we see that the following $(g) \sim (k)$ are equivalent.

(g) (σ, f) is a positive (resp. negative) pair.

(h) $\varepsilon\varepsilon' detB > 0$ (resp. < 0).

(i) $(\varepsilon' detA/|detA|)\varepsilon' det\Lambda detA > 0$ (resp. < 0).

(j) $det\Lambda > 0$ (resp. < 0).

(k) (s, f) is a positively (resp. negatively) fixed pair.

We claim that the following (l) and (m) are also equivalent.

(l) (σ, f) is π-balanced pair with respect to P.

(m) (s, f) is $|P|$-labelled under (φ, π).

If (l) holds, then $\varphi_f(\sigma)$ is π-balanced with respect to P and, by (5.6), (5.7) and (5.8), we have

$$s \subset A^P$$

so that, by (**L1**), (m) follows. Conversely, if (m) holds, then by (**L1**), there exists a $Q \in 2^N$ such that

(5.18) $$s \subset A^Q \text{ and } |Q| = |P|, \text{ and}$$

$\varphi_f(\sigma)$ is π-balanced with respect to Q, we have, by (5.18) and the affine independence of σ,

$$aff(\sigma) \subset aff(s) \subset aff(A^Q) \subset aff(\sigma)$$

thus

$$aff(\sigma) = aff(A^Q). \tag{5.19}$$

By (5.9), (5.18) and (5.19), we have

$$A^P = A^Q \quad \text{and} \quad |P| = |Q|. \tag{5.20}$$

Since a simplex determines its vertices, we have,by (5.20),

$$P = Q.$$

So $\varphi_f(\sigma)$ is π-balanced with respect to P and (l) holds. Recall that $\varphi^P_{|P|}$ is the number of positively fixed $|P|$-labelled pairs minus the number of negatively fixed $|P|$-labelled pairs under (φ, π) in H^P we have, by the equivalence of (g) and (k), by the equivalence of (l) and (m), and $s \subset A^P$, we have

$$\varphi^P_{|P|} = |\widetilde{K}^+_\pi(\varphi)| - |\widetilde{K}^-_\pi(\varphi)|, \tag{5.21}$$

and if $m \geq |P|$, by considering those injective f, we also have

$$\varphi^P_{|P|*} = |\widetilde{K}^+_\pi(\varphi)_*| - |\widetilde{K}^-_\pi(\varphi)_*|, \tag{5.22}$$

where *the balancedness in* (5.21) *and* (5.22) *is the* π*-balancedness with respect to* P.

To apply Theorem 1, we shall prove that

$$\varphi^{P\setminus\{p\}}_{|P|-1} = (-1)^{|P|-1}\{|\partial\widetilde{K}^+_\pi(\varphi)| - |\partial\widetilde{\mathcal{K}}^-_\pi(\varphi)|\} \tag{5.23}$$

and, if $m \geq |P|$,

$$\varphi^{P\setminus\{p\}}_{(|P|-1)*} = (-1)^{|P|-1}\{|\partial\widetilde{K}^+_\pi(\varphi)_*| - |\partial\widetilde{K}^-_\pi(\varphi)_*|\}, \tag{5.24}$$

where *the subbalancedness in* (5.23) *and* (5.24) *is the* π*-subbalancedness with respect to* (P, p).

Assume that $\tau \in \widetilde{K}$, $t \in K$ and τ spans t. And suppose $\tau \in \partial\widetilde{K}$ if and only if t is contained in some proper face of A^P or equivalently, $\tau \in \partial\widetilde{K}$ if and only if $t \subset A^Q$ for some $Q \subset P$ with $|Q| = |P| - 1$. We claim that the following (n) and (o) are equivalent.

(n) (t, g) is $(|P|-1)$-labelled under (φ, π) in $H^{P\setminus\{p\}}$.

(o) $(\tau, g) \in \partial\widetilde{K}_\pi(\varphi)$.

If (t, g) is $(|P|-1)$-labelled under (φ, π), then, by **(L1)**, there exists a $Q \in 2^N$ such that $|Q| = |P|-1$, $t \subset A^Q$ and $\varphi_g(\tau)$ is π-balabced with respect to Q, so that, by (Π1) and (Π5), we have

(5.25) $$|\tau| \geq |\varphi_g(\tau)| \geq |P|-1,$$

but $t \subset A^Q$ and $|Q| = |P|-1$ imply that

(5.26) $$|\tau| \leq |P|-1,$$

thus, (5.25) and (5.26) imply that

(5.27) $$|\tau| = |P|-1.$$

If (n) holds, then, by the definition of $H^{P\setminus\{p\}}$,

(5.28) $$t \subset A^{P\setminus\{p\}}$$

so that, by (5.27) and (5.28), we must have

(5.29) $$Q = P \setminus \{p\},$$

thus $\varphi_g(\tau)$ is π-balanced with respect to $P \setminus \{p\}$. Let $\mathcal{B} = \varphi_g(\tau)$. Replacing P by $P \setminus \{p\}$ in **(B1)** and **(B2)**, we have

(5.30) $$\mathcal{B} \subset 2^{P\setminus\{p\}}$$

and

(5.31) $$m_{P\setminus\{p\}} \in conv(\pi(\mathcal{B})).$$

(5.31) implies that

(5.32) $$conv(\pi(\mathcal{B})) \cap (m_p, m_{P\setminus\{p\}}] \neq \emptyset,$$

thus, by (5.30), (5.32), **(SB1)** and **(SB2)**, we have $\varphi_g(\tau)$ is π-subbalanced with respect to (P, p), and since $\tau \in \partial\widetilde{K}$ and (τ, g) is π-subbalanced pair with respect to (P, p) that is, (o) holds. This completes the proof of (n) implies (o).

Conversely, if (τ, g) is π-subbalanced pair with respect to (P, p), there exists a $v \in A^N$ such that

$$v \in conv(\pi(\mathcal{B})) \cap (m_p, m_{P\setminus\{p\}}], \tag{5.33}$$

thus, by the definition of $(m_p, m_{P\setminus\{p\}}]$ in (**SB2**), we may write

$$v = (1-\lambda)\sum_{i\in P}\frac{1}{|P|}a_{n_i} + \lambda\sum_{i\in P\setminus\{P\}}\frac{1}{|P|-1}a_{n_i} \quad \text{for some} \quad \lambda \in (0,1] \tag{5.34}$$

and since $(\tau, g) \in \partial\widetilde{K}_\pi(\varphi)$ thus $\tau \in \partial\widetilde{K}$ so that $\tau \subset A^Q$ for some $Q \in 2^N$ with $|Q| = |P| - 1$. On the other hand, from (**F2**), it follows that $\varphi_g(\tau) \subset 2^Q$ so that, by (**G1**) and the convexity of A^Q, we have

$$conv(\pi(\mathcal{B})) = conv(\pi(\varphi_g(\tau))) \subset A^Q, \tag{5.35}$$

thus, by (5.33) and (5.35), we may write

$$v = \sum_{i\in Q}\beta_i a_{n_i} \quad (\sum_{i\in Q}\beta_i = 1). \tag{5.36}$$

By the affine independence of $\{a_{n_1}, \ldots, a_{n_{|P|}}\}$, the vector $v \in A^N$ can be expressed as an affine combination of the vectors $a_{n_1}, \ldots, a_{n_{|P|}}$ in a unique way, thus (5.34) implies that v is not an affine combination of $|P| - 1$ distinct vectors of $\{a_{n_1}, \ldots, a_{n_{|P|}}\}$ if $\lambda \neq 1$, and (5.36) implies that v is an affine combination of $|P| - 1$ distinct vectors of $\{a_{n_1}, \ldots, a_{n_{|P|}}\}$. Thus $\lambda = 1$ in (5.34), we have

$$v = m_{P\setminus\{p\}} \quad \text{and} \quad Q = P \setminus \{p\}. \tag{5.37}$$

And since $\mathcal{B} = \varphi_g(\tau)$ and $\varphi_g(\tau) \subset 2^Q$ then $\mathcal{B} \subset 2^{P\setminus\{p\}}$, by (5.33), we have $m_{P\setminus\{p\}} \in conv(\pi(\mathcal{B}))$ thus $\varphi_g(\tau)$ is π-balanced with respect to $P \setminus \{p\}$ and $t \subset A^{P\setminus\{p\}}$, thus (n) holds. This completes the proof of (o) implies (n).
Let $g : \tau \to M$ be such $\tau \in \widetilde{K}$, τ spans t and (n) (o) hold. By (5.27), we may write

$$\tau = \{v_1, \ldots, v_{|P|-1}\}, \quad t = \overline{v_1 \ldots v_{|P|-1}}.$$

By (5.27), (5.28), (5.29) and (5.35), we have

$$\pi(\varphi_g(\tau)) \subset A^{P\setminus\{p\}} \subset aff(\tau) \tag{5.38}$$

so that, compare (5.38) with (F1), (t, g) is a fixed pair under (φ, π). By (5.38), we may write

$$\pi(\varphi_{g(v_i)}(v_i)) = \sum_{j=1}^{|P|-1}\lambda_{ij}v_j \quad (\sum_{j=1}^{|P|-1}\lambda_{ij} = 1) \quad \text{for} \quad i = 1, \ldots, |P| - 1. \tag{5.39}$$

By comparing (5.39) and (2.4), the following (p) and (q) are equivalent.

(p) (t, g) is positively (resp. negatively) fixed pair under (φ, π).

(q) $det\Lambda_1 > 0$ (resp. < 0).

where Λ_1 is the $(|P| - 1) \times (|P| - 1)$ matrix (λ_{ij}) in (5.39). Let (5.8) and (5.11) hold. Then, by (5.13), $\omega(\sigma)$ induces the orientation

(5.40) $$(-1)^{|P|-1}\varepsilon[v_1, \ldots, v_{|P-1|}] \quad (\varepsilon = \varepsilon' detA/|detA|)$$

on τ. Note that (5.38) implies that

(5.41) $$aff(\tau) = aff(A^{P\setminus\{p\}}) = aff(\{a_{n_1}, \ldots, a_{n_{|P|-1}}\}),$$

by the affine independence of $\{a_{n_1}, \ldots, a_{n_{|P|}}\}$ and by comparing (5.11) with (5.41), we have

(5.42) $$v_j = \sum_{k=1}^{|P|-1} \alpha_{jk} a_{n_k} \quad (\sum_{k=1}^{|P|-1} \alpha_{jk} = 1) \text{ for } j = 1, \ldots, |P| - 1.$$

this shows that the matrix $A = (\alpha_{jk})_{|P|\times|P|}$ in (5.11) is of the form

(5.43) $$A = \left\{ \begin{array}{c|c} & 0 \\ A_1 & \vdots \\ & 0 \\ \hline \cdots & \alpha_{|P||P|} \end{array} \right\}$$

where A_1 is the $(|P| - 1) \times (|P| - 1)$ matrix (α_{jk}) in (5.42). We claim that

(5.44) $$\alpha_{|P||P|} > 0.$$

By the affine independence of $\sigma = \{v_1, \ldots, v_{|P|}\}$, we have

(5.45) $$v_{|P|} \notin aff(\tau) \quad (\tau = \{v_1, \ldots, v_{|P|-1}\}).$$

By (5.41) and (5.45), we have

(5.46) $$v_{|P|} \notin aff(\{a_{n_1}, \ldots, a_{n_{|P|-1}}\}).$$

As $\sigma = \{v_1, \ldots, v_{|P|}\} \subset A^P$ and $A^P = \overline{\{a_{n_1} \ldots a_{n_{|P|}}\}}$, by (5.11) (with $j = |P|$) and (5.46), (5.44) follows. Note that (5.39) and (5.42) imply that

(5.47) $$\pi(\varphi_{g(v_i)}(v_i)) = \sum_{k=1}^{|P|-1} \gamma_{ik} a_{n_k} \quad (\sum_{k=1}^{|P|-1} \gamma_{ik} = 1) \text{ for } i = 1, \ldots, |P| - 1.$$

where

$$\gamma_{ik} = \sum_{j=1}^{|P|-1} \lambda_{ij}\alpha_{jk} \text{ for } i = 1, \dots, |P|-1 \text{ and } j = 1, \dots, |P|-1.$$

or equivalently

$$C = \Lambda_1 A_1 \tag{5.48}$$

where C is the $(|P|-1) \times (|P|-1)$ matrix (γ_{ik}) in (5.47). It follows from (2.3), (5.40), (5.43), (5.44) and (5.48) that the following $(r) \sim (u)$ are equivalent.

(r) (τ, g) is a positively (resp. negatively) boundary pair.

(s) $(-1)^{|P|-1}\varepsilon\varepsilon' detC > 0$ (resp. < 0) $(\varepsilon = \varepsilon' detA/|detA|)$.

(t) $(-1)^{|P|-1}(\varepsilon' detA_1/|detA|)\varepsilon' det\Lambda_1 detA_1 > 0$ (resp. < 0).

(u) $(-1)^{|P|-1}det\Lambda_1 > 0$ (resp. < 0).

Thus, by (4.54) and (4.56), by the definition of $\varphi^{P\setminus\{p\}}_{|P|-1}$, and the equivalences of (n) and (o) , (p) and (q), and (r) and (u), the formulae (5.23) and (5.24) are true. By (5.21), (5.22), (5.23), (5.24) and Theorem 1, we have

$$\varphi^{P}_{|P|} = m\varphi^{P\setminus\{p\}}_{|P|-1}, \tag{5.49}$$

and, if $m \geq |P|$ we have

$$\varphi^{P}_{|P|*} = (m - |P| + 1))\varphi^{P\setminus\{p\}}_{(|P|-1)*}. \tag{5.50}$$

Now, (5.3) and (5.49) imply (5.1), and (5.4) and (5.50) imply (5.2). This completes the inductive proof of Theorem 2.

Corollary 1. *Let T be a triangulation of an $(n-1)$-simplex $A^N = \overline{a_1 \dots a_n}$, let $\varphi : V(T) \to (2^N)^m$ and $\pi : 2^N \to A^N$, where m and n are positive integers, $N = \{1, \dots, n\}$ and π is a Shapley map, such that*

(F2) $\varphi(V(T) \cap A^S) \subset (2^S)^m$ *for all* $S \in 2^N$.

Then, there exist at least $m^{|P|}$ fixed $|P|$-labelled pairs under (φ, π) in H^P, and, if $m \geq |P|$, there exist at least $\frac{m!}{(m-|P|)!}$ fixed $|P|$-labelled pairs under (φ, π) in H^P_* for all $P \in 2^N$.

Proof. Since (**S1**) and (**F2**) imply (**F1**), all pairs are fixed under (φ, π). Now, the assertion follows from (Π3), (Π4) and Theorem 2.

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