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GROWTH ORDERS OF MEANS OF DISCRETE SEMIGROUPS OF OPERATORS IN BANACH SPACES

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. We study the growth orders of γ-th order Cesàro means $C_n^\gamma(T)$ ($\gamma \geq 0$) and Abel means $A_r(T)$ of the discrete semigroup $\{T^n : n \geq 0\}$ generated by a bounded linear operator T on a Banach space. Let T be of the form $T = -(I+N)$, where N is a nilpotent operator of order $k+1$ with $k \in \mathbb{N}$. Thus $N^{k+1} = 0$ and $N^k \neq 0$. Then we prove that (a) $\|C_n^\gamma(T)\| \sim n^{k-\gamma}$ $(n \to \infty)$ if $0 \leq \gamma \leq k+1$, and $\|C_n^\gamma(T)\| \sim n^{-1}$ $(n \to \infty)$ if $\gamma \geq k+1$; (b) $\|A_r(T)\| \sim 1-r$ $(r \uparrow 1)$. Here $a(n) \sim b(n)$ $(n \to \infty)$ [resp. $a(r) \sim b(r)$ $(r \uparrow 1)$] means that $0 < \liminf_{n\to\infty} a(n)/b(n) \leq \limsup_{n\to\infty} a(n)/b(n) < \infty$ [resp. $0 < \liminf_{r\uparrow 1} a(r)/b(r) \leq \limsup_{r\uparrow 1} a(r)/b(r) < \infty$].

1. Introduction and the Result

Let T be a bounded linear operator on a Banach space X. One of the important issues of the ergodic theory of T is concerned with convergence of various means of the discrete semigroup $\{T^n : n \geq 0\}$ generated by T. For $\gamma \in \mathbb{R} \setminus \{-1, -2, \ldots\}$, we define the *$\gamma$-th order Cesàro means* $C_n^\gamma(T)$ by

$$C_n^\gamma(T) := \frac{1}{\sigma_n^\gamma} \sum_{l=0}^{n} \sigma_{n-l}^{\gamma-1} T^l \qquad (n \geq 0), \tag{1}$$

where $\sigma_n^\gamma := \binom{\gamma+n}{n} = (\gamma+n)(\gamma+n-1)\ldots(\gamma+1)/n!$ for $n \geq 1$, and $\sigma_0^\gamma := 1$. The following two particular means are well-known: $C_n^0(T) = T^n$ and $C_n^1(T) = (n+1)^{-1}\sum_{l=0}^n T^l$ for $n \geq 0$. Here it should be noted that to treat means of $\{T^n : n \geq 0\}$ it would be natural to examine the case where the coefficients $\sigma_{n-l}^{\gamma-1}$ of T^l $(0 \leq l \leq n)$ are all nonnegative. Therefore we will restrict ourselves to

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considering $C_n^\gamma(T)$ with $\gamma \geq 0$. (In fact, there is a pathological phenomenon when we consider $C_n^\gamma(T)$ with $-1 < \gamma < 0$ (see [2, Theorem 4.1])).

We define the *Abel means* $A_r(T)$ by

$$A_r(T) := (1-r)\sum_{n=0}^{\infty} r^n T^n \qquad (0 < r < 1), \tag{2}$$

whenever the spectral radius $r(T) := \lim_{n\to\infty} \|T^n\|^{1/n}$ is less than or equal to 1. Since $(1-r)^{-\gamma} = \sum_{n=0}^{\infty} r^n \sigma_n^{\gamma-1}$ holds for all $r \in \mathbb{R}$ with $|r| < 1$, we have formally

$$\begin{aligned} \sum_{n=0}^{\infty} r^n T^n &= (1-r)^\gamma \Big(\sum_{n=0}^{\infty} r^n \sigma_n^{\gamma-1}\Big)\Big(\sum_{n=0}^{\infty} r^n T^n\Big) \\ &= (1-r)^\gamma \sum_{n=0}^{\infty} r^n \sum_{l=0}^{n} \sigma_{n-l}^{\gamma-1} T^l \\ &= (1-r)^\gamma \sum_{n=0}^{\infty} r^n \sigma_n^\gamma C_n^\gamma(T), \end{aligned} \tag{3}$$

so that if $\limsup_{n\to\infty} \|C_n^\gamma(T)\|^{1/n} \leq 1$, then $r(T) \leq 1$. The following result is well-known (cf. [4, Chapter 3]): If $0 < \gamma < \beta < \infty$, then

$$\sup_{n\geq 0} \|T^n\| \geq \sup_{n\geq 0} \|C_n^\gamma(T)\| \geq \sup_{n\geq 0} \|C_n^\beta(T)\| \geq \sup_{0<r<1} \|A_r(T)\|. \tag{4}$$

From now on, we consider T of the form $T = -(I+N)$, where N is a nilpotent operator of order $k+1$ with $k \in \mathbb{N}$. Thus $N^k \neq 0$ and $N^{k+1} = 0$. Then we have

$$T^n = (-1)^n (I+N)^n = (-1)^n \sum_{l=0}^{k} \binom{n}{l} N^l, \tag{5}$$

and

$$\binom{n}{l} \|N^l\| = \frac{n(n-1)\dots(n-l+1)}{l!} \|N^l\|. \tag{6}$$

Thus

$$\|C_n^0(T)\| = \|T^n\| \sim n^k \ \ (n \to \infty), \tag{7}$$

so that $r(T) = 1$. It was proved by Li, Sato and Shaw [2] that the operator $T = -(I+N)$ satisfies $\sup_{n\geq 0} \|C_n^\gamma\| = \infty$ if $0 \leq \gamma < k$, and $\sup_{n\geq 0} \|C_n^k(T)\| < \infty$.

The purpose of this paper is to refine on this result. That is, we prove the following

Theorem. *The above operator* $T = -(I+N)$ *satisfies*

$$\|C_n^\gamma(T)\| \sim \begin{cases} n^{k-\gamma} \ \ (n\to\infty) & \text{if } \ 0 \le \gamma \le k+1, \\ n^{-1} \ \ (n\to\infty) & \text{if } \ \gamma \ge k+1 \end{cases} \tag{8}$$

and

$$\|A_r(T)\| \sim 1-r \ \ (r \uparrow 1). \tag{9}$$

The proof is an adaptation of the argument in [2, Propositions 4.4]; the details will be given in the next section. We would like to note that the continuous analog of the above Theorem has been obtained in [3] (see also Chen, Sato and Shaw [1]).

2. Proof of the Theorem

The proof is divided into several steps.

Step I. In view of (7) we first consider the case $0 < \gamma < 1$. We write

$$\begin{aligned} C_n^\gamma(T) &= \frac{1}{\sigma_n^\gamma}\sum_{l=0}^n \sigma_{n-l}^{\gamma-1}(-1)^l(I+N)^l \\ &= \frac{1}{\sigma_n^\gamma}\sum_{l=0}^n(-1)^l\sigma_{n-l}^{\gamma-1}\sum_{s=0}^{k-1}\binom{l}{s}N^s + \frac{1}{\sigma_n^\gamma}\sum_{l=0}^n(-1)^l\sigma_{n-l}^{\gamma-1}\binom{l}{k}N^k \\ &=: I(n,\gamma) + II(n,\gamma). \end{aligned} \tag{10}$$

Putting $M(N) := \max\{\|N^s\| : 0 \le s \le k\}$, we have for all $n \ge k$

$$\begin{aligned} \|I(n,\gamma)\| &\le \frac{1}{\sigma_n^\gamma}\sum_{l=0}^n \sigma_{n-l}^{\gamma-1}\sum_{s=0}^{k-1}\binom{l}{s}M(N) \\ &\le \frac{1}{\sigma_n^\gamma}\sum_{l=0}^n \sigma_{n-l}^{\gamma-1}\cdot n(n-1)\dots(n-k+2)M(N) \\ &= n(n-1)\dots(n-k+2)M(N) \sim n^{k-1} \quad (n\to\infty). \end{aligned} \tag{11}$$

Next

$$\|II(n,\gamma)\| = \frac{1}{\sigma_n^\gamma}\left|\sum_{l=0}^n(-1)^l\sigma_{n-l}^{\gamma-1}\binom{l}{k}\right| \|N^k\|. \tag{12}$$

Since $0 < \sigma_n^{\gamma-1} \downarrow 0$ $(n\to\infty)$ for $0<\gamma<1$, and $0 \le \binom{l}{k} \le \binom{l+1}{k}$ for all $l \ge 0$, it follows that

$$0 \le \sigma_{n-l}^{\gamma-1}\binom{l}{k} \le \sigma_{n-(l+1)}^{\gamma-1}\binom{l+1}{k} \qquad (0 \le l \le n-1),$$

whence for all $n \geq k$

$$\begin{aligned} \sigma_0^{\gamma-1}\binom{n}{k} &\geq \left|\sum_{l=0}^{n}(-1)^l \sigma_{n-l}^{\gamma-1}\binom{l}{k}\right| \\ &\geq \sigma_0^{\gamma-1}\binom{n}{k} - \sigma_1^{\gamma-1}\binom{n-1}{k} \\ &= \frac{1}{k!}\Big\{n(n-1)\dots(n-k+1) - \gamma(n-1)\dots(n-k)\Big\} \\ &= \frac{1}{k!}\,n(n-1)\dots(n-k+1)\Big\{1-\frac{\gamma}{n}(n-k)\Big\} \\ &> \frac{1}{k!}\,n(n-1)\dots(n-k+1)(1-\gamma) \sim n^k \quad (n\to\infty). \end{aligned} \tag{13}$$

Thus, applying the known fact that $\sigma_n^\gamma \sim n^\gamma/\Gamma(\gamma+1)$ $(n \to \infty)$ (see e.g. [4, p. 77]), we obtain that

$$\|II(n,\gamma)\| \sim \frac{n^k\|N^k\|}{\sigma_n^\gamma} \sim n^{k-\gamma} \quad (n\to\infty). \tag{14}$$

Combining this with (11) we see that

$$\|C_n^\gamma(T)\| \sim n^{k-\gamma} \quad (n\to\infty). \tag{15}$$

Step II. Next suppose $1 \leq \gamma < k+1$. Then we use the fundamental equation

$$(T-I)C_n^\gamma(T) = \frac{\gamma}{n+1}\Big[C_{n+1}^{\gamma-1}(T) - I\Big] \qquad (\gamma \geq 1). \tag{16}$$

(This can be proved by an elementary calculation (cf. [4, Chapter 3]).) We already know from the above argument that if $0 \leq \beta < 1$, then $\|C_n^\beta(T)\| \sim n^{k-\beta}$ $(n\to\infty)$, so that $\|C_n^\beta(T)-I\| \sim n^{k-\beta}$ $(n\to\infty)$. Combining this with (16), we easily see that (15) holds for all $1 \leq \gamma < 2$. (Here we used the fact that $(T-I)^{-1} = -(2I+N)^{-1}$ exists, which follows from $\sigma(N) = \{0\}$.) This process can be repeated until $k \leq \gamma < k+1$, and hence (15) holds for all $1 \leq \gamma < k+1$.

Step III. Suppose $\gamma = k+1$. As in Step II it suffices to prove that $\|C_n^k(T) - I\| \sim 1$ $(n\to\infty)$. Since $\|C_n^k(T)\| \sim 1$ $(n\to\infty)$ by Step II, it follows that $\|C_n^k - I\| = O(1)$ $(n\to\infty)$. Thus it suffices to prove that $\liminf_{n\to\infty}\|C_n^k(T)-I\| > 0$. To do this, we write

$$(T-I)C_n^k(T) = \frac{k}{n+1}\Big[C_{n+1}^{k-1}(T) - I\Big] =: \frac{k}{n+1}C_{n+1}^{k-1}(T) + D_n^1(T),$$

where $\lim_{n\to\infty}\|D_n^1(T)\| = 0$; next

$$(T-I)^2 C_n^k(T) =: \frac{k(k-1)}{(n+1)(n+2)} C_{n+2}^{k-2}(T) + D_n^2(T),$$

where $\lim_{n\to\infty} \|D_n^2(T)\| = 0$; and finally

$$(T-I)^k C_n^k(T) =: \frac{k!}{(n+1)\dots(n+k)} C_{n+k}^0(T) + D_n^k(T),$$

where $\lim_{n\to\infty} \|D_n^k(T)\| = 0$. By (5) we then write

$$\begin{aligned} \frac{k!}{(n+1)\dots(n+k)} C_{n+k}^0 &= \frac{k!}{(n+1)\dots(n+k)} (-1)^{n+k} \sum_{l=0}^{k} \binom{n+k}{l} N^l \\ &=: (-1)^{n+k} N^k + E_n^k(T), \end{aligned}$$

where $\lim_{n\to\infty} \|E_n^k(T)\| = 0$. Consequently we have

$$C_n^k(T) = (T-I)^{-k}(-1)^{n+k}N^k + (T-I)^{-k}(E_n^k(T) + D_n^k(T)). \tag{17}$$

Now, take an $x \in X$ such that $\|x\| = 1$ and $N^k x = 0$. Then $\lim_{n\to\infty} \|C_n^k(T)x\| = 0$ by (17), and

$$\liminf_{n\to\infty} \|C_n^k(T) - I\| \geq \lim_{n\to\infty} \|C_n^k(T)x - x\| = \|-x\| = 1,$$

which is the desired result.

Step IV. Suppose $\gamma > k+1$. From Steps II and III we know that if $k < \beta \leq k+1$, then $\|C_n^\beta(T)\| \sim n^{k-\beta}$ $(n \to \infty)$, so that $\|C_n^\beta(T) - I\| \sim 1$ $(n \to \infty)$ because $\lim_{n\to\infty} n^{k-\beta} = 0$. Thus if $k+1 < \gamma \leq k+2$, then (16) implies

$$\|C_n^\gamma(T)\| \sim n^{-1} \qquad (n \to \infty). \tag{18}$$

This argument can be repeated by induction, and we see that (18) holds for all γ with $k+j < \gamma \leq k+j+1$, where $j \in \mathbb{N}$. This completes the proof of (8).

Step V. Using $(1-r)^{-1}A_r(T) = (I-rT)^{-1}$ $(0 < r < 1)$, we see that

$$\lim_{r\uparrow 1} (1-r)^{-1}A_r(T) = \lim_{r\uparrow 1} (I-rT)^{-1} = -(T-I)^{-1}. \tag{19}$$

Hence $\|A_r(T)\| \sim 1-r$ $(r \uparrow 1)$. This completes the proof. ∎

Remark. From (16) and (8) we see that

$$\lim_{n\to\infty} \frac{n+1}{\gamma} C_n^\gamma(T) = \lim_{n\to\infty} (T-I)^{-1}(C_{n+1}^{\gamma-1}(T) - I) = -(T-I)^{-1} \tag{20}$$

if $\gamma > k+1$. (It would be interesting to compare this with (19).) On the other hand, $\lim_{n\to\infty} \frac{n+1}{k+1} C_n^{k+1}(T)$ does not exist because $\lim_{n\to\infty} C_n^k(T)$ does not exist by (17), and $\frac{n+1}{k+1} \|C_n^{k+1}(T)\| \sim 1 \ (n \to \infty)$ by (8). If $k < \gamma < k+1$, then $\lim_{n\to\infty} \frac{n+1}{\gamma} \|C_n^\gamma(T)\| = \infty$, and $\lim_{n\to\infty} \|C_n^\gamma(T)\| = 0$. $\|C_n^k(T)\| \sim 1 \ (n \to \infty)$, and $\lim_{n\to\infty} C_n^k(T)$ does not exist. If $0 \le \gamma < k$, then $\lim_{n\to\infty} \|C_n^\gamma(T)\| = \infty$.

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