

PALINDROMIC EIGENVALUE PROBLEMS: A BRIEF SURVEY

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Abstract. The T-palindromic quadratic eigenvalue problem $(\lambda^2 B + \lambda C + A)x = 0$, with $A, B, C \in \mathbb{C}^{n \times n}$, $C^T = C$ and $B^T = A$, governs the vibration behaviour of trains. Other palindromic eigenvalue problems, quadratic or higher order, arise from applications in surface acoustic wave filters, optimal control of discrete-time systems and crack modelling. Numerical solution of palindromic eigenvalue problems is challenging, with unacceptably low accuracy from the basic linearization approach. In this survey paper, we shall talk about the history of palindromic eigenvalue problems, in terms of their history, applications, numerical solution and generalization. We shall also speculate on some future directions of research.

1. INTRODUCTION

Physical phenomena have been modelled in terms of differential equations ever since the invention of calculus, by Leibniz and Newton in the late seventeenth century. From Newton's second law, Kirchoff's law and the like, it is natural to model using second-order systems like

$$(1) \quad M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t)$$

where the state and input $x, f \in \mathbb{R}^n$, with the mass, damping and stiffness matrices $M, D, K \in \mathbb{R}^{n \times n}$. Despite of the equivalent first-order system of equations (in companion form):

$$(2) \quad \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix} \dot{z}(t) = \begin{bmatrix} 0 & I_n \\ -K & -D \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \quad z(t) \equiv \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

it is often more natural, sometimes even desirable, to work with the original matrices M, D and K from the original formulation (1). More importantly, applying blunt

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numerical tools to the companion form in (2) without considering or preserving its structure produces unnecessary errors.

In the field of numerical computation, an early account of matrix polynomials (under the name of λ -matrices) can be found in [36], followed by the authoritative book in [20] and the survey of applications and numerical methods in [56].

In general, we can analyze the solution or vibration of second-order systems through the quadratic eigenvalue problem (QEP)

$$(3) \quad Q(\lambda)x \equiv (\lambda^2 B + \lambda C + A)x = 0,$$

where $A, B, C \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$, $x \neq 0 \in \mathbb{C}^n$. Many QEPs from a variety of applications have extra structure that results in certain symmetry in the spectrum. One such QEP is the palindromic QEP with the property that reversing the order of the coefficients leads back to the original QEP, which explains the adjective “palindromic”.

If $B^T = A$ and $C^T = C$, then (3) is called a T-palindromic QEP:

$$(4) \quad Q(\lambda)x \equiv (\lambda^2 A^T + \lambda C + A)x = 0, \quad C^T = C$$

Transposing (4) implies the important *symplectic* or *reciprocity* property of the spectrum of palindromic eigenvalue problem, that

$$(5) \quad \lambda \in \sigma(Q(\lambda)) \Rightarrow 1/\lambda \in \sigma(Q(\lambda))$$

with $\sigma(\cdot)$ denoting the spectrum, and the convention that 0 and ∞ are considered to be mutually reciprocal.

In general, from [43], for a (possibly rectangular) matrix polynomial

$$Q(\lambda) \equiv \sum_{i=0}^k \lambda^i B_i, \quad B_i \in \mathbb{F}^{m \times n}$$

for some field \mathbb{F} , we can define the *adjoint* $Q^*(\lambda)$ and the *reversal* $\text{rev } Q(\lambda)$ by

$$Q^*(\lambda) \equiv \sum_{i=1}^k \lambda^i B_i^*, \quad \text{rev } Q(\lambda) \equiv \sum_{i=0}^k \lambda^{k-i} B_i$$

with $*$ = T or H . A matrix polynomial $Q(\lambda)$ is said to be **-palindromic* if $\text{rev } Q(\lambda) = Q^*(\lambda)$ and **-anti-palindromic* if $\text{rev } Q(\lambda) = -Q^*(\lambda)$. For a regular **-(anti-)palindromic* $Q(\lambda)$ (which is square with a discrete spectrum), we have the reciprocity property

$$(6) \quad \lambda \in \sigma(Q(\lambda)) \Rightarrow 1/\lambda^* \in \sigma(Q(\lambda))$$

similar to and more general than (5). The results in (5) and (6) are contained in the more general Theorem 5.1 later.

The T-palindromic QEP was first raised in a study of the vibration of fast trains in Germany [26, 27], associated with the company SFE GmbH in Berlin. Quadratic real and complex T-palindromic QEPs also arise in the mathematical modeling and numerical simulation of the behaviour of periodic surface acoustic wave (SAW) filters [58]. The computation of the Crawford number [25], associated with the perturbation analysis of symmetric generalized eigenvalue problems, produces an H-palindromic QEP, i.e., $B^H = A$ and $C^H = C$, where H stands for the conjugate transpose. The study of corner singularities in anisotropic elastic materials [2, 3, 34, 47, 51] and gyroscopic systems [56] leads to T-even QEPs, i.e., $B^T = B$, $C^T = -C$ and $A^T = A$. For other QEPs with symmetry in the spectrum, see [43].

A standard approach for solving the palindromic QEP is to transform it to a $2n \times 2n$ linearized eigenvalue problem and compute its generalized Schur form (see [56]). However, the symplectic property of eigenvalues of (3) is not preserved generally, producing large numerical errors [32]. Recently, some pioneering work [27, 23, 44] discovered that the T-palindromic QEP could be linearized into the form $\lambda Z^T + Z$, which preserves symplecticity to some extent. Later, a QR-like algorithm [52], a Jacobi-type method [27], a generalized Laub trick [45] and a URV-decomposition-based structured method of cubic complexity [53] have been proposed for the palindromic linear pencil $\lambda Z^T + Z$. However, the Jacobi-like method suffers from convergence problems and the QR-like method has quartic complexity [35]. Another T-symplectic linearization $\mathcal{M} - \lambda \mathcal{L}$ of the T-palindromic QEP has been developed in [29] and two structure-preserving methods based on Patel's and Arnoldi methods are proposed to solve the T-symplectic eigenvalue problem.

In [14], a structure-preserving doubling algorithm (SDA) was developed. The T-palindromic QEP can be rewritten as a factored form

$$Q(\lambda) = (\lambda A^T - X)X^{-1}(\lambda X - A)$$

with symmetries in the spectrum for some nonsingular X if and only if X is satisfied the following nonlinear matrix equation

$$(7) \quad A^T X^{-1} A + X + C = 0.$$

A SDA [14, 41] can then be applied to solve (7).

For perturbation and error analysis related to palindromic eigenvalue problems, see [15, 22, 24].

In this paper, we shall perform a survey of palindromic eigenvalue problems, in terms of their history, applications, numerical solution and generalization. After this introduction, we shall summarize the application of train vibration analysis and the

associated T-palindromic QEP in Section 2. Selected applications involving other palindromic eigenvalue problems will be presented in Section 3. A summary of numerical methods for palindromic eigenvalue problems is presented in Section 4. Some generalizations of palindromic eigenvalue problems and their solution are presented in Section 5 and the paper is concluded by some speculations on future research in Section 6.

2. TRAIN VIBRATION AND T-PALINDROMIC QEPs

We shall describe the train vibration problem, the associated T-palindromic QEP and the corresponding SDAs, quoting results from [14], without some details, proofs or numerical results.

We shall study the resonance phenomena of the track under high frequent excitation forces. Research in this area not only contributes to the safety of the operations of high-speed trains but also new designs of train bridges, embedded rail structures (ERS) and train suspension systems. Recently, the dynamic response of the vehicle-rails-bridge interaction system under different train speeds has been studied by Wu and Yang [57] and a procedure for designing an optimal ERS is proposed by Markine, de Man, Jovanovic and Esveld [46]. An accurate numerical estimation to the resonance frequencies of the rail plays an important role in both works. However, as mentioned by Ipsen [32], the classic finite element packages fail to deliver correct resonance frequencies for such problems [32]. Here, we would like to compare the method proposed by Mackey, Mackey, Mehl and Mehrmann [43] with the generalized SDA methods proposed by Chu, Fan, Lin and Wang [12] and Lin and Xu [41], in solving the palindromic eigenvalue problems arising from spectral modal analysis of the resonance of the rail under a periodic excitation force.

We assume that the rail sections between consecutive sleeper bays are identical, distances between consecutive wheels are the same and the wheel loads are equal. Figure 1 shows an example of the rail section we consider here.

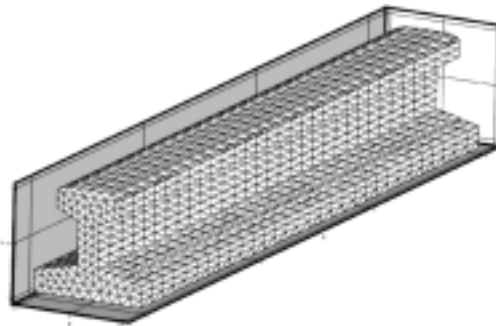


Fig. 1. A 3D rail model.

Base on our assumptions, we model the rail under cargo wheel loads by a section of rail between two sleepers. The external force is assumed to be periodic and the displacements of two boundary cross sections of the modelled rail are assumed to have a ratio κ , which is dependent on the excitation frequency of the external force. In the following, we consider the rail as a three dimensional isotropic elastic solid and a 3D finite element model of the solid with linear isoparametric tetrahedron elements is introduced.

From the element virtual work principle, the equilibrium state of the solid element e under external body forces satisfies the following equation:

$$(8) \quad \int_e (\delta\varepsilon^T) \mathcal{C} \varepsilon dV + \int_e (\delta q^T) \rho \ddot{q} dV = \int_e (\delta q^T) f dV.$$

Here, ρ is the mass density, f is the time-dependent body force, $q = [u, v, w]$ is the displacement vector, $\varepsilon = [\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}]$, and δq^T and $\delta\varepsilon^T$ are the virtual displacement and the corresponding virtual strain vectors, and

$$(9) \quad \mathcal{C} = \frac{E}{(1 + \nu)(1 - 2\nu)} \text{diag}(C_1, C_2),$$

is the well-known strain-stress relationship, where E is the Young’s modulus, ν is the Poisson ratio and

$$C_1 = \begin{bmatrix} 1 - \nu & \nu & \nu \\ \nu & 1 - \nu & \nu \\ \nu & \nu & 1 - \nu \end{bmatrix}, \quad C_2 = \left(\frac{1 - 2\nu}{2}\right) I_3.$$

Let ϕ_i and $[u_i, v_i, w_i]^T$ ($i = 1, \dots, 4$) be the linear nodal basis function and the nodal displacement vector associated with the i -th node of the element e , respectively, and let $X_e = [X_1^T, X_2^T, X_3^T, X_4^T]^T$, $B_e = [B_1, B_2, B_3, B_4]$ and $N_e = [N_1, N_2, N_3, N_4]$, where

$$X_i = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}, \quad N_i = \begin{bmatrix} \phi_i & 0 & 0 \\ 0 & \phi_i & 0 \\ 0 & 0 & \phi_i \end{bmatrix}, \quad B_i = \begin{bmatrix} \frac{\partial \phi_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \phi_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial \phi_i}{\partial z} \\ \frac{\partial \phi_i}{\partial y} & \frac{\partial \phi_i}{\partial x} & 0 \\ \frac{\partial \phi_i}{\partial z} & 0 & \frac{\partial \phi_i}{\partial x} \\ 0 & \frac{\partial \phi_i}{\partial z} & \frac{\partial \phi_i}{\partial y} \end{bmatrix}.$$

Equation (8) can now be discretized into the following linear equations

$$(10) \quad \sum_e \left(\int_e B_e^T \mathcal{C} B_e dV \right) X_e + \rho \left(\int_e N_e^T N_e dV \right) \ddot{X}_e = \sum_e \left(\int_e N_e^T f_e dV \right) F_e,$$

where $F_e = [F_1^T, F_2^T, F_3^T, F_4^T]^T$ and F_i ($i = 1, \dots, 4$) is the i -th nodal force vector acting on element e . In the following, we denote $K = \sum_e \int_e B_e^T \mathcal{C} B_e dV$, and $M = \sum_e \rho \int_e N_e^T N_e dV$. Equation (10) can now be written as

$$(11) \quad KX + M\ddot{X} = \rho^{-1}MF.$$

When considering the dynamic response of the solid, dissipative forces such as the force due to frictions have to be considered. Their effect is introduced in the form of the so-called viscous damping $D\dot{X}$ where D is the damping matrix. In this paper, proportional damping proposed by Strutt (Lord Rayleigh) [55] is employed where D is a linear combination of K and M . The equation of motion involving viscous damping can now be written as

$$KX + D\dot{X} + M\ddot{X} = \rho^{-1}MF.$$

Due to the given boundary conditions on a uniform mesh, K , D and M have the following form

$$\begin{bmatrix} G_{11} & G_{12} & 0 & \cdots & 0 & \frac{1}{\kappa}G_{m,m+1}^T \\ G_{12}^T & G_{22} & G_{23} & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & G_{m-2,m-1}^T & G_{m-1,m-1} & G_{m-1,m} \\ \kappa G_{m,m+1} & 0 & \cdots & 0 & G_{m-1,m}^T & G_{m,m} \end{bmatrix}$$

with $G_{ii} \in \mathbb{C}^{n_i \times n_i}$ for $i = 1, \dots, m$. Furthermore, from the spectral modal analysis, one considers $X = \hat{X}e^{i\omega t}$ where ω is the frequency of the external excitation force and \hat{X} is the corresponding eigenmode. Consequently, we arrive to a palindromic eigenvalue problem

$$(\kappa A_1 + A_0 + \kappa^{-1}A_1^T) \hat{X} = 0,$$

where $A_0, A_1 \in \mathbb{C}^{n \times n}$ with $n = n_1 + \dots + n_m$ and

$$(12) \quad [A_1]_{ij} = \begin{cases} K_{m,m+1} + i\omega D_{m,m+1} - \omega^2 M_{m,m+1} & (\text{if } i = m \text{ and } j = 1), \\ 0 & (\text{otherwise}), \end{cases}$$

$$(13) \quad [A_0]_{ij} = \begin{cases} K_{i,j} + i\omega D_{i,j} - \omega^2 M_{i,j} & (\text{if } i - 1 \leq j \leq i + 1), \\ 0 & (\text{otherwise}). \end{cases}$$

2.1. Deflation of T-palindromic QEPS

We shall consider the deflation of zero and infinite eigenvalues in this section. For the deflation of $\lambda = \pm 1$, consult [43].

From their definitions in (12), A_1 and A_0 can be partitioned as

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ L & 0 & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}, \quad A_0 = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12}^T & C_{22} & C_{23} \\ 0 & C_{23}^T & C_{33} \end{bmatrix} \in \mathbb{C}^{n \times n}$$

where $L \in \mathbb{C}^{n_m \times n_1}$, $C_{11} = C_{11}^T \in \mathbb{C}^{n_1 \times n_1}$, $C_{33} = C_{33}^T \in \mathbb{C}^{n_m \times n_m}$ and $C_{22} = C_{22}^T \in \mathbb{C}^{\ell \times \ell}$ with $\ell = n - n_1 - n_m$. Assume that C_{22} is nonsingular. We have observed that this assumption is generically valid from the numerical examples we have encountered. Otherwise, the preprocessing procedure in [27, 43] should be applied.

Let

$$\Theta = \begin{bmatrix} I_{n_1} & -C_{12}C_{22}^{-1} & 0 \\ 0 & I_\ell & 0 \\ 0 & -C_{23}^T C_{22}^{-1} & I_{n_m} \end{bmatrix}, \quad \Pi = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_{n_m} \\ 0 & I_\ell & 0 \end{bmatrix}.$$

Then, using a similarity transformation, $\mathcal{P}(\lambda)$ can be transferred to the following form

$$\begin{aligned} \Pi \Theta \mathcal{P}(\lambda) \Theta^T \Pi^T &= \begin{bmatrix} \lambda(C_{11} - C_{12}C_{22}^{-1}C_{12}^T) & L^T - \lambda C_{12}C_{22}^{-1}C_{23} & 0 \\ \lambda(\lambda L - C_{23}^T C_{22}^{-1}C_{12}^T) & \lambda(C_{33} - C_{23}^T C_{22}^{-1}C_{23}) & 0 \\ 0 & 0 & \lambda C_{22} \end{bmatrix} \\ (14) \quad &= \text{diag}(I_{n_1}, \lambda I_{n_m}, I_\ell) \begin{bmatrix} \mathcal{S}(\lambda) & 0 \\ 0 & \lambda C_{22} \end{bmatrix} \end{aligned}$$

where

$$\mathcal{S}(\lambda) = \begin{bmatrix} \lambda \tilde{C}_{11} & L^T - \lambda \tilde{C}_{12} \\ \lambda L - \tilde{C}_{12}^T & \tilde{C}_{22} \end{bmatrix}$$

with $\tilde{C}_{11} \equiv C_{11} - C_{12}C_{22}^{-1}C_{12}^T$, $\tilde{C}_{12} \equiv C_{12}C_{22}^{-1}C_{23}$ and $\tilde{C}_{22} \equiv C_{33} - C_{23}^T C_{22}^{-1}C_{23}$.

2.2. Structure-preserving doubling algorithms

We can obtain two different versions of SDA for the solution of T-palindromic QEPs arising from the train vibration problem. Other selected methods will be presented in Section 4.

After swapping the row-blocks, the pencil $\mathcal{S}(\lambda)$ is equivalent to

$$\lambda \begin{bmatrix} L & 0 \\ \tilde{C}_{11} & -\tilde{C}_{12} \end{bmatrix} + \begin{bmatrix} -\tilde{C}_{12}^T & \tilde{C}_{22} \\ 0 & L^T \end{bmatrix},$$

which is in a generalized standard symplectic form (GSSF) [30]. The structure-preserving doubling algorithm SDA1 in [30] can then be applied to solve the corresponding eigenvalue problem. During the iteration, some matrices are required to

be well-conditioned. When this does not hold, Cayley transforms can be applied to transform the corresponding symplectic matrix pair to an associated Hamiltonian matrix and then back, introducing free two parameters against which this condition can be optimized. For details, see [11].

In terms of accuracy and speed of convergence, SDA1 behaves similarly as SDA2 below. However, the operation count for SDA1 doubles that of SDA2. As a result, we shall not discuss SDA1 further. However, SDA1 is a weapon in reserve against difficult palindromic eigenvalue problems, when some assumptions for SDA2 are not satisfied.

For SDA2, assume that \tilde{C}_{22} is invertible. Define a new λ -matrix $\tilde{\mathcal{S}}(\lambda)$ as follows:

$$(15) \quad \begin{aligned} \tilde{\mathcal{S}}(\lambda) &\equiv \begin{bmatrix} I_{n_1} & -L^T \tilde{C}_{22}^{-1} \\ 0 & I_{n_3} \end{bmatrix} \mathcal{S}(\lambda) \begin{bmatrix} I_{n_1} & 0 \\ \tilde{C}_{22}^{-1} \tilde{C}_{12}^T & I_{n_3} \end{bmatrix} \\ &= \begin{bmatrix} \lambda(\tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12} \tilde{C}_{22}^{-1} \tilde{C}_{12}^T) + L^T \tilde{C}_{22}^{-1} \tilde{C}_{12}^T & -\lambda \tilde{C}_{12} \\ \lambda L & \tilde{C}_{22} \end{bmatrix} \end{aligned}$$

and let $[\tilde{x}^T, \tilde{y}^T]^T$ be an eigenvector of $\tilde{\mathcal{S}}(\lambda)$; i.e.,

$$(16a) \quad \lambda \left[(\tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12} \tilde{C}_{22}^{-1} \tilde{C}_{12}^T) \tilde{x} - \tilde{C}_{12} \tilde{y} \right] + L^T \tilde{C}_{22}^{-1} \tilde{C}_{12}^T \tilde{x} = 0,$$

$$(16b) \quad \lambda L \tilde{x} + \tilde{C}_{22} \tilde{y} = 0.$$

Since \tilde{C}_{22} is invertible, from (16b) \tilde{y} can be represented as

$$(17) \quad \tilde{y} = -\lambda \tilde{C}_{22}^{-1} L \tilde{x}.$$

Substituting (17) into (16a), we get the following new small size palindromic quadratic eigenvalue problem:

$$(18) \quad \mathcal{P}_d(\lambda) \tilde{x} \equiv (\lambda^2 A_{d1} + \lambda A_{d0} + A_{d1}^T) \tilde{x} = 0,$$

where

$$(19) \quad A_{d1} = \tilde{C}_{12} \tilde{C}_{22}^{-1} L \in \mathbb{C}^{n_1 \times n_1},$$

$$(20) \quad A_{d0} = \tilde{C}_{11} - L^T \tilde{C}_{22}^{-1} L - \tilde{C}_{12} \tilde{C}_{22}^{-1} \tilde{C}_{12}^T \in \mathbb{C}^{n_1 \times n_1}.$$

Suppose that X is nonsingular. Rewrite $\mathcal{P}_d(\lambda)$ in (18) as

$$\mathcal{P}_d(\lambda) = (\lambda A_{d1} - X) X^{-1} (\lambda X - A_{d1}^T) + \lambda (A_{d1} X^{-1} A_{d1}^T + X + A_{d0}).$$

It follows that $\mathcal{P}_d(\lambda)$ can be factorized (or square-rooted) as

$$\mathcal{P}_d(\lambda) = (\lambda A_{d1} - X)X^{-1}(\lambda X - A_{d1}^T)$$

for some nonsingular X if and only if X is satisfied following nonlinear matrix equation with the plus sign (NME):

$$(21) \quad A_{d1}X^{-1}A_{d1}^T + X + A_{d0} = 0.$$

We can easily prove the following lemma on the existence of the solutions of the NME:

Lemma 2.1. *Let $(\Lambda_1 \oplus \Lambda_2, [Y_1, Y_2])$ be an eigenpair of $\mathcal{P}_d(\lambda)$ in the sense that*

$$A_{d1}Y_i\Lambda_i^2 + A_{d0}Y_i\Lambda_i + A_{d1}^TY_i = 0 \quad (i = 1, 2),$$

where $Y_i \in \mathbb{C}^{n_1 \times n_1}$ for $i = 1, 2$. Suppose that A_{d1} and Y_i ($i = 1, 2$) are invertible. Then the corresponding NME (21) has the solutions $X = A_{d1}^TY_i\Lambda_i^{-1}Y_i^{-1}$ ($i = 1, 2$).

Evidently, there are many solutions to the NME, each will facilitate the factorization of $\mathcal{P}_d(\lambda)$ we aim for. Assume that there are no eigenvalues on the unit circle. Consequently, we can partition the spectrum into $\Lambda_s \oplus \Lambda_s^{-1}$, with Λ_s containing the stable eigenvalues (inside the unit circle). The SDA will seek a stable solution $X_s \equiv A_{d1}^TY_s\Lambda_s^{-1}Y_s^{-1}$, where Y_s contains the eigenvectors corresponding to Λ_s . Note that X_s is unique as it is independent of the order of the eigenvalues in Λ_s .

The structure-preserving doubling algorithm SDA2 in [41] can then be applied to solve the NME, and subsequently the palindromic eigenvalue problem. We require the invertibility of the matrices $Q_k - P_k$. This is the case for large values of k , as indicated by Corollary 2.1.

2.3. Convergence of SDA

The behaviour of the SDAs are well-documented in [12, 21, 30, 31, 41]. However, these results are mostly written for real problem with real variables and have to be modified for our situation. Following the development in [41], let $\mathcal{M} - \lambda\mathcal{L} \in \mathbb{C}^{2n \times 2n}$ be a T -symplectic pencil, in the sense that

$$(22) \quad \mathcal{M}J\mathcal{M}^T = \mathcal{L}J\mathcal{L}^T, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Define the nonempty null set

$$\begin{aligned} &\mathcal{N}(\mathcal{M}, \mathcal{L}) \\ &\equiv \left\{ [\mathcal{M}_*, \mathcal{L}_*] : \mathcal{M}_*, \mathcal{L}_* \in \mathbb{C}^{2n \times 2n}, \text{rank} [\mathcal{M}_*, \mathcal{L}_*] = 2n, [\mathcal{M}_*, \mathcal{L}_*] \begin{bmatrix} \mathcal{L} \\ -\mathcal{M} \end{bmatrix} = 0 \right\}. \end{aligned}$$

For any given $[\mathcal{M}_*, \mathcal{L}_*] \in \mathcal{N}(\mathcal{M}, \mathcal{L})$, define

$$\widehat{\mathcal{M}} = \mathcal{M}_* \mathcal{M}, \quad \widehat{\mathcal{L}} = \mathcal{L}_* \mathcal{L}.$$

The transformation $\mathcal{M} - \lambda \mathcal{L} \rightarrow \widehat{\mathcal{M}} - \lambda \widehat{\mathcal{L}}$ is a *doubling* transformation. Below is an adaptation of [41, Theorem 2.1]:

Theorem 2.1. *Let $\widehat{\mathcal{M}} - \lambda \widehat{\mathcal{L}}$ be a doubling transformation of a T -symplectic pencil $\mathcal{M} - \lambda \mathcal{L}$. Then we have:*

(a) *The pencil $\widehat{\mathcal{M}} - \lambda \widehat{\mathcal{L}}$ is still T -symplectic.*

(b) *If $\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} = \mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} S$, where $U, V \in \mathbb{C}^{n \times m}$ and $S \in \mathbb{C}^{m \times m}$, then*

$$\widehat{\mathcal{M}} \begin{bmatrix} U \\ V \end{bmatrix} = \widehat{\mathcal{L}} \begin{bmatrix} U \\ V \end{bmatrix} S^2.$$

(c) *If the pencil $\mathcal{M} - \lambda \mathcal{L}$ has the Kronecker canonical form*

$$(23) \quad W \mathcal{M} Z = \begin{bmatrix} J_r & 0 \\ 0 & I_{2n-r} \end{bmatrix}, \quad W \mathcal{L} Z = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r} \end{bmatrix}$$

where W, Z are nonsingular; J_r a Jordan matrix corresponding to the finite eigenvalues of $\mathcal{M} - \lambda \mathcal{L}$ and N_{2n-r} a nilpotent Jordan matrix corresponding to the infinite eigenvalues of $\mathcal{M} - \lambda \mathcal{L}$, then there exists a nonsingular matrix \widehat{W} such that

$$(24) \quad \widehat{W} \widehat{\mathcal{M}} Z = \begin{bmatrix} J_r^2 & 0 \\ 0 & I_{2n-r} \end{bmatrix}, \quad \widehat{W} \widehat{\mathcal{L}} Z = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r}^2 \end{bmatrix}.$$

It is easy to verify that NME (21) has a symmetric nonsingular solution X if and only if X satisfies

$$\mathcal{M} \begin{bmatrix} I \\ X \end{bmatrix} = \mathcal{L} \begin{bmatrix} I \\ X \end{bmatrix} S$$

for some $S \in \mathbb{C}^{n \times n}$, where

$$\mathcal{M} \equiv \begin{bmatrix} A_{d1}^T & 0 \\ -A_{d0} & -I \end{bmatrix}, \quad \mathcal{L} \equiv \begin{bmatrix} 0 & I \\ A_{d1} & 0 \end{bmatrix}.$$

Note that $\mathcal{M} - \lambda \mathcal{L}$ is in the second standard symplectic form (SSF-2) [41].

For the convergence of the SDA, we have the following adaptation of [41, Theorem 4.1]:

Theorem 2.2. *Let X be a symmetric invertible solution of (21) and let $S = X^{-1}A_{d1}$. Then the matrix sequences $\{R_k\}$, $\{Q_k\}$ and $\{P_k\}$ generated by SDA2 satisfy*

- (a) $R_k = (X - P_k)S^{2^k}$;
- (b) $Q_k - P_k = (X - P_k) + R_k^T(X - P_k)^{-1}R_k$;
- (c) $Q_k - X = (S^T)^{2^k}(X - P_k)S^{2^k}$;

provided that all the required inverses of $Q_k - P_k$ exist.

Note that Theorem 2.2 provides only the algebraic expressions for R_k , $Q_k - P_k$ and $Q_k - X$. Convergence to the unique symmetric stable solution X_s , which the SDA seeks, is summarized in the following Corollary.

Corollary 2.1. *When S is stable, $R_k \rightarrow 0$ and $Q_k \rightarrow X$ quadratically as $k \rightarrow \infty$.*

3. OTHER APPLICATIONS

There are many applications involving palindromic eigenvalue problems and we shall describe selected ones in this Section. For other examples, please consult [42].

3.1. Surface acoustic wave filters

We shall present a summary of the treatment of surface acoustic wave (SAW) filters in [58].

SAW filters are important in the telecommunication industry. These filters are built on the physical property of piezoelectric materials, that electrical charges induce mechanical deformations and vice versa. The main component (or cell) of an SAW filter composes of a piezoelectric substrate and the input and output interdigital transducers (IDT). An input electrical signal from the input IDT produces a surface acoustic wave, travelling through a gap and the output IDT picks up the output electrical signal. Depending on the material used and geometry of the gap, some frequencies are then stopped or filtered off. An associated palindromic QEP characterizes this filtering process.

Assuming that large number of equally spaced cells are placed along a straight line. Because of symmetry, a 2-dimensional model can be applied. We want to solve the (undamped) 2-D harmonic wave equation with periodic coefficients in the direction of the wave propagation x_1 , for $u(x, t)$ in

$$\operatorname{div}_x(a(x)\nabla_x u(x, t)) = u_{tt}(x, t), \quad a(x_1 + p, x_2) = a(x_1, x_2)$$

The positive function a describes the periodic properties of the material in the x_1 -direction. With separation of variables, assuming the form of the solution $u(x)e^{i\omega t}$, we aim to solve

$$-\operatorname{div}(a\nabla u(x)) = \omega^2 u(x)$$

Applying the Floquet-Bloch theory [58, Theorem 4.4.1], the equation is solved in a domain containing only one cell in the domain $\Omega_p = [0, p]$, with appropriate periodic boundary conditions on the boundaries Γ_L and Γ_R on the left and the right. The corresponding weak or variational formulation has the form, for some test function $v(x)$:

$$\int_{\Omega_p} \operatorname{div}_x(a(x)\nabla u(x)) \cdot v(x) dx - \int_{\Omega_p} \omega^2 u(x)v(x) dx = 0$$

The usual process of integration-by-parts, incorporating the boundary conditions, yields the formulation for u :

$$(25) \quad a_1(u, v) - \omega^2 a_0(u, v) = \langle \bar{w}, v \rangle$$

where the bilinear forms a_0 and a_1 and the inner product $\langle \bar{w}, v \rangle$ are defined by

$$\begin{aligned} a_0 &\equiv \int_{\Omega_p} u(x)v(x) dx, \quad a_1 \equiv \int_{\Omega_p} a(x)\nabla u(x)\nabla^T v(x) dx, \quad \langle \bar{w}, v \rangle \\ &\equiv \left(\int_{\Gamma_L} \tilde{w}_l(x)v(x) dx \right) \cdot \left(\int_{\Gamma_R} \tilde{w}_r(x)v(x) dx \right) \end{aligned}$$

After applying the quasi-periodic condition

$$(26) \quad u_r = \gamma u_l$$

a finite element discretization of the domain Ω_p produces the system of equations

$$(27) \quad (K - \omega^2 M)u = w$$

where

$$K = \begin{bmatrix} K_{ii} & K_{il} & K_{ir} \\ K_{li} & K_{ll} & K_{lr} \\ K_{ri} & K_{rl} & K_{rr} \end{bmatrix}, \quad M = \begin{bmatrix} M_{ii} & M_{il} & M_{ir} \\ M_{li} & M_{ll} & M_{lr} \\ M_{ri} & M_{rl} & M_{rr} \end{bmatrix}, \quad u = \begin{bmatrix} u_i \\ u_l \\ \gamma u_l \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ w_l \\ -\gamma w_l \end{bmatrix}$$

Here the subscripts i, l and r referred the interior, the left and the right of the domain Ω_p . It is easy to see that mass and stiffness matrices M and K are respectively constructed using the bilinear forms a_0 and a_1 . Consequently, M, K, M_{jj} and K_{jj} ($j = i, l, r$) are symmetric and positive definite. Furthermore, the lack of direct interaction between the left and the right boundaries Γ_L and Γ_R implies that $K_{lr} = K_{rl}$ and $M_{lr} = M_{rl}$ vanishes.

We are interested in the relationship between the frequency ω and the propagation factor γ . For a given γ , (27) can be considered a rectangular constrained

eigenvalue problem for (ω^2, u) , with w the result of the boundary conditions (see also Section 3.2 below). For a given ω , denote

$$\hat{K}(\omega) \equiv K - \omega^2 M = \left[\begin{array}{c|cc} \hat{K}_{ii} & \hat{K}_{il} & \hat{K}_{ir} \\ \hline \hat{K}_{li} & \hat{K}_{ll} & \hat{K}_{lr} \\ \hline \hat{K}_{ri} & \hat{K}_{rl} & \hat{K}_{rr} \end{array} \right] = \left[\begin{array}{cc} \hat{K}_{ii} & \hat{K}_{ib} \\ \hat{K}_{bi} & \hat{K}_{bb} \end{array} \right]$$

with the subscript b referring to the boundary $\Gamma_L \cup \Gamma_R$.

There are three possible approaches in [58]. To obtain a T-palindromic QEP, Approach 1 (Schur complement method) eliminates u_i from (27) to produce

$$[\gamma^2 S_{12} + \gamma(S_{11} + S_{22}) + S_{12}^T] u_l = 0$$

where the symmetric matrix

$$S = \left[\begin{array}{cc} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{array} \right] = \hat{K}_{bb} - \hat{K}_{bi} \hat{K}_{ii}^{-1} \hat{K}_{ib}$$

With a small amount of attenuation (with a small α), the propagation factor $\gamma = e^{\alpha+i\beta} \approx e^{i\beta}$, we are interested in the eigenvalues $\lambda = \gamma$ near the unit circle. Without attenuation, we are looking for eigenvalues $\lambda = \gamma = e^{i\beta}$ on the unit circle. Note that the inversion of \hat{K}_{ii} can be expensive as there are usually more internal variables in u_i than other variables.

From the quasi-periodic condition (26), the multiplication of γ and γ^{-1} respectively represent the propagation of signals to the right or the left. This, and similar considerations in other applications, correspond to the reciprocity in (5) and (6).

In Approach 2, with $\gamma = e^{i\beta}$ and recognizing that

$$u = \left[\begin{array}{c} u_i \\ u_l \\ \gamma u_l \end{array} \right] = T \tilde{u}, \quad T \equiv \left[\begin{array}{cc} I_i & 0 \\ 0 & I_l \\ 0 & \gamma I_l \end{array} \right], \quad \tilde{u} \equiv \left[\begin{array}{c} u_i \\ u_l \end{array} \right]$$

and $T^H w = 0$, we obtain the generalized eigenvalue problem

$$(T^H K T - \omega^2 T^H M T) \tilde{u} = 0$$

Here both $T^H K T$ and $T^H M T$ are complex, Hermitian, positive definite and dependent on $\gamma \in \mathbb{C}$. For the general attenuated case, the right-hand-side w in (27) is annihilated by the pre-multiplication of $T_1^H = I_i \oplus [I_l, \gamma^{-1} I_l]$ to produce a non-Hermitian eigenvalue problem $(T_1^H K T - \omega^2 T_1^H M T) \tilde{u} = 0$.

Approach 3 (inner-node-matrix or INM method) starts from an equivalent form of (27):

$$\tilde{K}(\omega) = \begin{bmatrix} \hat{K}_{ii} & \hat{K}_{il} + \gamma \hat{K}_{ir} \\ \hat{K}_{li} & \hat{K}_{ll} + \gamma \hat{K}_{lr} \\ \hat{K}_{ri} & \hat{K}_{rl} + \gamma \hat{K}_{rr} \end{bmatrix} \tilde{u} = w$$

Similar to Approach 2, pre-multiplication by $I_i \oplus [\gamma I_l, I_l]$ annihilates w and yields the generalized eigenvalue problem

$$\left(\gamma \begin{bmatrix} 0 & \hat{K}_{ir} \\ \hat{K}_{il}^T & \hat{K}_{ll} + \hat{K}_{rr} \end{bmatrix} + \begin{bmatrix} \hat{K}_{ii} & \hat{K}_{il} \\ \hat{K}_{ir}^T & 0 \end{bmatrix} \right) \tilde{u} = 0$$

as $\hat{K}_{lr} = 0$. The pencil is in the form $A - \lambda B$ where

$$(28) \quad A = \begin{bmatrix} M_1 & G \\ F^T & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -F \\ -G^T & -M_2 \end{bmatrix}$$

with symmetric M_1 and M_2 . The form of the pencil in (28) is new and can easily be proved to possess a spectrum satisfying the reciprocity property (5).

For the damped problem, (27) is modified, by Rayleigh damping or otherwise, to

$$(K - i\omega C + \omega^2 M)u = w$$

with C being symmetric. Subsequent development will then produce a complex symmetric $\tilde{K}(\omega)$.

For the purpose of filtering, the eigen-curves or dispersion diagram $\beta(\omega)$ are required, making our problem more than the task of solving the eigenvalue problems. For the continuation of the curve $\beta_k(\omega)$, we have to calculate the derivatives $\frac{\partial \beta_k}{\partial \omega}$ ([1]), in a structure-preserving way.

3.2. Rectangular eigenvalue problems

Some eigenvalue problem for differential operators with various boundary conditions, similar to the one in Section 3.1, can be discretized to an algebraic eigenvalue problem with constraints. These in turn can be rewritten as rectangular eigenvalues [10].

Projecting onto the orthogonal complement of w , (27) can be written as a rectangular eigenvalue problem (REVP) [5, 54]

$$(29) \quad (A - \lambda B)x = 0$$

with rectangular A and B . REVPs are closely related to QEPs. For example, a necessary condition for (29) is $v(B^H + \lambda A^H)(A - \lambda B)v = 0$, involving a generalized

palindromic QEP (H-odd-anti-palindromic, with $Q(\lambda)^H = -\text{rev } Q(-\lambda)$). Similarly, another necessary condition $(A - \lambda B)^T(\lambda A - B)x = 0$ is a T-palindromic QEP. More interestingly, a necessary condition for the minimal perturbation approach in the treatment of (29) [5] involved the H-palindromic QEP $(B^H + \lambda A)(A - \lambda B)v = 0$. Note that a infinitesimally small perturbation to A or B changes various invariant subspaces and the nature or existence of the solutions of (29). Consequently, REVPs can only be considered sensibly by looking for the nearest neighbouring problem to (29) for which solutions exist. Another possibility is to look for (λ, x) so that (29) is satisfied in the least squares or minimal residual sense [10]. Using Lagrange multipliers, a necessary condition for the solution will be

$$x^H \left[-\bar{\alpha}^2 A^H B + \bar{\alpha} \bar{\beta} (A^H A + B^H B) - \bar{\beta}^2 B^H A \right] x = 0$$

involving a H-palindromic QEP. For other applications involving REVPs, see [5, 10, 54].

3.3. Crack modelling

We shall summarize how palindromic QEPs can be derived from the problem of crack modelling, following the approach in [17] (see also [2, 4, 6, 37, 50] for similar formulations).

We are interested in the solution of a linear-elasticity problem in non-smooth domains, like cracks or sudden changes of the type of boundary conditions and material properties. In the simplest case of a canonical point in \mathbb{R}^3 the solution is given as an asymptotical expansion:

$$u = \sum_i \sum_{k=0}^{k_i} K_{ik} r^{\lambda_i} \ln^k(r) f_{ik}(\theta, \varphi)$$

where (r, θ, φ) are the spherical coordinates at the singular point. The singularity exponents λ_i and the angular functions f_{ik} only depend on the local geometry and the material properties, but not on the applied loads, unlike the corner intensity factors K_{ik} . To estimate or understand the behaviour of the solution u , we require the orders and the corresponding modes λ_i and $f_i = (f_{i1}, \dots, f_{i,k_i})^T$, or the eigenvalues and eigenvectors of the associated nonlinear eigenvalue problem. For example, for crack propagation, $u_{\theta\theta}$ and $u_{\varphi\varphi}$ has to be maximized.

The eigenvalue problem can be approximated, after the separation of spatial and time variables, through a finite element discretization for (θ, φ) on the unit sphere in \mathbb{R}^2 . In the domain Ω_O^ϵ (typically a wedge or cone of the unit sphere), we are seeking u which satisfies

$$\mathcal{L}(u) \equiv \Delta u + \frac{1}{1 - 2\nu} \nabla(\nabla \cdot u) = 0$$

together with various boundary condition. In its variation or weak formulation, we have to solve

$$B(u, v) \equiv \int_{\Omega_\varepsilon} \sigma^T(u) \varepsilon(v) d\Omega = 0$$

for some stress and strain tensors ε and σ , respectively, and test function v . By separation of variables, we choose $u = r^\lambda U(\theta, \varphi)$ and $v = \Phi(r)V(\theta, \varphi)$ where $\Phi(r)$ has a compact support in r . Approximated by finite elements, we have

$$(30) \quad B(u^h, v^h) = 0$$

where

$$u_i^h(r, \theta, \varphi) = r^\lambda N(\theta, \varphi) L_i^{-1} d_i, \quad v_i^h(r, \theta, \varphi) = \Phi(r) N(\theta, \varphi) L_i^{-1} g_i$$

with $N(\theta, \varphi) = (1, \theta, \varphi) \oplus (1, \theta, \varphi) \oplus (1, \theta, \varphi)$ and

$$L_i = \begin{bmatrix} N(\theta_1, \varphi_1) \\ N(\theta_2, \varphi_2) \\ N(\theta_3, \varphi_3) \end{bmatrix}$$

The corresponding strain vectors can be obtained by

$$(31) \quad \begin{aligned} \varepsilon(u_i^h) &= r^{\lambda-1} T_\varepsilon [F_0(\theta, \varphi) + \lambda F_1(\theta, \varphi)] L_i^{-1} d_i, \quad \varepsilon(v_i^h) \\ &= T_\varepsilon \left[\frac{\Phi(r)}{r} F_0(\theta, \varphi) + \Phi'(r) F_1(\theta, \varphi) \right] L_i^{-1} g_i \end{aligned}$$

with the Boolean matrix

$$T_\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and $F_0 = [\hat{F}_0, \hat{F}_0, \hat{F}_0]$, $F_1 = [\hat{F}_1, \hat{F}_1, \hat{F}_1]$ and

$$\hat{F}_0 = \begin{bmatrix} 0 & \cos \varphi \cos \theta & -\frac{\sin \varphi}{\sin \theta} \\ 0 & \sin \varphi \cos \theta & \frac{\cos \varphi}{\sin \theta} \\ 0 & -\sin \theta & 0 \end{bmatrix}, \quad \hat{F}_1 = \begin{bmatrix} \sin \theta \cos \varphi & \theta \sin \theta \cos \varphi & \varphi \sin \theta \cos \varphi \\ \sin \theta \sin \varphi & \theta \sin \theta \sin \varphi & \varphi \sin \theta \sin \varphi \\ \cos \theta & \theta \cos \theta & \varphi \cos \theta \end{bmatrix}$$

From the relationship between stresses and strains given by $\sigma(u_i^h) = \mathcal{C} \varepsilon(u_i^h)$ with \mathcal{C} from (9), we obtain the stress vector

$$(32) \quad \sigma(u_i^h) = r^{\lambda-1} \mathcal{C} T_\varepsilon [F_0(\theta, \varphi) + \lambda F_1(\theta, \varphi)] L_i^{-1} d_i$$

Substituting (31) and (32) into (30), we obtain

$$\sum_{i=1}^N d_i^T k_i g_j = 0, \quad \forall g_j, \quad j = 1, \dots, M$$

which is equivalent to

$$(33) \quad \sum_{i=1}^N k_i^T d_i = 0$$

with

$$(34) \quad k_i = L_i^{-1} B_i L_i^{-1}, \quad Z = T_\epsilon^T C T_\epsilon$$

and

$$\begin{aligned} B_i &= \int_{\Omega_i} r^{\lambda-1} (F_0 + \lambda F_1)^T Z \left[\frac{\Phi(r)}{r} F_0 + \Phi'(r) F_1 \right] r^2 \sin \theta \, dr \, d\theta \, d\varphi \\ &= \int_{\Omega_i} r^{\lambda+1} \Phi'(r) (F_0 + \lambda F_1)^T Z F_1 \sin \theta \, dr \, d\theta \, d\varphi \\ &\quad + \int_{\Omega_i} r^\lambda \Phi(r) (F_0 + \lambda F_1)^T Z F_0 \sin \theta \, dr \, d\theta \, d\varphi \end{aligned}$$

The integration is carried out over $\Omega_\epsilon = \{(r, \theta, \varphi) \in [0, \epsilon) \times \Delta_i\}$. Integrating the first integral by parts with respect to r , we have

$$\begin{aligned} \mathcal{I} \int_0^\epsilon r^{\lambda+1} \Phi'(r) \, dr &= \mathcal{I} \left[r^{\lambda+1} \Phi(r) \right]_0^\epsilon - \mathcal{I} \int_0^\epsilon (\lambda + 1) r^\lambda \Phi(r) \, dr, \\ \mathcal{I} &\equiv \int_{\Delta_i} (F_0 + \lambda F_1)^T Z F_1 \sin \theta \, d\theta \, d\varphi \end{aligned}$$

Because $\Phi(r)$ has a local support in $[0, \epsilon)$, the integral-free term vanishes and we obtain

$$\begin{aligned} B_i &= -c \int_{\Delta_i} [(\lambda + 1)(F_0 + \lambda F_1)^T Z F_1 + (F_0 + \lambda F_1)^T Z F_0] \sin \theta \, d\theta \, d\varphi, \\ c &\equiv \int_0^\epsilon r^\lambda \Phi(r) \, dr \end{aligned}$$

Substituting into (34), we obtain

$$k_i = L_i^{-1} B_i L_i^{-1} = c [(K_i - D_i) + \lambda(D_i^T - D_i - M_i) - \lambda^2 M_i]$$

with

$$\begin{aligned} L_i^T M_i L_i &\equiv \int_{\Delta_i} F_1^T Z F_1 \sin \theta \, d\theta \, d\varphi, \\ L_i^T D_i L_i &\equiv \int_{\Delta_i} F_0^T Z F_1 \sin \theta \, d\theta \, d\varphi, \\ L_i^T K_i L_i &\equiv \int_{\Delta_i} F_0^T Z F_0 \sin \theta \, d\theta \, d\varphi \end{aligned}$$

Substituting back into (33) and ignoring the constant c , we obtain the QEP

$$[\lambda^2 M + \lambda(M - D + D^T) + (D^T - K)] d = 0$$

where $M \equiv \sum_{i=1}^N M_i$, $D \equiv \sum_{i=1}^N D_i$ and $K \equiv \sum_{i=1}^N K_i$, with M and K are symmetric.

Finally, with the shift $\mu = \lambda + \frac{1}{2}$, we have the QEP

$$\tilde{Q}(\lambda)d = 0, \quad \tilde{Q}(\lambda) \equiv \mu^2 R + \mu Q + P$$

with

$$R = -M = R^T, \quad Q = D - D^T = -Q^T, \quad P = K + \frac{1}{4}M - \frac{1}{2}(D + D^T) = P^T$$

Note that $\tilde{Q}(\mu)$ is T-even [43] or $\tilde{Q}^T(-\mu) = \tilde{Q}(\mu)$, with $\mu \in \sigma(\tilde{Q}(\lambda))$ implying $-\mu \in \sigma(\tilde{Q}(\lambda))$. Two T-palindromic QEPs are produced from the Cayley transforms

$$\hat{Q}_-(\mu) \equiv (\mu + 1)^2 \tilde{Q} \left(\frac{\mu - 1}{\mu + 1} \right), \quad \hat{Q}_+(\mu) \equiv (1 - \mu)^2 \tilde{Q} \left(\frac{1 + \mu}{1 - \mu} \right)$$

3.4. Optimal control of discrete-time systems

Consider the discrete-time optimal control problem

$$\min_{\{u_j\}} \sum_{j=0}^{\infty} \begin{bmatrix} x_j \\ u_j \end{bmatrix}^H \begin{bmatrix} Q & S \\ S^H & R \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}$$

with $Q^H = Q$, $M_i \in \mathbb{F}^{n \times n}$, $S, B \in \mathbb{F}^{n \times n}$ and $R^H = R \in \mathbb{F}^{m \times m}$, for some scalar field \mathbb{F} , subject to the discrete-time control

$$\sum_{i=0}^{2l} M_i x_{i+l} = B u_i$$

for given $x_0, x_1, \dots, x_{2l-1}$. There are more than one way to produce some higher order *-palindromic polynomials [7, 28] and we shall present an old approach in

[42]. The discrete-time control system or difference equation associated with our problem corresponds to the matrix polynomial

$$P_s(\lambda) = \lambda^{2l} \begin{bmatrix} 0 & M_0 & 0 \\ M_{2l}^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda^{2l-1} \begin{bmatrix} 0 & M_1 & 0 \\ M_{2l-1}^* & Q & 0 \\ 0 & S^* & 0 \end{bmatrix} + \lambda^{2l-2} \begin{bmatrix} 0 & M_2 & 0 \\ M_{2l-2}^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ + \dots + \lambda^2 \begin{bmatrix} 0 & M_{2l-2} & 0 \\ M_2^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & M_{2l-1} & 0 \\ M_1^* & 0 & 0 \\ -B & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & M_{2l} & -B \\ M_0^* & 0 & S \\ 0 & 0 & R \end{bmatrix}$$

Multiply $P_s(\lambda)$ on the left and the right, respectively, by $\text{diag}(\lambda^{l-1}I_n, I_n, \lambda^l I_m)$ and $\text{diag}(I_n, \lambda^{1-l}I_n, I_m)$, we produce the *-palindromic polynomial

$$P_p(\lambda) = \lambda^{2l} \begin{bmatrix} 0 & M_0 & 0 \\ M_{2l}^* & 0 & 0 \\ 0 & S^* & 0 \end{bmatrix} + \lambda^{2l-1} \begin{bmatrix} 0 & M_1 & 0 \\ M_{2l-1}^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ + \dots + \lambda^{l+2} \begin{bmatrix} 0 & M_{l-2} & 0 \\ M_{l+2}^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda^{l+1} \begin{bmatrix} 0 & M_{l-1} & 0 \\ M_{l+1}^* & 0 & 0 \\ -B & 0 & 0 \end{bmatrix} \\ + \lambda^l \begin{bmatrix} 0 & M_l & 0 \\ M_l^* & Q & 0 \\ 0 & 0 & R \end{bmatrix} + \lambda^{l-1} \begin{bmatrix} 0 & M_{l+1} & -B \\ M_{l-1}^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ + \lambda^{l-2} \begin{bmatrix} 0 & M_{l+2} & 0 \\ M_{l-2}^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots + \lambda \begin{bmatrix} 0 & M_{2l-1} & 0 \\ M_1^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & M_{2l} & 0 \\ M_0^* & 0 & S \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\det P_p(\lambda) = \lambda^{lm} \det P_s(\lambda)$, the two polynomials share the same finite eigenvalues (counting multiplicities) except for the additional lm zero eigenvalues for $P_p(\lambda)$.

3.5. Computation of Crawford numbers

Consider the Crawford number [16]

$$\gamma(A, B) \equiv \min_{z \in \mathbb{C}^n, \|z\|_2=1} \sqrt{(z^H A z)^2 + (z^H B z)^2}$$

for two Hermitian matrices $A, B \in \mathbb{C}^{n \times n}$. From [42], the problem of computing the Crawford number is reduced to computing

$$\max \{ \lambda_{\min}(M(z)) : |z| = 1 \}$$

where $M(z) \equiv (z^{-1}C + zC^H)/2$ is Hermitian for z on \mathbb{D} and $C = A + iB$. With

$$\det(M(z) - \xi I) = 0 \Leftrightarrow \det(C - 2\xi zI + z^2C^H) = 0$$

for $\xi \in \mathbb{R}$ and $z \in \mathbb{D}$, a bisection search search can be devised. For any $\xi \in \mathbb{R}$, calculate all the unimodular eigenvalues z_i of the H-palindromic quadratic pencil $C - 2\xi z_i I + z_i^2 C^H$ and the corresponding $\lambda_{\min}(M(z_i))$. If $\lambda_{\min}(M(z_i)) = \xi$ then $\gamma(A, B) \geq \lambda_{\min}(M(z_i))$; otherwise $\gamma(A, B) < \lambda_{\min}(M(z_i))$.

4. OTHER NUMERICAL METHODS

We have presented the SDAs in Section 2.2 (which will be generalized in Section 5) for the T-palindromic QEPs arisen from the train vibration problems. We shall describe other selected methods in this Section. The linearization approach in Section 4.1 is easy to apply but there is a lack of structure-preserving methods for the resulting (palindromic) linearizations. For some palindromic QEPs, a generalized Patel method is applicable and will be described in Section 4.2. For a particular palindromic eigenvalue problem, it may have to be quadratized [39, 40] before SDAs or generalized Patel methods can be applied. Ultimately, methods have to be selected or designed appropriate to the particular structures of the application at hand and the amount of information sought from the associated palindromic eigenvalue problem.

4.1. Linearization

It is well-known that QEPs can be solved using various linearizations, such as

$$\begin{bmatrix} 0 & I \\ -A_1 & -A_0 \end{bmatrix} \begin{bmatrix} x \\ \lambda x \end{bmatrix} = \lambda \begin{bmatrix} I & 0 \\ 0 & A_1^T \end{bmatrix} \begin{bmatrix} x \\ \lambda x \end{bmatrix}$$

or

$$\begin{bmatrix} -A_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \lambda x \end{bmatrix} = \lambda \begin{bmatrix} A_0 & A_1^T \\ I & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda x \end{bmatrix}$$

The standard QZ algorithms can then be applied. However, QZ cannot preserve the symplectic structure, producing no significant figures in numerical experiments.

In [7, 23, 43, 44, 45, 42], a palindromic linearization of the form

$$(35) \quad \lambda Z + Z^T$$

with

$$Z = \begin{bmatrix} A_1^T & A_0 - A_1 \\ A_1^T & A_1^T \end{bmatrix}$$

was discovered for the palindromic QEP. This linearization preserves symplecticity of our problem to some extent but the accuracy of the eigenpairs from the non-structure-preserving QZ algorithm was still not good enough for application purposes. Scaling [18] improves the accuracy slightly but is not effective for general large ill-conditioned problems. Note that the pencil is $2n \times 2n$ and the QZ algorithm requires $O((2n)^3)$ flops. Relatively cheaper and numerically much more accurate and better behaved, the doubling algorithm in [14] and Section 2.2 works with $n \times n$ matrices and performs well for the T-palindromic QEP arisen from the train vibration problem.

For structure-preserving algorithms, attempts have been made to transform Z in (35) to an anti-triangular Schur form (Mackey/Mackey/Mehl/Mehrmann 2007, Schröder 2007)

$$\lambda Z + Z^T \longrightarrow \lambda \begin{bmatrix} & & & \times \\ & & \times & \times \\ & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} + \begin{bmatrix} & & & \times \\ & & \times & \times \\ & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$$

Congruence transformations in terms of unitary matrices have been used. However, the Jacobi-like method is inefficient, suffering from convergence problem. The QR-like method has n^4 complexity and is not competitive in efficiency, unless Z is given in Hessenberg form. The efficient transformation of Z to Hessenberg form is still an open problem. Also, a backward stable URV-based method has been proposed. See [29] for the details on the equivalence of the URV-method and the generalized Patel method for palindromic QEPs.

As for other palindromic linearizations other than (35), start from the companion linearization

$$\lambda L - M \equiv \lambda \begin{bmatrix} I & 0 \\ A_0 & A_1^T \end{bmatrix} - \begin{bmatrix} 0 & I \\ -A_1 & 0 \end{bmatrix}$$

One step of doubling produces

$$\tau \widehat{L} - \widehat{M} = \tau \begin{bmatrix} A_1^T & 0 \\ A_0 & A_1^T \end{bmatrix} + \begin{bmatrix} A_1 & A_0 \\ 0 & A_1 \end{bmatrix}$$

with $\tau = \lambda^2$ and the same eigenvector $(x^T, \lambda x^T)^T$.

Scaling the first row-blocks by $-A_1^{-1}$ and the second column-blocks by A_1^{-T} , then swap the roles of \widehat{L} and \widehat{M} , we have a SSF form

$$\tau \begin{bmatrix} I & G \\ 0 & A^T \end{bmatrix} - \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix}$$

with $A \equiv -A_1^{-1} A_1^T$, $H \equiv -A_0 = H^T$, $G \equiv A_1^{-1} A_0 A_1^{-T} = G^T$.

The SDA can then be applied.

Similarly, another doubling step produces

$$(\tau Z + Z^T)y = 0, \quad \tau = \lambda^4$$

with

$$Z = \begin{bmatrix} -A_1^T A_1^{-1} A_0 & -A_1^T A_1^{-1} A_1^T \\ A_1^T - A_0 A_1^{-1} A_0 & -A_0 A_1^{-1} A_1^T \end{bmatrix}$$

4.2. Generalized Patel algorithms

From [49, 29], the Patel algorithm [48] was generalized for the T-palindromic QEP, resulting in a backward-stable algorithm.

First, the matrices

$$\mathcal{M} = \begin{bmatrix} A_1 & 0 \\ -A_0 & -I \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 0 & I \\ A_1^T & 0 \end{bmatrix}$$

define a T -symplectic pencil $\mathcal{M} - \lambda\mathcal{L}$, with

$$\mathcal{M}\mathcal{J}\mathcal{M}^T = \mathcal{L}\mathcal{J}\mathcal{L}^T, \quad \mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Next we define the $\mathcal{S} + \mathcal{S}^{-1}$ transformation:

$$\widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}} \equiv (\mathcal{M}\mathcal{J}\mathcal{L}^T + \mathcal{L}\mathcal{J}\mathcal{M}^T) - \lambda\mathcal{L}\mathcal{J}\mathcal{L}^T$$

A matrix $\mathcal{H} \in \mathbb{C}^{2n \times 2n}$ is T-skew-Hamiltonian, if $(\mathcal{H}\mathcal{J})^T = -\mathcal{H}\mathcal{J}$. We have

$$\begin{aligned} \widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}} &= \begin{bmatrix} A_1 - A_1^T & A_0 \\ -A_0 & A_1 - A_1^T \end{bmatrix} - \lambda \begin{bmatrix} 0 & -A_1 \\ A_1^T & 0 \end{bmatrix} \\ &= \left(\begin{bmatrix} A_0 & A_1^T - A_1 \\ A_1 - A_1^T & A_0 \end{bmatrix} - \lambda \begin{bmatrix} -A_1 & 0 \\ 0 & -A_1^T \end{bmatrix} \right) \mathcal{J} \\ &\equiv (\mathcal{K} - \lambda\mathcal{N})\mathcal{J} \end{aligned}$$

Both \mathcal{K} and \mathcal{N} are T -skew-Hamiltonian. In other words, the upper-right and lower-left blocks of both \mathcal{K} or \mathcal{N} are anti-symmetric and negative of each other, and the transpose of the upper-left block equal the lower-right block. To maintain the skew-Hermitian structure in both \mathcal{K} and \mathcal{N} , the transformations applied have the form $\mathcal{U} = \begin{bmatrix} U_1 & \overline{U}_2 \\ -U_2 & \overline{U}_1 \end{bmatrix}$. Consequently, the diagonals of the upper-right and lower-left blocks in the transformed pencil remain zero.

See [29] for the details on how the generalized Patel method is generalized for large and sparse palindromic QEPs, coupled with Arnoldi-type techniques in the TSHIRA and G^TSHIRA algorithms.

We shall explain how the generalized Patel method works using an $n = 4$ example, in which the changing pattern in \mathcal{K} and \mathcal{N} during the algorithm is presented.

At the beginning of the algorithm, we have the T -skew-Hamiltonian pencil \mathcal{K} and \mathcal{N} :

$$\begin{array}{c} \mathcal{K} \qquad \qquad \qquad \mathcal{N} \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & \times & \times & \times \\ \times & \times & \times & \times & \times & 0 & \times & \times \\ \times & \times & \times & \times & \times & \times & 0 & \times \\ \times & \times & \times & \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & \times & \times & \times & \times \\ \times & 0 & \times & \times & \times & \times & \times & \times \\ \times & \times & 0 & \times & \times & \times & \times & \times \\ \times & \times & \times & 0 & \times & \times & \times & \times \end{array} \right] \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \end{array} \right] \end{array}$$

Annihilate the strictly lower triangular part of $\mathcal{N}(1 : 4, 1 : 4)$:

$$\begin{array}{c} \mathcal{K} \leftarrow Q\mathcal{K}Z \qquad \qquad \qquad \mathcal{N} \leftarrow Q\mathcal{N}Z \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & \times & \times & \times \\ \times & \times & \times & \times & \times & 0 & \times & \times \\ \times & \times & \times & \times & \times & \times & 0 & \times \\ \times & \times & \times & \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & \times & \times & \times & \times \\ \times & 0 & \times & \times & \times & \times & \times & \times \\ \times & \times & 0 & \times & \times & \times & \times & \times \\ \times & \times & \times & 0 & \times & \times & \times & \times \end{array} \right] \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \end{array} \right] \end{array}$$

Annihilate $\mathcal{K}(6, 1)$ using the rotations of columns (2,3) and rows (6,7):

$$\begin{array}{c} \mathcal{K} \leftarrow Q\mathcal{K}Z \qquad \qquad \qquad \mathcal{N} \leftarrow Q\mathcal{N}Z \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & \times & \times & \times \\ \times & \times & \times & \times & \times & 0 & \times & \times \\ \times & \times & \times & \times & \times & \times & 0 & \times \\ \times & \times & \times & \times & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times & \times & \times \\ \times & \times & 0 & \times & \times & \times & \times & \times \\ \times & \times & \times & 0 & \times & \times & \times & \times \end{array} \right] \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & \otimes & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \otimes & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \end{array} \right] \end{array}$$

The zeroes at the (3,2) and (6,7) positions in \mathcal{N} , indicated by \otimes are filled in but are annihilated again similar to the earlier steps.

Similarly, annihilate $\mathcal{K}(7, 1)$ using the rotations of columns (3,4) and rows (7,8):

$$\begin{array}{c} \mathcal{K} \leftarrow Q\mathcal{K}Z \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & \times & \times & \times \\ \times & \times & \times & \times & \times & 0 & \times & \times \\ \times & \times & \times & \times & \times & \times & 0 & \times \\ \times & \times & \times & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times & \times & \times \\ 0 & \times & 0 & \times & \times & \times & \times & \times \\ \times & \times & \times & 0 & \times & \times & \times & \times \end{array} \right] \end{array} \quad \begin{array}{c} \mathcal{N} \leftarrow Q\mathcal{N}Z \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & \otimes & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & \otimes \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \end{array} \right] \end{array}$$

The zeroes at the (4,3) and (7,8) positions in \mathcal{N} , indicated by \otimes are filled in but are annihilated again similar to the earlier steps.

Similarly, annihilate $\mathcal{K}(8, 1)$ using the rotations of columns and rows (4,8):

$$\begin{array}{c} \mathcal{K} \leftarrow Q\mathcal{K}Z \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & \times & \times & \times \\ \times & \times & \times & \times & \times & 0 & \times & \times \\ \times & \times & \times & \times & \times & \times & 0 & \times \\ \times & \times & \times & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times & \times & \times \\ 0 & \times & 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & 0 & \times & \times & \times & \times \end{array} \right] \end{array} \quad \begin{array}{c} \mathcal{N} \leftarrow Q\mathcal{N}Z \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & 0 & 0 & \times \\ 0 & \times & \times & \times & 0 & 0 & 0 & \times \\ 0 & 0 & \times & \times & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \end{array} \right] \end{array}$$

Notice that the upper-right block starts to fill in but it will be of no consequence.

Then annihilate $\mathcal{K}(4, 1)$ using the rotations of rows (3,4) and columns (7,8):

$$\begin{array}{c} \mathcal{K} \leftarrow Q\mathcal{K}Z \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & \times & \times & \times \\ \times & \times & \times & \times & \times & 0 & \times & \times \\ \times & \times & \times & \times & \times & \times & 0 & \times \\ 0 & \times & \times & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & \times & \times & \times \\ 0 & \times & 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & 0 & \times & \times & \times & \times \end{array} \right] \end{array} \quad \begin{array}{c} \mathcal{N} \leftarrow Q\mathcal{N}Z \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & 0 & \times & \times & \times \\ 0 & \times & \times & \times & \times & 0 & \times & \times \\ 0 & 0 & \times & \times & \times & \times & 0 & \times \\ 0 & 0 & \otimes & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & \otimes \\ 0 & 0 & 0 & 0 & \times & \times & \times & \times \end{array} \right] \end{array}$$

Then annihilate $\mathcal{K}(3, 1)$ using the rotations of rows (2,3) and columns (5,6):

$$\begin{array}{cc}
 \mathcal{K} \leftarrow Q\mathcal{K}Z & \mathcal{N} \leftarrow Q\mathcal{N}Z \\
 \left[\begin{array}{cccccccc}
 \times & \times & \times & \times & 0 & \times & \times & \times \\
 \times & \times & \times & \times & \times & 0 & \times & \times \\
 0 & \times & \times & \times & \times & \times & 0 & \times \\
 0 & \times & \times & \times & \times & \times & \times & 0 \\
 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\
 0 & 0 & \times & \times & \times & \times & \times & \times \\
 0 & \times & 0 & \times & \times & \times & \times & \times \\
 0 & \times & \times & 0 & \times & \times & \times & \times
 \end{array} \right] & \left[\begin{array}{cccccccc}
 \times & \times & \times & \times & 0 & \times & \times & \times \\
 0 & \times & \times & \times & \times & 0 & \times & \times \\
 0 & \otimes & \times & \times & \times & \times & 0 & \times \\
 0 & 0 & 0 & \times & \times & \times & \times & 0 \\
 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \times & \times & \otimes & 0 \\
 0 & 0 & 0 & 0 & \times & \times & \times & 0 \\
 0 & 0 & 0 & 0 & \times & \times & \times & \times
 \end{array} \right]
 \end{array}$$

Finally, repeat the above process for the smaller blocks, ignoring their first rows and columns, we obtain the final form:

$$\begin{array}{cc}
 \mathcal{K} \leftarrow Q\mathcal{K}Z & \mathcal{N} \leftarrow Q\mathcal{N}Z \\
 \left[\begin{array}{cccccccc}
 \times & \times & \times & \times & 0 & \times & \times & \times \\
 \times & \times & \times & \times & \times & 0 & \times & \times \\
 0 & \times & \times & \times & \times & \times & 0 & \times \\
 0 & 0 & \times & \times & \times & \times & \times & 0 \\
 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\
 0 & 0 & 0 & 0 & \times & \times & \times & 0 \\
 0 & 0 & 0 & 0 & \times & \times & \times & \times \\
 0 & 0 & 0 & 0 & \times & \times & \times & \times
 \end{array} \right] & \left[\begin{array}{cccccccc}
 \times & \times & \times & \times & 0 & \times & \times & \times \\
 0 & \times & \times & \times & \times & 0 & \times & \times \\
 0 & 0 & \times & \times & \times & \times & 0 & \times \\
 0 & 0 & 0 & \times & \times & \times & \times & 0 \\
 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\
 0 & 0 & 0 & 0 & \times & \times & \times & 0 \\
 0 & 0 & 0 & 0 & \times & \times & \times & \times
 \end{array} \right]
 \end{array}$$

The palindromic QEP is thus square-rooted, with the pencils on the upper-left and lower-right corners in Hessenberg-triangular form for reciprocal eigenvalues. Note that the Patel method is numerical backward stable.

5. *g*-PALINDROMIC QEPs

In this section, we first present the *g*-palindromic QEP, a unified framework including several palindromic QEPs. Results are quoted without proofs from [13].

Definition 5.1. A function $g : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ is called a $(*, \varepsilon)$ -homomorphism if $g(\alpha_1\Phi_1 + \alpha_2\Phi_2) = \alpha_1^*g(\Phi_1) + \alpha_2^*g(\Phi_2)$ and $g(\Phi_1\Phi_2) = \varepsilon g(\Phi_2)g(\Phi_1)$, for all $\Phi_1, \Phi_2 \in \mathbb{C}^{n \times n}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. Furthermore, g preserves the singularity, i.e., $\det(\Phi) = 0 \Leftrightarrow \det(g(\Phi)) = 0$. Here “*” denotes “H” (Hermition/conjugate transpose) or “T” (transpose) and $\varepsilon = \pm 1$.

Proposition 5.1. *Let g be a $(*, \varepsilon)$ -homomorphism. Then it holds (i) $g(0) = 0$ (ii) $g(I) = \varepsilon I$ (iii) $g(\Phi^{-1}) = g(\Phi)^{-1}$.*

Definition 5.2. The QEP in (3) is called a g -palindromic QEP if there is a $(*, \varepsilon)$ -homomorphism g such that $g(B) = A$, $g(C) = C$ and $g(A) = B$. Moreover, C is called g -symmetric and A and B are said to be g -related, denoted by $A \stackrel{g}{\sim} B$.

Under the above definitions, we have the following property of symmetry in the spectrum for the above g -palindromic quadratic pencil $Q(\lambda)$.

Theorem 5.1. *Let $Q(\lambda)$ be a g -palindromic quadratic pencil. We have $\lambda \in \sigma(Q(\lambda))$ if and only if $1/\lambda^* \in \sigma(Q(\lambda))$. Here we follow the convention that 0 and ∞ are reciprocal to each other.*

A (g -palindromic) quadratic pencil can be rewritten as

$$\lambda^2 B + \lambda C + A = (\lambda B + X)X^{-1}(\lambda X + A) = \lambda^2 B + \lambda(X + BX^{-1}A) + A.$$

Then we introduce a g -nonlinear matrix equation (g -NME)

$$(36) \quad X + BX^{-1}A = C, \quad A \stackrel{g}{\sim} B, \quad g(C) = C.$$

If we can find a solution X for (36) structurally, then the g -palindromic QEP is factorized. We then need only to solve the eigenvalue problem for the factor $\lambda B + X$, with the reciprocal eigenvalues for $\lambda X + A$ obtained free.

For a given g -NME (36), we define

$$(37) \quad \mathcal{M} = \begin{bmatrix} A & 0 \\ C & -I \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} -D & I \\ B & 0 \end{bmatrix}.$$

It is easy to see that the pencil $\mathcal{M} - \lambda \mathcal{L}$ is a linearization of g -palindromic QEP (3) with $D = 0$. Based on the SDA algorithm proposed in [41], one can also develop a g -SDA algorithm for solving the g -NME.

For $\mathcal{M} - \lambda \mathcal{L}$ given in (37), we compute

$$(38) \quad \mathcal{M}_* = \begin{bmatrix} A(C - D)^{-1} & 0 \\ -B(C - D)^{-1} & I \end{bmatrix}, \quad \mathcal{L}_* = \begin{bmatrix} I & -A(C - D)^{-1} \\ 0 & B(C - D)^{-1} \end{bmatrix}$$

which satisfies $\mathcal{M}_* \mathcal{L} = \mathcal{L}_* \mathcal{M}$. Direct calculations give rise to

$$(39) \quad \widehat{\mathcal{M}} \equiv \mathcal{M}_* \mathcal{M} = \begin{bmatrix} \widehat{A} & 0 \\ \widehat{C} & -I \end{bmatrix}, \quad \widehat{\mathcal{L}} \equiv \mathcal{L}_* \mathcal{L} = \begin{bmatrix} -\widehat{D} & I \\ \widehat{B} & 0 \end{bmatrix},$$

where

$$(40a) \quad \widehat{A} = A(C - D)^{-1}A, \quad \widehat{B} = B(C - D)^{-1}B,$$

$$(40b) \quad \widehat{C} = C - B(C - D)^{-1}A, \quad \widehat{D} = D + A(C - D)^{-1}B.$$

Theorem 5.2. (i) *The pencil $\widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}}$ has the doubling property, i.e., if*

$$\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} = \mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} S,$$

where $U, V \in \mathbb{C}^{n \times m}$ and $S \in \mathbb{C}^{m \times m}$, then

$$\widehat{\mathcal{M}} \begin{bmatrix} U \\ V \end{bmatrix} = \widehat{\mathcal{L}} \begin{bmatrix} U \\ V \end{bmatrix} S^2.$$

(ii) *The quadratic pencil $\lambda^2\widehat{B} + \lambda\widehat{C} + \widehat{A}$ corresponding to $\widehat{\mathcal{M}} - \lambda\widehat{\mathcal{L}}$ is still a g-palindromic quadratic pencil.*

5.1. Convergence of g-SDA

First consider the special case that $B = A^T \in \mathbb{R}^{n \times n}$ and C is real symmetric positive definite in the g-NME (36). If $(\mathcal{M}, \mathcal{L})$ in (37) has no unimodular eigenvalues and the g-NME (36) has a symmetric positive solution, then the convergence of the g-SDA is quadratic [41]. If $(\mathcal{M}, \mathcal{L})$ has unimodular eigenvalues with even partial multiplicities and the g-NME (36) has a symmetric positive solution, then the convergence is globally linear with rate $1/2$ [9].

In this section, we shall discuss the convergence of the g-SDA for the general case of the g-NME. The quadratic convergence of the g-SDA algorithm, when no eigenvalues of the matrix pair $(\mathcal{M}, \mathcal{L})$ in (37) lies on the unit circle, follows from Theorem 5.2. We shall concentrate on the more general case, assuming the following:

(H) The partial multiplicities of $(\mathcal{M}, \mathcal{L})$ corresponding to each unimodular eigenvalue are all even with the same sizes.

Definition 5.3. A solution X for the g-NME (36) is called to have property (P), if (i) $\rho(X^{-1}A) \leq 1$, (ii) if $\rho(X^{-1}A) = 1$, then the partial multiplicities of each unimodular eigenvalue of $X^{-1}A$ is half of that of the corresponding unimodular eigenvalue of the associated pair $(\mathcal{M}, \mathcal{L})$ in (37).

If X_d is a solution for the dual g-NME satisfying

$$(41) \quad X_d + AX_d^{-1}B = C.$$

Then we have

$$(42) \quad \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} \begin{bmatrix} I \\ Y \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & -I \end{bmatrix} \begin{bmatrix} I \\ Y \end{bmatrix} R_d,$$

where $Y \equiv C - X_d$. That is $\text{span}\{[I, Y^T]^T\}$ forms a deflating subspace of $(\mathcal{L}, \mathcal{M})$ corresponding to $R_d \equiv X_d^{-1}A$.

The convergence of the g-SDA algorithm, iterating as indicated by (40a) and (40b) with $(A_0, B_0, C_0, D_0) = (A, B, C, 0)$, is summarized in the following theorem.

Theorem 5.3. *Assume that the g-NME (36) and the dual g-NME (41) have the solutions X and Y with properties (P), respectively. Suppose the sequence $\{A_k, B_k, C_k, D_k\}$ generated by g-SDA is well-defined. Then it holds*

- (i) $\|A_k\| = O(\rho(J_s)^{2^k}) + O(2^{-k}) \rightarrow 0$, as $k \rightarrow \infty$,
- (ii) $\|B_k\| = O(\rho(J_s)^{2^k}) + O(2^{-k}) \rightarrow 0$, as $k \rightarrow \infty$,
- (iii) $\|C_k - X\| = O(\rho(J_s)^{2^{k+1}}) + O(2^{-k}) \rightarrow 0$, as $k \rightarrow \infty$,
- (iv) $\|D_k - Y\| = O(\rho(J_s)^{2^{k+1}}) + O(2^{-k}) \rightarrow 0$, as $k \rightarrow \infty$.

Furthermore, X and Y are g-symmetric, i.e., $g(X) = X$ and $g(Y) = Y$.

We have shown that the convergence of the g-SDA is quadratic when no eigenvalues of the matrix pair $(\mathcal{M}, \mathcal{L})$ in (37) lies on the unit circle.

Remark 5.1. Comparing the results in [9] with Theorem 5.3, we have the following comments.

- (i) For the existence of the unique weakly stable deflating subspace of $(\mathcal{M}, \mathcal{L})$ in (37), we have assumed that (a) the partial multiplicities of $(\mathcal{M}, \mathcal{L})$ corresponding to each unimodular eigenvalues are all even, and (b) if $\{\lambda_1, \lambda_2\}$ are unimodular eigenvalues of $(\mathcal{M}, \mathcal{L})$ with $\lambda_1 = \lambda_2$, then the partial multiplicity of λ_1 must be equal to that of λ_2 . In [9], only assumption (a) was required.
- (ii) If the NME in [9] has a symmetric positive solution, then the matrices sequence produced by the SDA algorithm are well-defined and the convergence is globally linear with rate $1/2$. To guarantee the linear convergence of the g-SDA algorithm in Theorem 5.3, we need to add the assumptions that the solution of the dual g-NME exists and the iterates from the g-SDA algorithm are well-defined.

5.2. Application to g-palindromic QEP

5.2.1. T- and H-(anti-)palindromic QEPs

If the $(*, \varepsilon)$ -homomorphism is defined by $g(\Phi) = \Phi^*$. Then the g-palindromic QEP (3) becomes

(i) T-palindromic QEP (* = "T"):

$$(43) \quad (\lambda^2 A^T + \lambda C + A)x = 0 \quad \text{with } C^T = C.$$

(ii) H-palindromic QEP (* = "H"):

$$(44) \quad (\lambda^2 A^H + \lambda C + A)x = 0 \quad \text{with } C^H = C.$$

If the $(*, \varepsilon)$ -homomorphism is defined by $g(\Phi) = -\Phi^*$. Then the g-palindromic QEP becomes

(iii) T-anti-palindromic QEP (* = "T"):

$$(45) \quad (\lambda^2 A^T + \lambda C - A)x = 0 \quad \text{with } C = -C^T.$$

(iv) H-anti-palindromic QEP (* = "H"):

$$(46) \quad (\lambda^2 A^H + \lambda C - A)x = 0 \quad \text{with } C = -C^H.$$

For cases (i) and (ii), the g-SDA can be simplified to

$$\begin{aligned} A_0 &= A, \quad C_0 = C = C^*, \quad D_0 = 0, \\ A_{k+1} &= A_k(C_k - D_k)^{-1}A_k, \\ C_{k+1} &= C_k - A_k^*(C_k - D_k)^{-1}A_k, \\ D_{k+1} &= D_k + A_k(C_k - D_k)^{-1}A_k^*. \end{aligned}$$

For cases (iii) and (iv), the g-SDA can be simplified to

$$\begin{aligned} A_0 &= A, \quad C_0 = C = -C^*, \quad D_0 = 0, \\ A_{k+1} &= A_k(C_k - D_k)^{-1}A_k, \\ C_{k+1} &= C_k + A_k^*(C_k - D_k)^{-1}A_k, \\ D_{k+1} &= D_k - A_k(C_k - D_k)^{-1}A_k^*. \end{aligned}$$

5.2.2. *-palindromic_2 QEPs

We now consider the *-palindromic_2 QEP

$$(47) \quad (\lambda^2 A^* + \lambda C - A)x = 0 \quad \text{with } C^* = C,$$

and the *-anti-palindromic_2 QEP

$$(48) \quad (\lambda^2 A^* + \lambda C + A)x = 0 \quad \text{with } C^* = -C.$$

The quadratic pencil in (47) can be factorized by

$$\lambda A^* + \lambda C - A = (\lambda A^* + X)X^{-1}(\lambda X - A)$$

where X satisfies

$$(49) \quad X - A^*X^{-1}A = C, \quad C^* = C.$$

The quadratic pencil in (48) can be factorized by

$$\lambda A^* + \lambda C + A = (\lambda A^* + X)X^{-1}(\lambda X + A)$$

where X satisfies

$$(50) \quad X + A^*X^{-1}A = C, \quad C^* = -C.$$

If we perform one step of g-SDA on (49), then X in (49) satisfies

$$(51) \quad X + \hat{A}^*X^{-1}\hat{A} = \hat{C}, \quad \hat{C}^* = \hat{C},$$

where

$$(52) \quad \hat{A} = AC^{-1}A, \quad \hat{C} = C - A^*C^{-1}A, \quad \hat{D} = AC^{-1}A^*.$$

The g-NME in (51) corresponds to the *-palindromic QEP

$$(53) \quad (\lambda^2\hat{A}^* + \lambda\hat{C} + \hat{A})x = 0, \quad \hat{C}^* = \hat{C}.$$

If we perform one step of g-SDA on (50), then X in (50) satisfies

$$(54) \quad X - \hat{A}^*X^{-1}\hat{A} = \hat{C}, \quad \hat{C}^* = -\hat{C},$$

where

$$(55) \quad \hat{A} = AC^{-1}A, \quad \hat{C} = C + A^*C^{-1}A, \quad \hat{D} = -AC^{-1}A^*.$$

The g-NME in (54) corresponds to the *-anti-palindromic QEP

$$(56) \quad (\lambda^2\hat{A}^* + \lambda\hat{C} - \hat{A})x = 0, \quad \hat{C}^* = -\hat{C}.$$

The g-SDA can then be applied to (53) and (56) as in Section 4.1.

5.2.3. *-even and odd QEPs

We now consider the *-even and *-odd QEPs, respectively:

$$(57) \quad Q(\lambda)x \equiv (\lambda^2M + \lambda G + K)x = 0, \quad M^* = M, \quad K^* = K, \quad G^* = -G,$$

and

$$(58) Q(\lambda)x \equiv (\lambda^2 M + \lambda G + K)x = 0, \quad M^* = -M, \quad K^* = -K, \quad G^* = G.$$

It is well-known that $Q(\lambda)$ has the factorization

$$(59) \quad Q(\lambda) = (\lambda M + MS + G)(\lambda I - S)$$

if and only if S is a solution of the quadratic matrix equation

$$(60) \quad MS^2 + GS + K = 0.$$

If $\lambda \in \sigma(Q(\lambda))$, then $-\lambda^* \in \sigma(Q(\lambda))$. If x_i and y_i are, respectively, the right and left eigenvectors corresponding to λ_i of S , i.e.,

$$(61) \quad Sx_i = \lambda_i x_i, \quad y_i^* S = \lambda_i y_i^*,$$

then x_i and $(\lambda_i M + MS + G)^{-*} y_i$ are eigenvectors corresponding to λ_i and $-\lambda_i^*$, respectively, of the QEP (57) or (58).

It seems difficult to find the solution S of (60) directly whose eigenvalues are on the right half-plane. Instead, the Cayley transformation $S = (I + Y)(I - Y)^{-1}$ is used. Equation (60) then becomes

$$(62) \quad \varepsilon A^* Y^2 + CY + A = 0,$$

where $A = M + K + G$, $C = 2(M - K)$, $\varepsilon = 1$ for (57) and $\varepsilon = -1$ for (58). With $Y = -X^{-1}A$ in (62), we have the NME:

$$(63) \quad X + \varepsilon A^* X^{-1} A = C,$$

to which the g-SDA or Algorithm 2.1 can be applied.

5.3. Other palindromic QEPs

Interestingly, the simplest palindromic QEP $(\lambda^2 A + \lambda B + A)x = 0$, a g-palindromic QEP with $g(\Phi) = \Phi$ and property (5), has not been considered by anyone after being defined in [43]. Of course, the problem can be solved by g-SDA as indicated earlier. Similar to the palindromic linearization approach in [44] (see also Section 4.1 earlier), the QEP can be solved via the structure-preserving linearization

$$(\lambda Z + PZP)v = 0, \quad Z = \begin{bmatrix} A & A \\ B - A & A \end{bmatrix}, \quad P = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

with $PZP = \begin{bmatrix} A & B - A \\ A & A \end{bmatrix}$. More generally, a similar and even more general P-conjugate P-palindromic (PCP) QEP [19]:

$$(\lambda^2 A + \lambda B + \Pi A \Pi)x = 0, \quad \Pi B \Pi = B, \quad \Pi^2 = \Pi$$

has property (5) and a similar structure-preserving linearization

$$(\lambda Z + PZP)v = 0, \quad Z = \begin{bmatrix} A & A \\ B - \Pi A \Pi & A \end{bmatrix}, \quad P = \begin{bmatrix} 0 & \Pi \\ \Pi & 0 \end{bmatrix}$$

With the notation $\tilde{A} \equiv \Pi A \Pi$, $\tilde{B} \equiv \Pi B \Pi$ and $\tilde{Z} \equiv PZP$, we have

$$\lambda Z + \tilde{Z} = \lambda \begin{bmatrix} A & A \\ B - \tilde{A} & A \end{bmatrix} + \begin{bmatrix} \tilde{A} & \tilde{B} - A \\ \tilde{A} & \tilde{A} \end{bmatrix}$$

With $\Pi = I$, the PCP-QEP degenerates back into the palindromic QEP. For more results on the structure-preserving linearizations and an associated Schur-like decomposition constructed using PQZ [12], see [19]. Also, see [33] for an application of PCP-QEPs in delay-differential equations.

It is not clear how the structure-preserving linearization or the corresponding PCP-QEP can be solved in a structure-preserving manner. Some preliminary results for a generalized SDA for the PCP-QEP can be found in [38].

For g-palindromic QEPs, several numerical examples can be found in [13], showing that the g-SDA algorithm converges to the desired solutions efficiently and reliably.

6. FUTURE WORK

There is no lack of possibilities in terms of future work involving palindromic eigenvalue problems and we shall speculate on a few in this Section.

For the train vibration problems, more accurate 3-D finite element models can be constructed. The resulting large and sparse T-palindromic QEP has to be solved using SDAs and the generalized Patel method, possibly coupled with an adaptation of some Araldi technique. Design parameters for the train and rail system can then be optimized and the associated model reduction problem can be attempted. Notice that the eigenvalue problem has to be solved for many values of the speed parameter ω and the associated refinement and updating problems will be of interest.

For the SAW filter problem, finite element models have to be constructed and refined, and the associated T-palindromic QEPs attempted for various frequencies ω so that the dispersion diagram $\beta(\omega)$ can be sketched. For practical purposes, we are interested in the ‘‘ill-conditioned’’ eigenvalues near the unit circle, which will

present difficulties and challenges for any numerical methods. Similar challenges also exist for the crack propagation and prediction problem.

For discrete-time optimal control problems and the associated higher-order palindromic eigenvalue problems, different applications present different structures and properties. Much work has to be done before an appropriate approach can be found for their solution, although the quadratization approach [39] is a promising possibility.

For the simplest first order palindromic eigenvalue problem associated with $\lambda Z + Z^*$, it is not clear how it can be solved in a structure-preserving manner. One possibility in terms of the *-Riccati equation is discussed in [8].

Finally, we are still looking for structured numerical methods for the class of PCP-QEPs [38] from Section 5.3.

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