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ASYMPTOTIC PERTURBATION OF PALINDROMIC EIGENVALUE PROBLEMS

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Abstract. We investigate the perturbation of the palindromic eigenvalue problem for the matrix quadratic $P(\lambda) \equiv \lambda^2 A_1^T + \lambda A_0 + A_1$, with $A_0,\ A_1 \in \mathbb{C}^{n\times n}$ and $A_0^T = A_0$. The perturbation of eigenvalues and eigenvectors, in terms of palindromic matrix polynomials and palindromic linearizations, are discussed using Sun's implicit function approach.

1. INTRODUCTION

Consider the matrix quadratic

$$P(\lambda) \equiv \lambda^2 A_1^T + \lambda A_0 + A_1$$

where $A_0,\ A_1 \in \mathbb{C}^{n\times n}$ and $A_0^T = A_0$, and the corresponding *palindromic* quadratic eigenvalue problem

$$P(\lambda)x = 0\ , \quad x \neq 0\ . \tag{1}$$

In this paper, we shall consider only ***regular*** matrix polynomials $P(\lambda)$, in the sense that $\det P(\lambda) \not\equiv 0$.

From the transpose of (1), a palindromic eigenvalue problem possesses a spectrum $\sigma(P)$ containing both λ and its reciprocal λ^{-1} (with 0 and ∞ considered to be reciprocal to each other). When $\lambda \neq -1$, the eigenvalue problem of the original matrix polynomial $P(\lambda)$ has a palindromic linearization [5, 8, 14] of the form $\lambda Z \pm Z^T$. (We can transform $\lambda Z - Z^T$ to the form $\nu(-Z) + (-Z)^T$ with $\nu = -\lambda$. Similarly, $\lambda^2 A_1^T + \lambda A_0 + A_1$ and $\nu^2 A_1^T - \nu A_0 + A_1$ define equivalent palindromic eigenvalue problems).

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A great foundation for the solution of palindromic eigenvalue problems has been laid by Hilliges, Mackey, Mehl and Mehrmann [11, 14, 15]. Alternative approaches in tackling the problem with structure-preserving doubling algorithms and generalized Patel/Analdi method can be found in [5, 12]. There has been much recent interest in quadratic eigenvalue problems [20]. An important example of palindromic eigenvalue problems can be found in the vibration analysis of fast trains; see [11, 13] for general introductions and [10] for details. For general perturbation of eigenvalues for polynomial eigenvalue problems, see [4, 19]. On results for general matrix polynomials, see the masterpiece [8].

This paper is organized as follows. The perturbation result for a simple eigenvalue and its corresponding eigenvector is obtained by differentiation in Section 2. Sun's implicit function approach [18] is then applied to palindromic linearizations, general matrix quadratics and palindromic eigenvalue problems in Section 3, to obtain perturbation results for (simple) eigenvalues and the corresponding eigenvectors/deflating subspaces. An illustrative example is given in Section 4 and the paper is concluded in Section 5.

Some related results for perturbation of arbitrary size, in terms of Bauer-Fike type theorems, can be found in [6]. Although we have only considered the T-paldindromic problems here, related H-palindromic, anti-palindromic and odd/even problems can be treated similarly.

2. Perturbation by Differentiation

Without establishing the differentiability or the existence of asymptotic expansions (which can be achieved using the implicit function approach in the next section), perturbation results can be obtained by simple differentiation. See [1] for more details on this approach.

For some fixed $z \neq 0$, consider the palindromic eigenvalue problem

$$P(\lambda, \rho)x(\rho) = 0 \ , \quad P(\lambda, \rho) \equiv \lambda(\rho)^2 A_1^T(\rho) + \lambda(\rho)A_0(\rho) + A_1(\rho)$$

with the scaling $z^T x(\rho) - 1 = 0$, where ρ is the perturbation parameter, $A_0(0) = A_0$ and $A_1(0) = A_1$. We shall use the subscripts $(\cdot)_\rho$ and $(\cdot)_\lambda$ to denote the corresponding partial derivatives. For a simple eigenvalue λ, differentiation produces, at $\rho = 0$:

$$\lambda_\rho = -\frac{y^T P_\rho x}{y^T P_\lambda x} = -\frac{y^T P_\rho x}{y^T (2\lambda A_1^T + A_0)x} \tag{2}$$

and

$$P x_\rho = -(\lambda_\rho P_\lambda + P_\rho)x \ , \quad z^T x_\rho = 0 \ .$$

Choosing $z = y(0)$ (the left-eigenvector corresponding to $\lambda(0)$), we have, at $\rho = 0$:

$$x_\rho = -P^\dagger(\lambda_\rho P_\lambda + P_\rho)x$$

where $P^\dagger$ denotes the Penrose generalized inverse [9].

The usual conclusion can be drawn — the right-eigenvector x will be rotated through a big angle, even for a small perturbation, when $\|P^\dagger\|$ is large, i.e., when the separation between λ and other eigenvalues is fine. This happens, of course, when the assumption of simplicity for the eigenvalue is near collapsing.

Note that for palindromic eigenvalues problems, $\lambda = \pm 1$ may be multiple and non-differentiable, and a more sophisticated approach, like the one in [3], is required. This comment is also applicable to Sun's approach in the next Section.

3. Sun's Implicit Function Approach

In this Section, we shall apply Sun's approach [7, 18] where the implicit function theorem is applied. Asymptotic perturbation series for the eigenvalues and the corresponding deflating subspaces are obtained.

3.1 Palindromic pemcils

From [16], we have the following anti-triangular canonical form:

Theorem 1. *Let $Z - \lambda Z^T$ be a regular $n \times n$ palindromic linearization. There exists a unitary $U \in \mathbb{C}^{n\times n}$ such that $U^T Z U = (m_{ij})$ with $m_{ij} = 0$ $(i+j \leq n+1)$ (i.e., $U^T Z U$ is anti-triangular, with zero elements in the upper left corner).*

The palindromic eigenvalues are:

$$\frac{m_{1n}}{m_{n1}}, \frac{m_{2,n-1}}{m_{n-1,2}}, \cdots, \frac{m_{i,n-i+1}}{m_{n-i+1,i}}, \cdots, \frac{m_{n-i+1,i}}{m_{i,n-i+1}}, \cdots, \frac{m_{n-1,2}}{m_{2,n-1}}, \frac{m_{n1}}{m_{1n}} .$$

Using the anti-triangular form in Theorem 1, Sun's approach [7] can be applied to obtain the power series of eigenvalues and deflating subspaces. Without loss of generality, we shall use an upper-triangular form in this section (with the help of the order-reversing permutation matrix $\mathcal{P}_r \equiv [e_r, e_{r-1}, \cdots, e_1] \in \mathbb{R}^{r\times r}$ with e_i being the ith column of the r-dimensional identity matrix I_r, so that Sun's approach can be followed faithfully).

Using $\mathcal{P}_n$, the anti-triangular form (in Theorem 1) is turned into a upper triangular form and is organized to reflect the symmetry of the eigenvalue pairs $\{\lambda_j,\ \lambda_j^{-1}\}$:

$$\mathcal{P}_n U^T (Z - \lambda Z^T) U$$

$$= \begin{bmatrix} \Lambda_{\alpha_1} & * \\ 0 & \Lambda_{\alpha 2} \end{bmatrix} - \lambda \begin{bmatrix} \Lambda_{\beta_1} & * \\ 0 & \Lambda_{\beta 2} \end{bmatrix}$$
$$= \begin{bmatrix} \Lambda_{\alpha 1} & & * \\ & \Lambda_{\alpha 0} & \\ 0 & & \Lambda_{\alpha,-1} \end{bmatrix} - \lambda \begin{bmatrix} \Lambda_{\beta 1} & & * \\ & \Lambda_{\beta 0} & \\ 0 & & \Lambda_{\beta,-1} \end{bmatrix} .$$

With $2p < n$, $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$ contains the p eigenvalues whose perturbation we are interested in, $(\Lambda_{\alpha,-1},\ \Lambda_{\beta,-1})$ their reciprocals, $(\Lambda_{\alpha 0},\ \Lambda_{\beta 0})$ the rest of the spectrum, and $(\Lambda_{\alpha 2},\ \Lambda_{\beta 2})$ the complement of those in $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$. Consequently, we have

$$\Lambda_{\beta,-1} = \mathcal{P}_p \Lambda_{\alpha 1} \mathcal{P}_p\ ,\quad \Lambda_{\beta,1} = \mathcal{P}_p \Lambda_{\alpha,-1} \mathcal{P}_p\ ,\quad \Lambda_{\beta 0} = \mathcal{P}_{2n-p} \Lambda_{\alpha 0} \mathcal{P}_{2n-p}$$

and $\{U_{-1},\ U_1\}$ spanning the deflating subspaces of $Z - \lambda Z^T$ corresponding to $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$. With similar organization of the deflating subspaces, we have

$$U \equiv [U_1, U_0, U_{-1}] = [U_1, U_2] = [U_{-2}, U_{-1}], U_2 \equiv [U_0, U_{-1}], U_{-2} \equiv [U_1, U_0]$$

and

$$U\mathcal{P}_n = [U_{-1}\mathcal{P}_p,\ U_0\mathcal{P}_{n-2p},\ U_1\mathcal{P}_p] = [U_{-1}\mathcal{P}_p,\ U_{-2}\mathcal{P}_{n-p}] = [U_2\mathcal{P}_{n-p},\ U_1\mathcal{P}_p]\ .$$

Here U_1 (corresponding to $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$) and U_{-1} (corresponding to the "reciprocal" pencil $(\Lambda_{\alpha,-1},\ \Lambda_{\beta,-1})$) are respectively the first and last p columns of U.

Assume that $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$ and $(\Lambda_{\alpha 2},\ \Lambda_{\beta 2})$ have nonitersecting spectra. (This excludes the possibility of $\lambda = \pm 1$ appearing in both $(\Lambda_{\alpha,-1},\Lambda_{\beta,-1})$ and $(\Lambda_{\alpha 1},\Lambda_{\beta 1})$. All these unimodular eigenvalues have to be grouped together inside $(\Lambda_{\alpha 1},\Lambda_{\beta 1})$, with the necessary adjustment of the Us.) We assume that $Z(\rho)$ is analytic with respect to the perturbation parameter ρ. (Frequently, the dependence on ρ is not written explicitly to avoid messy expressions.) We then have

$$\mathcal{M} \equiv \mathcal{P}_n U^T Z(\rho) U = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix} \equiv \begin{bmatrix} \mathcal{P}_p U_1^T Z U_{-1} & \mathcal{P}_p U_1^T Z U_{-2} \\ \mathcal{P}_{n-p} U_2^T Z U_{-1} & \mathcal{P}_{n-p} U_2^T Z U_{-2} \end{bmatrix}$$

and

$$\mathcal{L} \equiv \mathcal{P}_n U^T Z(\rho)^T U = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \equiv \begin{bmatrix} \mathcal{P}_p U_1^T Z^T U_{-1} & \mathcal{P}_p U_1^T Z^T U_{-2} \\ \mathcal{P}_{n-p} U_2^T Z^T U_{-1} & \mathcal{P}_{n-p} U_2^T Z^T U_{-2} \end{bmatrix} .$$

From the (2,1)-block of

$$\begin{bmatrix} I_p & 0 \\ -\Psi & I_{n-p} \end{bmatrix} (\mathcal{M} - \lambda \mathcal{L}) \begin{bmatrix} I_p & 0 \\ \Phi & I_{n-p} \end{bmatrix}$$

with $\Psi(\rho)$ and $\Phi(\rho)$ being functions of ρ, we construct

$$\begin{bmatrix} F(\Phi,\Psi,\rho) \\ G(\Phi,\Psi,\rho) \end{bmatrix} = \begin{bmatrix} -\Psi\mathcal{P}_p U_1^T Z U_{-1} - \Psi\mathcal{P}_p U_1^T Z U_{-2}\Phi + \mathcal{P}_{n-p} U_2^T Z U_{-1} + \mathcal{P}_{n-p} U_2^T Z U_{-2}\Phi \\ -\Psi\mathcal{P}_p U_1^T Z^T U_{-1} - \Psi\mathcal{P}_p U_1^T Z^T U_{-2}\Phi + \mathcal{P}_{n-p} U_2^T Z^T U_{-1} + \mathcal{P}_{n-p} U_2^T Z^T U_{-2}\Phi \end{bmatrix}.$$

At $\rho = 0$, we have $\Phi = 0 = \Psi$ and $F(0,0,0) = 0 = G(0,0,0)$. The appropriate implicit function theorem can then be applied to $[F^T,\, G^T]^T$. Differentiation of F and G respect to Ψ and Φ at $\rho = 0$ yields (after stacking columns and apply Kronecker products)

$$\begin{bmatrix} F \\ G \end{bmatrix}_{(\Psi,\Phi)} = \begin{bmatrix} -\Lambda_{\alpha 1}^T \otimes I_{n-p} & I_p \otimes \Lambda_{\alpha 2} \\ -\Lambda_{\beta 1}^T \otimes I_{n-p} & I_p \otimes \Lambda_{\beta 2} \end{bmatrix} .$$

It is easy to see that the above operator is invertible when the spectra of $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$ and $(\Lambda_{\alpha 2},\ \Lambda_{\beta 2})$ do not intersect [2] (and it will be ill-conditioned when the separation of the spectra is small). Consequently, we have proved that the power series exist for $\Phi(\rho)$ and $\Psi(\rho)$ within some small neighbourhood of $\rho = 0$, and

$$\Phi(\rho) = \Phi_\rho(0)\rho + \cdots , \quad \Psi(\rho) = \Psi_\rho(0)\rho + \cdots .$$

Furthermore, the perturbed deflation subspaces [7] are

$$\text{(3)} \qquad \begin{aligned} &\left\{ \text{span}\left(\overline{U}\mathcal{P}_n \begin{bmatrix} I_p \\ \Psi(p) \end{bmatrix} \right),\ \text{span}\left(U \begin{bmatrix} I_p \\ \Phi(p) \end{bmatrix} \right) \right\} \\ &= \left\{ \text{span}\left(\overline{U}_1\mathcal{P}_n + \overline{U}_2\mathcal{P}_{n-p}\Psi(p) \right),\ \text{span}\left(U_{-1} + U_{-2}\Phi(p) \right) \right\} . \end{aligned}$$

Differentiation of

$$\mathcal{M}_{11} \equiv \mathcal{P}_p U_1^T Z U_{-1} , \quad \mathcal{L}_{11} \equiv \mathcal{P}_p U_1^T Z^T U_{-1}$$

also yields, at $\rho = 0$:

$$\frac{\partial \mathcal{M}_{11}}{\partial \rho} = \mathcal{P}_p U_1^T Z_\rho U_{-1} , \quad \frac{\partial \mathcal{L}_{11}}{\partial \rho} = \mathcal{P}_p U_1^T Z_\rho^T U_{-1} .$$

Applying the trace operator to the above derivatives will produce the derivatives of the averages of multiple eigenvalues [3]. Alternatively, subgradients can be applied for such analysis. Note that the multiple eigenvalues $\lambda = \pm 1$ are of particular interest in the study of palindromic eigenvalue problems and their perturbation analysis can be performed using the above formulae [5, 14, 15, 16].

When $p = 1$ and $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1}) = (\lambda_{\alpha 1},\ \lambda_{\beta 1})$ represents the finite eigenvalue $\lambda_1 = \lambda_{\alpha 1}/\lambda_{\beta 1}$, the above derivatives translate to

$$\frac{\partial \lambda_1}{\partial \rho} = \frac{\frac{\partial \lambda_{\alpha 1}}{\partial \rho}\lambda_{\beta 1} - \frac{\partial \lambda_{\beta 1}}{\partial \rho}\lambda_{\alpha 1}}{\lambda_{\beta 1}^2} = \frac{\frac{\partial \lambda_{\alpha 1}}{\partial \rho}}{\lambda_{\beta 1}} - \frac{\lambda_1 \frac{\partial \lambda_{\beta 1}}{\partial \rho}}{\lambda_{\beta 1}} = \frac{y_1^T Z_\rho x_1 - \lambda_1 y_1^T Z_\rho^T x_1}{y_1^T Z^T x_1} ,$$

producing a result analogous to (2).

Lastly, differentiating F and G with respect to ρ at $\rho = 0$ produces

$$F_\rho \,=\, \Lambda_{\alpha 2}\Phi_\rho - \Psi_\rho \Lambda_{\alpha 1} + \mathcal{P}_{n-p} U_2^T Z_\rho U_{-1} = 0 \ ,$$
$$G_\rho \,=\, \Lambda_{\beta 2}\Phi_\rho - \Psi_\rho \Lambda_{\beta 1} + \mathcal{P}_{n-p} U_2^T Z_\rho^T U_{-1} = 0 \ .$$

The derivatives $\Phi_\rho(0)$ and $\Psi_\rho(0)$ (required in (**??**) when calculating the perturbed deflating subspaces) can then be retrieved from the above equations, when $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$ and $(\Lambda_{\alpha 2},\ \Lambda_{\beta 2})$ have nonintersecting spectra [2].

3.2 General matrix quadratics

We now apply Sun's approach [7] to the general matrix quadratic

$$Q_2(\lambda) = \lambda^2 M + \lambda D + K$$

similar to the development in the previous subsection. Assume the $n \times 2n$ matrices $X = [x_j]$ and $Y = [y_j]$ contain, respectively, the right- and left-eigenvectors x_j and y_j corresponding to $\lambda_j = \alpha_j/\beta_j$. For the companion linearization

(4) $$\mathcal{L}(\lambda) \equiv \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} - \lambda \begin{bmatrix} I & \\ & M \end{bmatrix} ,$$

it is easy to check that the right- and left-eigenvectors corresponding to $\lambda_j = \alpha_j/\beta_j$ are respectively

$$\begin{bmatrix} \beta_j x_j \\ \alpha_j x_j \end{bmatrix} , \left[\beta_j y_j^T D + \alpha_j y_j^T M,\ \beta_j y_j^T\right]^T .$$

Notice that by using $(\alpha_j,\ \beta_j)$ to represent λ_j, we have avoided the difficulties involving infinite eigenvalues. We can then scale the eigenvectors to satisfy the biorthogonality equations

(5) $$\Lambda_\alpha Y^T M X \Lambda_\beta + \Lambda_\beta Y^T M X \Lambda_\alpha + \Lambda_\beta Y^T D X \Lambda_\beta \,=\, \Lambda_\beta \ ,$$

(6) $$\Lambda_\alpha Y^T M X \Lambda_\alpha - \Lambda_\beta Y^T K X \Lambda_\beta \,=\, \Lambda_\alpha \ .$$

Here Λ_α and Λ_β are block-diagonal matrices, with blocks being the usual identity or Jordan matrices for the generalized eigenvalue sub-problems.

Assume further that X, Y, Λ_α and Λ_β are organized as follows:

$$X = [X_1, X_2],\ Y = [Y_1, Y_2];\ \Lambda_\alpha = \mathrm{diag}\{\Lambda_{\alpha 1}, \Lambda_{\alpha 2}\},\ \Lambda_\beta = \mathrm{diag}\{\Lambda_{\beta 1}, \Lambda_{\beta 2}\}$$

where the spectra of $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$ and $(\Lambda_{\alpha 2},\ \Lambda_{\beta 2})$ do not intersect.

We assume the matrices M, D and K are differentiable with respect to the perturbation parameter ρ. We first transform the linearization in (4) by pre-(post-)multiplying with its left-(right-)eigenvectors:

$$\mathcal{M} \equiv \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix} \equiv \left[\Lambda_\beta Y^T D + \Lambda_\alpha Y^T M,\ \Lambda_\beta Y^T\right] \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \begin{bmatrix} X\Lambda_\beta \\ X\Lambda_\alpha \end{bmatrix}$$

and

$$\mathcal{L} \equiv \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \equiv \left[\Lambda_\beta Y^T D + \Lambda_\alpha Y^T M,\ \Lambda_\beta Y^T\right] \begin{bmatrix} I & \\ & M \end{bmatrix} \begin{bmatrix} X\Lambda_\beta \\ X\Lambda_\alpha \end{bmatrix} .$$

From the (2,1)-block of

$$\begin{bmatrix} I_p & 0 \\ -\Psi & I_{n-p} \end{bmatrix} (\mathcal{M} - \lambda\mathcal{L}) \begin{bmatrix} I_p & 0 \\ \Phi & I_{n-p} \end{bmatrix}$$

with $\Psi(\rho)$ and $\Phi(\rho)$ being functions of ρ, we construct

$$\begin{aligned} F(\Phi, \Psi, \rho) \equiv\ & -\Psi(\Lambda_{\alpha 1} Y_1^T M X_1 \Lambda_{\alpha 1} - \Lambda_{\beta 1} Y_1^T K X_1 \Lambda_{\beta 1}) \\ & -\Psi(\Lambda_{\alpha 1} Y_1^T M X_2 \Lambda_{\alpha 2} - \Lambda_{\beta 1} Y_1^T K X_2 \Lambda_{\beta 2})\Phi \\ & +(\Lambda_{\alpha 2} Y_2^T M X_1 \Lambda_{\alpha 1} - \Lambda_{\beta 2} Y_2^T K X_1 \Lambda_{\beta 1}) \\ & +(\Lambda_{\alpha 2} Y_2^T M X_2 \Lambda_{\alpha 2} - \Lambda_{\beta 2} Y_2^T K X_2 \Lambda_{\beta 2})\Phi\ , \\ G(\Phi, \Psi, \rho) \equiv\ & -\Psi(\Lambda_{\beta 1} Y_1^T D X_1 \Lambda_{\beta 1} + \Lambda_{\alpha 1} Y_1^T M X_1 \Lambda_{\beta 1} + \Lambda_{\beta 1} Y_1^T M X_1 \Lambda_{\alpha 1}) \\ & -\Psi(\Lambda_{\beta 1} Y_1^T D X_2 \Lambda_{\beta 2} + \Lambda_{\alpha 1} Y_1^T M X_2 \Lambda_{\beta 2} + \Lambda_{\beta 1} Y_1^T M X_2 \Lambda_{\alpha 2})\Phi \\ & +(\Lambda_{\beta 2} Y_2^T D X_1 \Lambda_{\beta 1} + \Lambda_{\alpha 2} Y_2^T M X_1 \Lambda_{\beta 1} + \Lambda_{\beta 2} Y_2^T M X_1 \Lambda_{\alpha 1}) \\ & +(\Lambda_{\beta 2} Y_2^T D X_2 \Lambda_{\beta 2} + \Lambda_{\alpha 2} Y_2^T M X_2 \Lambda_{\beta 2} + \Lambda_{\beta 2} Y_2^T M X_2 \Lambda_{\alpha 2})\Phi\ . \end{aligned}$$

At $\rho = 0$, we have $\Phi = 0 = \Psi$ and $F(0,0,0) = 0 = G(0,0,0)$. The complex implicit function theorem can then be applied to $(F^T, G^T)^T$. Differentiation of F and G respect to Ψ and Φ at $\rho = 0$ yields (after stacking columns and apply Kronecker products)

$$\begin{bmatrix} F \\ G \end{bmatrix}_{(\Psi,\Phi)} = \begin{bmatrix} -\Lambda_{\alpha 1}^T \otimes I_p & I_{n-p} \otimes \Lambda_{\alpha 2} \\ -\Lambda_{\beta 1}^T \otimes I_p & I_{n-p} \otimes \Lambda_{\beta 2} \end{bmatrix} .$$

It is easy to see that the above operator is invertible when the spectra of $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$ and $(\Lambda_{\alpha 2},\ \Lambda_{\beta 2})$ do not intersect [2] (and it will be ill-conditioned when the separation of the spectra is small). Consequently, we have proved that the power series exists for $\Phi(\rho)$ and $\Psi(\rho)$ within some small neighbourhood of $\rho = 0$, and

$$\Phi(\rho) = \Phi_\rho(0)\rho + \cdots , \quad \Psi(\rho) = \Psi_\rho(0)\rho + \cdots .$$

Furthermore, the perturbed deflation subspaces [7] of the companion linearization equal

$$\left\{ \text{span}\left(\tilde{Y}^{-T} \left[\begin{array}{c} I_p \\ \Psi(p) \end{array} \right] \right),\ \text{span}\left(\tilde{X} \left[\begin{array}{c} I_p \\ \Phi(p) \end{array} \right] \right) \right\}$$

with

$$\tilde{Y} = \left[\Lambda_\beta Y^T D + \Lambda_\alpha Y^T M,\ \Lambda_\beta Y^T \right]^T , \quad \tilde{X} = \left[\begin{array}{c} X\Lambda_\beta \\ X\Lambda_\alpha \end{array} \right] .$$

Differentiation of

$$\mathcal{M}_{11} \equiv \Lambda_{\alpha 1} Y_1^T M X_1 \Lambda_{\alpha 1} - \Lambda_{\beta 1} Y_1^T K X_1 \Lambda_{\beta 1} ,$$
$$\mathcal{L}_{11} \equiv \Lambda_{\beta 1} Y_1^T D X_1 \Lambda_{\beta 1} + \Lambda_{\alpha 1} Y_1^T M X_1 \Lambda_{\beta 1} + \Lambda_{\beta 1} Y_1^T M X_1 \Lambda_{\alpha 1}$$

also yields, at $\rho = 0$:

$$\frac{\partial \mathcal{M}_{11}}{\partial \rho} = \Lambda_{\alpha 1} Y_1^T M_\rho X_1 \Lambda_{\alpha 1} - \Lambda_{\beta 1} Y_1^T K_\rho X_1 \Lambda_{\beta 1} ,$$
$$\frac{\partial \mathcal{L}_{11}}{\partial \rho} = \Lambda_{\beta 1} Y_1^T D_\rho X_1 \Lambda_{\beta 1} + \Lambda_{\alpha 1} Y_1^T M_\rho X_1 \Lambda_{\beta 1} + \Lambda_{\beta 1} Y_1^T M_\rho X_1 \Lambda_{\alpha 1} .$$

Again for multiple eigenvalues (like $\lambda = \pm 1$), applying the trace operator to the above derivatives will produce the derivatives of the averages of multiple eigenvalues [3]; otherwise subgradients will be required.

When $p = 1$ and $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1}) = (\lambda_{\alpha 1},\ \lambda_{\beta 1})$ represents the finite eigenvalue $\lambda_1 = \lambda_{\alpha 1}/\lambda_{\beta 1}$, the above derivatives translate to

$$\begin{aligned} &\frac{\partial \lambda_1}{\partial \rho} \\ &= \frac{\frac{\partial \lambda_{\alpha 1}}{\partial \rho}\lambda_{\beta 1} - \frac{\partial \lambda_{\beta 1}}{\partial \rho}\lambda_{\alpha 1}}{\lambda_{\beta 1}^2} = \frac{\frac{\partial \lambda_{\alpha 1}}{\partial \rho}}{\lambda_{\beta 1}} - \frac{\lambda_1 \frac{\partial \lambda_{\beta 1}}{\partial \rho}}{\lambda_{\beta 1}} \\ &= \frac{-\lambda_{\beta 1}^2 y_1^T K_\rho x_1 + \lambda_{\alpha 1}^2 y_1^T M_\rho x_1 - \lambda_1(\lambda_{\beta 1}^2 y_1^T D_\rho x_1 + 2\lambda_{\alpha 1}\lambda_{\beta 1} y_1^T M_\rho x_1)}{\lambda_{\beta 1}^2 y_1^T D x_1 + 2\lambda_{\alpha 1}\lambda_{\beta 1} y_1^T M x_1} \\ &= -\frac{y_1^T(\lambda_1^2 M_\rho + \lambda_1 D_\rho + K_\rho)x_1}{y_1^T(2\lambda_1 M + D)x_1} , \end{aligned} \tag{7}$$

producing a formula similar to (2).

Lastly, differentiating F and G with respect to ρ at $\rho = 0$ produces

$$F_\rho = \Lambda_{\alpha 2}\Phi_\rho - \Psi_\rho\Lambda_{\alpha 1} + \Lambda_{\alpha 2}Y_2^T M_\rho(0)X_1\Lambda_{\alpha 1} - \Lambda_{\beta 2}Y_2^T K_\rho(0)X_1\Lambda_{\beta 1} = 0 \ ,$$

$$G_\rho = \Lambda_{\beta 2}\Phi_\rho - \Psi_\rho\Lambda_{\beta 1} + \Lambda_{\alpha 2}\Lambda_{\beta 2}Y_2^T D_\rho(0)X_1\Lambda_{\beta 1} + \Lambda_{\alpha 2}Y_2^T M_\rho(0)X_1\Lambda_{\beta 1} + \Lambda_{\beta 2}Y_2^T M_\rho(0)X_1\Lambda_{\alpha 1} = 0 \ .$$

The derivatives $\Phi_\rho(0)$ and $\Psi_\rho(0)$ can then be retrieved from the above equations, when $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$ and $(\Lambda_{\alpha 2},\ \Lambda_{\beta 2})$ have nonintersecting spectra [2].

3.3 Palindromic case

For palindromic eigenvalue problems, the perturbation results can be obtained from those for general matrix quadratics, utilizing $M = A_1^T = K^T$, $D = A_0 = D^T$ and the palindromic properties of the eigenvalues and eigenvectors. For the spectrum, we have $(\Lambda_{\alpha 1},\ \Lambda_{\beta 1})$ and $(\Lambda_{\alpha 2},\ \Lambda_{\beta 2}) = (\Lambda_{\beta 1},\ \Lambda_{\alpha 1})$ representing, respectively, eigenvalues on/inside and on/outside the unit circle, and $[Y_1,\ Y_2] = [X_2,\ X_1]$.

For the simple palindromic eigenvalues λ_1 and λ_1^{-1}, **(??)** implies, at $\rho = 0$:

$$\frac{\partial\lambda_1}{\partial\rho} = -\frac{y_1^T\left[\lambda_1^2(A_1)_\rho^T + \lambda_1(A_0)_\rho + (A_1)_\rho\right]x_1}{y_1^T(2\lambda_1A_1^T + A_0)x_1} \ ,$$

$$\frac{\partial\left\{\lambda_1^{-1}\right\}}{\partial\rho} = \frac{y_1^T\left[(A_1)_\rho^T + \lambda_1^{-1}(A_0)_\rho + \lambda_1^{-2}(A_1)_\rho\right]x_1}{y_1^T(2\lambda_1A_1^T + A_0)x_1}$$

from $\frac{\partial\{\lambda_1^{-1}\}}{\partial\rho} = -\lambda_1^{-2}\frac{\partial\lambda_1}{\partial\rho}$ or $(2\lambda_1A_1 + A_0)x_1 = -(2\lambda_1^{-1}A_1^T + A_0)x_1$. Interestingly, the asymptotic relative errors of the pair of reciprocal eigenvalues equal to, at $\rho = 0$:

$$\frac{\rho}{\lambda_1^{\pm 1}}\frac{\partial\lambda_1^{\pm 1}}{\partial\rho} = \mp\rho\frac{y_1^T\left[\lambda_1(A_1)_\rho^T + (A_0)_\rho + \lambda_1^{-1}(A_1)_\rho\right]x_1}{y_1^T(2\lambda_1A_1^T + A_0)x_1} \ . \tag{8}$$

These are identical except of the opposite signs. Equation (8) can easily be understood through

$$\frac{1}{\lambda + \delta\lambda} = \frac{1}{\lambda\left(1 + \frac{\delta\lambda}{\lambda}\right)} \approx \frac{1}{\lambda}\left(1 - \frac{\delta\lambda}{\lambda}\right) \Rightarrow \frac{(\lambda + \delta\lambda)^{-1} - \lambda^{-1}}{\lambda^{-1}} \approx -\frac{\delta\lambda}{\lambda} \ .$$

After applying inequalities of norms, a condition number for both $\lambda_1^{\pm 1}$ can then be produced:

$$\kappa \equiv \frac{\|y_1\|_2\,\|x_1\|_2}{|y_1^T(2\lambda_1A_1^T + A_0)x_1|}\sqrt{|\lambda_1|^2 + 1 + |\lambda_1|^{-2}} \ . \tag{9}$$

Table 1: Perturbation of a palindromic eigenvalue problem

i	1	2, 3	4
λ_i	-0.5488554937	0.6854143467 $\pm$ 0.7281532623i	-1.8219731997
$\tilde{\lambda}_i$	-0.5488554724	0.6854143626 $\pm$ 0.7281532473i	-1.8219732705
$\delta\lambda_i$	2.1312649867e-08	1.5930742392e-08 $\pm$ 1.4995688913e-08i	$-$7.0749180514e-08
r_i	$-$3.8831076873e-08	$-$4.1288419705e-16 $\pm$ 2.1878282362e-08i	3.8831076399e-08
$\|r_i\|$	3.8831076873e-08	2.1878282362e-08	3.8831076399e-08
$r_i^{(e)}$	$-$3.8831078433e-08	4.7865922782e-16 $\pm$ 2.1878282332e-08i	3.8831078433e-08
$\|r_i^{(e)}\|$	3.8831078433e-08	2.1878282332e-08	3.8831078433e-08
κ_i	1.2701114034	1.5072107111	1.2701114034
$r_i^{(\kappa)}$	8.8772549782e-08	1.0534425368e-07	8.8772549782e-08

With appropriate scaling of the eigenvectors, κ can be interpreted as proportional to the product of the norms of the left- and the right-eigenvectors, or in terms of the angle between the eigenvectors, as in other algebraic eigenvalue problems.

4. Numerical Example

We shall consider the following small example to illustrate the results in Section 7.3:

$$A_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}.$$

The parameter ω can be used to vary the condition of the eigenvalue problem, but will be fixed to be 0.5 in the following calculations. The matrices are perturbed randomly to the magnitude of 0.5×10^{-7}, with $\delta = \|[\delta A_1^T, \delta A_0, \delta A_1]\| = 0.6989351449543007 \times 10^{-7}$. The eigenvalues are λ_i $(i = 1, \cdots, 4)$ with $\lambda_1 = \lambda_4^{-1}$ and $\lambda_2 = \lambda_3^{-1}$. The numerical results are summarized in Table 1, with $\tilde{\lambda}_i$ denoting the perturbed eigenvalues, $\delta\lambda_i \equiv \tilde{\lambda}_i - \lambda_i$, $r_i \equiv \delta\lambda_i/\lambda_i$ (the relative error in $\tilde{\lambda}_i$), $r_i^{(e)}$ estimating r_i using (8), κ_i the individual condition numbers as in (9), and $\left|r_i^{(e)}\right| \leq r_i^{(\kappa)} \equiv \kappa_i \delta$ estimating $|r_i|$. All calculations were carried out using MATLAB 7.1 (R14) on a Apple MacIntosh G4 Powerbook, with $eps \approx 2.2204 \times 10^{-16}$.

It is obvious from Table 1 that (8) provides accurate approximations to the relative errors of palindromic eigenvalues for small perturbations and the condition number κ in (9) produces tight upper bounds for the (relative) errors. Also, the fact that the relative errors for a reciprocal pairs of eigenvalues are negative of each other is confirmed by the example.

5. Conclusions

Results for eigenvalues and the corresponding deflating subspaces of palindromic linearizations and (palindromic) matrix quadratics are obtained using Sun's implicit function approach. Consistent results for simple eigenvalues and the corresponding eigenvetors/deflating subspaces are obtained using simple differentiation. These results indicate, not surprisingly, that the perturbations of an eigenvalue λ and its corresponding deflating subspace $\mathcal{S}_\lambda$, respectively, are proportional to the size of the perturbation and the reciprocal of the gap between $\mathcal{S}_\lambda$ and other deflating subspaces. Condition numbers are typically proportional the products of the norms of the left- and right-eigenvectors or deflating subspaces.

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