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THE FIXED POINT PROPERTY AND UNBOUNDED SETS IN BANACH SPACES

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Abstract. Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty closed convex subset of E. Then, a mapping $S : C \to C$ is said to be nonspreading [23] if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. In this paper, we prove that every nonspreading mapping of C into itself has a fixed point in C if and only if C is bounded. This theorem extends Ray's theorem [27] in a Hilbert space to that in a Banach space.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $F : C \to C$ is also said to be firmly nonexpansive if $\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$ for all $x, y \in C$; see, for instance, Browder [6], Goebel and Kirk [10], Goebel and Reich [11], and Takahashi [34]. Ray [27] proved the following theorem.

Theorem 1.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then, the following are equivalent:*

(i) *Every nonexpansive mapping of C into itself has a fixed point in C;*

(ii) *C is bounded.*

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Sine [33] gave a simple proof of Ray's theorem by using that the metric projection is nonexpansive in a Hilbert space. We know that a nonexpansive mapping is deduced from a firmly nonexpansive mapping. Recently, the first author [38] defined the following nonlinear mapping $S : C \to C$ called hybrid which is also deduced from a firmly nonexpansive mapping:

$$3\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|x - Sy\|^2 + \|y - Sx\|^2$$

for all $x, y \in C$. Using Ray's theorem, he proved the following theorem.

Theorem 1.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then, the following are equivalent:*

(i) *Every hybrid mapping of C into itself has a fixed point in C;*

(ii) *C is bounded.*

However, such theorems have not been extended to those of a Banach space. Recently, Kohsaka and Takahashi [23] introduced the following nonlinear mapping in a Banach space. Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty closed convex subset of E. Then, a mapping $S : C \to C$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. They proved a fixed point theorem for such mappings. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping S in a Hilbert space H is defined as follows:

$$2\,\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|Sy - x\|^2$$

for all $x, y \in C$.

In this paper, motivated by these results, we try to extend Ray's theorem to that in a Banach space by the theory of convex analysis. We prove that if C is a closed convex subset of a smooth, strictly convex and reflexive Banach space, then every nonspreading mapping of C into itself has a fixed point in C if and only if C is bounded.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in

E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a closed convex subset of E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [15].

Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J ia a single valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection.

Theorem 2.1. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \ge 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a reflexive, strictly convex and smooth Banach space. The function ϕ: $E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [19]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\|^2 - \|y\|^2) \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Further, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.3}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

A multi-valued operator $A\colon E \to 2^{E^*}$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. Let E be a Banach space and let f be a function of E into $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is proper if $f(x) \in \mathbb{R}$ for some $x \in E$. f is convex if for $x, y \in E$ and $t \in (0, 1)$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

f is lower semicontinuous if for every $\alpha \in \mathbb{R}$, $\{x \in E : f(x) \leq \alpha\}$ is closed. The following is Rockafellar's theorem.

Theorem 2.2. [30, 31]. *Let E be a real Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is as follows:*

$$\partial f(z) = \{v^* \in E^* : f(y) \geq f(z) + \langle y - z, v^*\rangle,\ \forall y \in E\}, \quad \forall z \in E.$$

Then, $\partial f\colon E \to 2^{E^*}$ is a maximal monotone operator.

The following theorem is well known; see Browder [8], Rockafellar [32] and Takahashi [35].

Theorem 2.3. [8, 32]. *Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a reflexive, strictly convex and smooth Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. For $r > 0$ and $x \in E$, consider

$$J_r x = \{z \in E : Jx \in Jz + rA(z)\}.$$

We know from [35] that $J_r x$ is a singleton. We denote J_r by $J_r = (J + rA)^{-1}J$. We call J_r the resolvent of A for $r > 0$. For all $r > 0$, the Yosida approximation A_r is also defined by

$$A_r = \frac{1}{r}(J - JJ_r).$$

3. A Generalization of Ray's Theorem

In this section, we try to extend Ray's theorem in a Hilbert space to that in a Banach space. Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let J be the duality mapping from E into E^*. Then, we say that $T : C \to C$ is of firmly nonexpansive type [22] if

$$\langle Tx - Ty, JTx - JTy\rangle \leq \langle Tx - Ty, Jx - Jy\rangle$$

for all $x, y \in C$. We have from (2.3) that for any $x, y \in C$,

$$\begin{aligned} &\langle Tx - Ty, JTx - JTy\rangle \leq \langle Tx - Ty, Jx - Jy\rangle \\ \Longleftrightarrow\ & 2\langle Tx - Ty, JTx - JTy\rangle \leq 2\langle Tx - Ty, Jx - Jy\rangle \\ \Longleftrightarrow\ & \phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) - \phi(Tx, x) - \phi(Ty, y) \\ \Longrightarrow\ & \phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x). \end{aligned}$$

This means that a firmly nonexpansive type mapping is nonspreading. The following theorem extends Ray's theorem in a Hilbert space to that of a Banach space.

Theorem 3.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Then, the following conditions are equivalent:*

- (i) *Every firmly nonexpansive type mapping of C into itself has a fixed point in C;*
- (ii) *C is bounded.*

Proof. We know from [22] that if C is bounded, then every firmly nonexpansive type mapping of C into itself has a fixed point in C. So, we have that (ii) implies (i). We will show that (i) implies (ii). Suppose that C is not bounded. Then the uniform boundedness theorem ensures the existence of $x^* \in E^*$ such that $\sup_{x\in C} |x^*(x)| = \infty$. Since E is a real Banach space, we have

$$\begin{aligned} |x^*(x)| &= \max\{x^*(x), -x^*(x)\} \\ &\leq \max\Big\{\sup_{z\in C} x^*(z), \sup_{z\in C}\big\{-x^*(z)\big\}\Big\} \end{aligned}$$

for all $x \in C$. Hence we have that $\sup_{x\in C} x^*(x) = \infty$ or $\sup_{x\in C}\big\{-x^*(x)\big\} = \infty$ and hence there exists $y^* \in E^*$ such that

$$\sup_{x\in C} y^*(x) = \infty. \tag{3.1}$$

Let us define a function g of E into $(-\infty, \infty]$ as follows:

$$g(x) = \begin{cases} -y^*(x), & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Then, g is obviously a proper lower semicontinuous convex function of E into $(-\infty, \infty]$. Further, it follows from (3.1) that

$$\inf_{x \in E} g(x) = \inf_{x \in C} \{-y^*(x)\} = -\sup_{x \in C} y^*(x) = -\infty.$$

This implies that g does not have a minimizer in E. For the proper lower semicontinuous convex function $g : E \to (-\infty, \infty]$, the subdifferential ∂g of g is defined as follows:

$$\partial g(x) = \{x^* \in E^* : g(x) + \langle y - x, x^* \rangle \leq g(y),\ \forall y \in E\}$$

for all $x \in E$. We know from Rockafellar's theorem (Theorem 2.2) that the subdifferential ∂g of g is a maximal monotone operator of E into E^*. Since g does not have a minimizer in E, we have that $(\partial g)^{-1}0 = \emptyset$. Further, from the definition of g, we have that

$$D(\partial g) \subset C \subset J^{-1}R(J + \partial g) = E.$$

We can also define the resolvent J_1 of ∂g as follows:

$$J_1(x) = \{z \in E : Jx \in Jz + \partial g(z)\},\ \forall x \in E.$$

We know from [34, 35] that J_1 is a single-valued mapping of E into C. Further, for $x, y \in C$, we have $(J_1x, A_1x), (J_1y, A_1y) \in \partial g$. Since ∂g is monotone, we have

$$\langle J_1x - J_1y, Jx - JJ_1x - (Jy - JJ_1y) \rangle \geq 0.$$

Thus, we have

$$\langle J_1x - J_1y, JJ_1x - JJ_1y \rangle \leq \langle J_1x - J_1y, Jx - Jy \rangle.$$

Then, J_1 is a firmly nonexpansive type mapping of C into itself. We know that J_1 is also as follows:

$$J_1(x) = \arg\min_{y \in E} \{g(y) + \frac{1}{2}(\|y\|^2 - 2\langle y, Jx \rangle)\},\ \forall x \in E.$$

Further, we have that

$$\begin{aligned} 0 \in \partial g(z) &\iff Jz \in Jz + \partial g(z) \\ &\iff z = J_1z. \end{aligned}$$

From $(\partial g)^{-1}0 = F(J_1)$ and $(\partial g)^{-1}0 = \emptyset$, we know that J_1 does not have a fixed point. This means that (i) implies (ii). ■

Using Theorem 3.1 and Kohsaka and Takahashi [23], we obtain the following theorem.

Theorem 3.2. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Then, the following conditions are equivalent:*

(i) *Every nonspreading mapping of C into itself has a fixed point in C;*

(ii) *C is bounded.*

Proof. It follows from Kohsaka and Takahashi [23] that (ii) implies (i). Since a firmly nonexpansive type mapping is nonspreading, we have from Theorem 3.1 that (i) implies (ii). ■

Using Theorem 3.1, we obtain the following result in a Hilbert space.

Theorem 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then, the following conditions are equivalent:*

(i) *Every firmly nonexpansive mapping of C into itself has a fixed point in C;*

(ii) *C is bounded.*

Proof. Since $J = I$ in a Hilbert space, every firmly nonexpansive type mapping of C into itself is firmly nonexpansive. From Theorem 3.1, we get the desired result. ■

Since a nonexpansive mapping and a hybrid mapping in a Hilbert space are deduced from a firmly nonexpansive mapping, we have Theorems 1.1 and 1.2 from Theorem 3.3.

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