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# ROTATION HYPERSURFACES IN LORENTZ-MINKOWSKI SPACE WITH CONSTANT MEAN CURVATURE

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**Abstract.** We give explicit parameterizations of rotation hypersurfaces in Lorentz-Minkowski space $L^{n+1}$. Then we obtain rotation hypersurfaces in Lorentz-Minkowski space $L^{n+1}$ with constant mean curvature. In particular, we determine nonplanar rotation hypersurfaces with zero mean curvature, namely, generalized catenoids of $L^{n+1}$. In the case the rotation axis is light-like, the generalized catenoids generalize Enneper's surfaces of the 2nd and 3rd kind.

## 1. Introduction

In an old paper [3], Delaunay proved that the profile curve of a rotation surface with nonzero constant mean curvature in Euclidean 3-space can be described as the locus of a focus when a quadratic curve is rolled along the axis of revolution. This result was generalized in various directions by Hsiang and Yu [5]. In [9], Pinl and Ziller proved that the only minimal rotation hypersurface (except the hyperplane) of Euclidean space is the generalized catenoid. In [4], Carmo and Dajczer defined rotation hypersurfaces in space of constant curvature and gave a local characterization of such hypersurfaces. Also they studied some special cases of rotation hypersurfaces with constant mean curvature in hyperbolic space.

On the other hand, in Lorentz-Minkowski 3-space there are several different kinds of rotation surfaces depending on the rotation axis: rotations about space-like, time-like, and light-like axes. Rotation surfaces of constant mean curvature in Minkowski 3-space $L^3$ has been studied by a number of differential geometers. For

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instance, in [4], Hano and Nomizu studied the Delaunay's problem in the Lorentz-Minkowski 3-space $L^3$, restricting themselves to the space-like surfaces such that the rotation axis is either space-like, time-like or light-like. In the case of revolution about the space-like and time-like axis, the profile curve is obtained by revolving the focus of a quadratic curve along the axis of rotation, similarly to the Delaunay surface in Euclidean 3-space. They also studied for the profile curves when the rotation axis is light-like. The completeness of the surfaces obtained in [4] was investigated in [2]. Recently in [7], Lee and Varnado studied various nonlinear ordinary differential equations that characterize space-like constant mean curvature rotation surfaces in Minkowski 3-space. They solved the differential equations by using some numerical methods to obtain examples of space-like constant mean curvatures.

In particular, there are various type of catenoids, that is, nonplanar rotation surfaces with zero mean curvature, in Lorentz-Minkowski 3-space depending on the rotation axis. In [6], Kobayashi classified maximal space-like rotation surfaces in Minkowski space $L^3$. However, McNertney [8] and Van de Woestijne [10] independently classified catenoids, that is, nonplaner rotation surfaces with zero mean curvature, in Lorentz-Minkowski space $L^3$. In [10], Van de Woestijne calls the space-like catenoid and time-like catenoid with light-like axis the surface of Enneper of the 2nd kind and 3rd kind, respectively.

In this paper we extend the notion of rotation surfaces of the 3-dimensional Lorentz-Minkowski space $L^3$ to hypersurfaces of an $(n+1)$-dimensional Lorentz -Minkowski space $L^{n+1}$. We firstly give explicit parametrization of rotation hypersurfaces in $L^{n+1}$ according to the rotation axis is time-like, space-like or light-like. Especially, when the rotation axis is light-like we determine the orthogonal transformations of $L^{n+1}$ that leaves the rotation axis fixed. We then compute the principal curvatures of each rotation hypersurface in $L^{n+1}$ to determine the differential equation of the hypersurface with constant mean curvature. By solving the differential equations we obtain profile curves of rotation hypersurfaces of constant mean curvature. As a consequence we determine the generalized catenoids of $L^{n+1}$. In the case the rotation axis is light-like, the generalized catenoids generalize Enneper's surfaces of the 2nd and 3rd kind.

## 2. Preliminaries

Let $L^{n+1}$ denotes the (n + 1)-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric $\langle,\rangle = (dx_1)^2 + \cdots + (dx_n)^2 - (dx_{n+1})^2$, where $(x_1, \ldots, x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$. A vector $x$ of $L^{n+1}$ is said to be space-like if $\langle x, x\rangle > 0$ or $x = 0$, time-like if $\langle x, x\rangle < 0$ or light-like (or null) if $\langle x, x\rangle = 0$ and $x \neq 0$.

An immersed hypersurface $M_q$ of $L^{n+1}$ with index $q$ ($q = 0, 1$) is called space-like (Riemannian) or time-like (Lorentzian) if the induced metric, which, as usual, is also denoted by $\langle , \rangle$ on $M_q$ has the index 0 or 1, respectively. The de Sitter $n$-space $\mathbb{S}_1^n(x_0, c)$ centered at $x_0 \in L^{n+1}$, $c > 0$, is a Lorentzian hypersurface of $L^{n+1}$ defined by

$$\mathbb{S}_1^n(x_0, c) = \{x \in L^{n+1} | \ \langle x - x_0, x - x_0 \rangle = c^2\},$$

and the hyperbolic space $\mathbb{H}^n(x_0, -c)$ centered at $x_0 \in L^{n+1}$, $c > 0$, is a space-like hypersurface of $L^{n+1}$ defined by

$$\mathbb{H}^n(x_0, -c) = \{x \in L^{n+1} | \ \langle x - x_0, x - x_0 \rangle = -c^2 \text{ and } x_{n+1} - x_{n+1}^0 > 0\},$$

where $x_{n+1} - x_{n+1}^0$ is the $(n + 1)$-th component of $x - x_0$. The de Sitter $n$-space $\mathbb{S}_1^n(x_0, c)$ and the hyperbolic space $\mathbb{H}^n(x_0, -c)$, $c > 0$, are both totally umbilical hypersurfaces of the Lorentzian space $L^{n+1}$.

Let $e_1, \ldots, e_n$ be an orthonormal local tangent frame on a hypersurface $M_q$ of $L^{n+1}$ with signatures $\varepsilon_i = \langle e_i, e_i \rangle = \mp 1$, and $A_N$ denotes the shape operator of $M_q$ in a chosen unit normal direction $N$. Then the mean curvature $\alpha$ of $M_q$ is defined by

$$\alpha = \frac{1}{n}\text{tr}A_N = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \langle A_N(e_i), e_i \rangle .$$

A space-like hypersurface of $L^{n+1}$ with vanishing mean curvature is called maximal.

Let $\Pi$ be a 2-dimensional subspace of $L^{n+1}$ passing through the origin. We will say that $\Pi$ is non-degenerate if the metric $\langle , \rangle$ restricted to $\Pi$ is a non-degenerate quadratic form. A curve in $L^{n+1}$ is called space-like, time-like or light-like if the tangent vector at any point is space-like, time-like or light-like, respectively. An orthogonal transformation of $L^{n+1}$ is a linear map that preserves the metric.

Here we will define non-degenerate rotation hypersurfaces in $L^{n+1}$ with time-like, space-like or light-like axis. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \to \Pi$ be a regular smooth curve in a non-degenerate 2-plane $\Pi$ of $L^{n+1}$ and let $\ell$ be a line in $\Pi$ that does not meet the curve $\gamma$. A rotation hypersurface $M_q$ with index $q$ in $L^{n+1}$ with a rotation axis $\ell$ is defined as the orbit of a curve $\gamma$ under the orthogonal transformations of $L^{n+1}$ with positive determinant that leaves the rotation axis $\ell$ fixed. When the rotation axis $\ell$ is space-like or time-like it is easy to write the orthogonal transformations of $L^{n+1}$ that leaves the rotation axis $\ell$ fixed. However, if the rotation axis $\ell$ is light-like we will give the orthogonal transformations of $L^{n+1}$ that leaves the axis $\ell$ fixed. The curve $\gamma$ is called profile curve of the rotation hypersurface. As we consider non-degenerate rotation hypersurfaces it is sufficient to consider the case that the profile curve is space-like or time-like.

We will give explicit parameterizations for non-degenerate rotation hypersurfaces $M_q$ in $L^{n+1}$ according to the axis $\ell$ is time-like, space-like or light-like. Let

$\{\eta_1, \ldots, \eta_{n+1}\}$ be the standard orthonormal basis of $L^{n+1}$, that is, $\langle \eta_i, \eta_j \rangle = \delta_{ij}$, $\langle \eta_{n+1}, \eta_{n+1} \rangle = -1,\ \langle \eta_i, \eta_{n+1} \rangle = 0,\ i, j = 1, 2, \ldots, n$.

Let $\Theta(u_1, \ldots u_{n-2})$ denotes an orthogonal parametrization of the unit sphere $\mathbb{S}^{n-2}(1)$ in the Euclidean space $E^{n-1}$ generated by $\{\eta_1, \ldots, \eta_{n-1}\}$:

$$\begin{aligned} \Theta(u_1, \ldots u_{n-2}) &= \cos u_1 \eta_1 + \sin u_1 \cos u_2 \eta_2 + \cdots + \sin u_1 \cdots \sin u_{n-3} \\ &\quad \cos u_{n-2} \eta_{n-2} + \sin u_1 \cdots \sin u_{n-3} \sin u_{n-2} \eta_{n-1}, \end{aligned} \tag{2.1}$$

where $0 < u_i < \pi\ (i = 1, \ldots, n-3),\ 0 < u_{n-2} < 2\pi$.

**Remark.** When $n = 2$, the term $\Theta(u_1, \ldots, u_{n-2})$ in the following definitions of rotation hypersurfaces is replaced by $\eta_1$.

**Case 1.** $\ell$ is time-like. In this case the plane $\Pi$ that contains the line $\ell$ and a profile curve $\gamma$ is Lorentzian. Without lose of generality, we may suppose that $\ell$ is the $x_{n+1}$-axis and $\Pi$ is the $x_n x_{n+1}$-plane which is Lorentzian. Since every time-like line is transformed to the $x_{n+1}$-axis by a Lorentz transformation, and then every time-like plane containing the $x_{n+1}$-axis is transformed to the $x_n x_{n+1}$-plane.

Let $\gamma(t) = \varphi(t)\eta_n + \psi(t)\eta_{n+1}$ be a parametrization of $\gamma$ in the plane $\Pi$ with $x_n = \varphi(t) > 0,\ t \in I \subset \mathbb{R}$. The curve is space-like if $\varepsilon = \mathrm{sgn}(\varphi'^2 - \psi'^2) = 1$ and time-like if $\varepsilon = \mathrm{sgn}(\varphi'^2 - \psi'^2) = -1$.

So we can give a parametrization of a rotation hypersurface $M_{q,T}$ of $L^{n+1}$ with time-like axis as

$$\begin{aligned} f_T(u_1, \ldots, u_{n-1}, t) &= \varphi(t) \sin u_{n-1} \Theta(u_1, \ldots, u_{n-2}) \\ &\quad + \varphi(t) \cos u_{n-1}\, \eta_n + \psi(t)\eta_{n+1}, \end{aligned} \tag{2.2}$$

where $0 < u_{n-1} < \pi$. The second index in $M_{q,T}$ stands for the time-like axis. The hypersurface $M_{q,T}$ is also called a spherical rotation hypersurface of $L^{n+1}$ as parallels of $M_{q,T}$ are spheres $\mathbb{S}^{n-1}(0, \varphi(t))$.

**Case 2.** $\ell$ is space-like. In this case the plane $\Pi$ which contains a profile curve is Lorentzian or Riemannian. So there are rotation hypersurfaces of the first and second kind labeled by $M_{q,S_1}$ and $M_{q,S_2}$ in $L^{n+1}$ with space-like axis.

**Subcase 2.1.** The plane $\Pi$ is Lorentzian. Without losing generality we may suppose that $\ell$ is the $x_n$-axis, that is, the vector $\eta_n = (0, \ldots, 0, 1, 0)$ is the direction of the rotation axis, and $\Pi$ is the $x_n x_{n+1}$-plane. Let $\gamma(t) = \psi(t)\eta_n + \varphi(t)\eta_{n+1}$ be a parametrization of $\gamma$ in the plane $\Pi$ with $x_{n+1} = \varphi(t) > 0,\ t \in I \subset \mathbb{R}$. Thus we can give a parametrization of a rotation hypersurface of the first kind $M_{q,S_1}$ of $L^{n+1}$ with space-like axis as

$$\begin{aligned} f_{S_1}(u_1, \ldots, u_{n-1}, t) &= \varphi(t) \sinh u_{n-1} \Theta(u_1, \ldots, u_{n-2}) + \psi(t)\eta_n \\ &\quad + \varphi(t) \cosh u_{n-1} \eta_{n+1}, \end{aligned} \tag{2.3}$$

$0 < u_{n-1} < \infty$, which is also called a hyperbolic rotation hypersurface of $L^{n+1}$ as parallels of $M_{q,S_1}$ are hyperbolic spaces $\mathbb{H}^{n-1}(0, -\varphi(t))$.

**Subcase 2.2.** The plane $\Pi$ is Riemannian. We may suppose that $\ell$ is the $x_n$-axis and $\Pi$ is the $x_{n-1}x_n$-plane without lose of generality. Let $\gamma(t) = \varphi(t)\eta_{n-1}+\psi(t)\eta_n$ be a parametrization of $\gamma$ in the plane $\Pi$ with $x_{n-1} = \varphi(t) > 0,\ t \in I \subset \mathbb{R}$. In this case the curve $\gamma$ is space-like. Similarly we give a parametrization of a rotation hypersurface of the second kind $M_{q,S_2}$ of $L^{n+1}$ with space-like axis as

$$\begin{aligned} f_{S_2}(u_1,\ldots,u_{n-1},t) =&\varphi(t)\cosh u_{n-1}\Theta(u_1,\ldots,u_{n-2}) + \psi(t)\eta_n \\ &+ \varphi(t)\sinh u_{n-1}\eta_{n+1}, \end{aligned} \tag{2.4}$$

$-\infty < u_{n-1} < \infty$, which is called a pseudo-spherical rotation hypersurface of $L^{n+1}$ as parallels of $M_{q,S_2}$ are pseudo-spheres $\mathbb{S}_1^{n-1}(0,\varphi(t))$ when $n > 2$. (If $n = 2$, then $S_1^1 \equiv H^1$.) Later we will show that $M_{q,S_2}$ has the index 1, that is, $q = 1$.

**Case 3.** $\ell$ is light-like. Let $\{\hat\eta_1,\ldots,\hat\eta_{n+1}\}$ be a pseudo-Lorentzian basis of $L^{n+1}$, that is, $<\hat\eta_i,\hat\eta_j>=\delta_{ij},\ i,j = 1,\ldots,n-1,\ \ <\hat\eta_i,\hat\eta_n>=<\hat\eta_i,\hat\eta_{n+1}>=0,\ \ i = 1,2,\ldots,n-1,\ \ <\hat\eta_n,\hat\eta_{n+1}>=1,\ <\hat\eta_n,\hat\eta_n>=0,\ <\hat\eta_{n+1},\hat\eta_{n+1}>=0$. We can choose $\hat\eta_1 = (1,0,\ldots,0),\ldots,\hat\eta_{n-1} = (0,\ldots,1,0,0)$, $\hat\eta_n = \frac{1}{\sqrt2}(0,\ldots,0,1,-1)$, $\hat\eta_{n+1} = \frac{1}{\sqrt2}(0,\ldots,0,1,1)$. We may suppose that $\ell$ is the line spanned by the null vector $\hat\eta_{n+1}$ and $\Pi$ is the $x_nx_{n+1}$-plane without lose of generality. Let $\gamma(t) = \sqrt2\varphi(t)\hat\eta_n + \sqrt2\psi(t)\hat\eta_{n+1}$ be a parametrization of $\gamma$ in the plane $\Pi$ with $x_n = \varphi(t) > 0,\ t \in I \subset \mathbb{R}$.

Let $\Theta_1(u_1,\ldots,u_{n-2}),\ldots,\Theta_{n-1}(u_1,\ldots,u_{n-2})$ be the components of the orthogonal parameterization $\Theta(u_1,\ldots,u_{n-2})$ given by (2.1) of the unit sphere $\mathbb{S}^{n-2}(1)$ in the basis $\{\hat\eta_1,\ldots,\hat\eta_{n-1}\}$. We consider the subgroup of Lorentz group which fixes the direction $\hat\eta_{n+1}$ of the light-like axis $\ell$ is given by

$$\{B(u_1,\ldots,u_{n-1}) : u_1,\ldots,u_{n-3} \in (0,\pi),\ u_{n-2} \in (0,2\pi),\ u_{n-1} \in \mathbb{R}\},$$

where $B$ is the $(n+1)\times(n+1)$ matrix of the form

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & u_{n-1}\Theta_1 & -u_{n-1}\Theta_1 \\ 0 & 1 & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & u_{n-1}\Theta_{n-1} & -u_{n-1}\Theta_{n-1} \\ u_{n-1}\Theta_1 & \cdots & \cdots & u_{n-1}\Theta_{n-1} & 1-\frac{u_{n-1}^2}{2} & \frac{u_{n-1}^2}{2} \\ u_{n-1}\Theta_1 & \cdots & \cdots & u_{n-1}\Theta_{n-1} & -\frac{u_{n-1}^2}{2} & 1+\frac{u_{n-1}^2}{2} \end{pmatrix},$$

which has determinant one. When we apply the transformation $B$ to the vectors $\hat{\eta}_1, \ldots, \hat{\eta}_{n-1}$ we can have

$$B(\hat{\eta}_i) = \hat{\eta}_i + \sqrt{2}u_{n-1}\Theta_i\hat{\eta}_{n+1}, \quad i = 1, \ldots, n-1, \tag{2.5}$$

$$B(\hat{\eta}_n) = \sum_{i=1}^{n-1} \sqrt{2}u_{n-1}\Theta_i\hat{\eta}_i + \hat{\eta}_n - u_{n-1}^2\hat{\eta}_{n+1} \quad \text{and} \quad B(\hat{\eta}_{n+1}) = \hat{\eta}_{n+1}. \tag{2.6}$$

If we write $x = \sum_{i=1}^{n+1} x_i\hat{\eta}_i$, then by using (2.5) and (2.6) it can easily be shown that $B$ preserves the metric, that is, $\langle B(x), B(x)\rangle = \langle x, x\rangle$.

Hence, writing $\gamma(t) = (0, \ldots, 0, \psi(t) + \varphi(t), \psi(t) - \varphi(t))$ the rotation hypersurface $M_{q,L}$ of $L^{n+1}$ with light-like axis is defined as

$$\begin{aligned} f_L(u_1, \ldots, u_{n-1}, t) &= B(\gamma(t)) \\ &=(2\varphi(t)u_{n-1}\Theta_1, \ldots, 2\varphi(t)u_{n-1}\Theta_{n-1}, (\psi(t) + \varphi(t) - \varphi(t)u_{n-1}^2), \\ &\quad (\psi(t) - \varphi(t) - \varphi(t)u_{n-1}^2)), \quad u_{n-1} \neq 0 \end{aligned} \tag{2.7}$$

or equivalently by using $\gamma(t) = \sqrt{2}\varphi(t)\hat{\eta}_n + \sqrt{2}\psi(t)\hat{\eta}_{n+1}$ and (2.6) in the pseudo-Lorentzian basis we can write

$$\begin{aligned} f_L(u_1, \ldots, u_{n-1}, t) &= B(\gamma(t)) \\ &= 2\varphi(t)u_{n-1}\Theta(u_1, \ldots, u_{n-2}) + \sqrt{2}\varphi(t)\hat{\eta}_n \\ &\quad +\sqrt{2}(\psi(t) - \varphi(t)u_{n-1}^2)\hat{\eta}_{n+1}, \ u_{n-1} \neq 0. \end{aligned} \tag{2.8}$$

Note that in the third case if $\varphi(t) = \varphi_0$ or $\psi(t) = \psi_0$ is a constant, then the profile curve is degenerate. However, in the other cases if $\varphi(t) = \varphi_0 > 0$ is a constant and $\psi(t) = t$, then the rotation hypersurface $M_{1,T}$ is the Lorentz cylinder $\mathbb{S}^{n-1}(0, \varphi_0) \times L^1$, $M_{0,S_1}$ is the hyperbolic cylinder $\mathbb{H}^{n-1}(0, -\varphi_0) \times \mathbb{R}$, and $M_{1,S_2}$ is the pseudo-spherical cylinder $\mathbb{S}_1^{n-1}(0, \varphi_0) \times \mathbb{R}$. If $\varphi(t) = t$ and $\psi(t) = \psi_0$ is a constant, then $M_{0,T}$ is a space-like hyperplane of $L^{n+1}$, and $M_{1,S_1}$ and $M_{1,S_2}$ are time-like hyperplanes of $L^{n+1}$. Therefore all these are rotational hypersurfaces of $L^{n+1}$ with constant mean curvature.

## 3. Rotation Hypersurfaces with Time-like Axis

In this section we determine rotation hypersurfaces $M_{q,T}$ of $L^{n+1}$ with time-like axis and constant mean curvature. Especially we determine maximal space-like rotation hypersurface and Lorentzian rotation hypersurface with zero mean curvature of $L^{n+1}$ as a generalization of catenoids of the first kind and third kind, respectively.

**Proposition 3.1.** *Let $M_{q,T}$ be a rotation hypersurface of $L^{n+1}$ with the index $q$ and time-like axis parameterized by* (2.2). *Then the directions of parameters are principal directions, and the principal curvatures along the coordinate curves $u_i$, $i = 1, \ldots, n-1$, are all equal and given by*

$$\lambda = -\frac{\psi'}{\varphi\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}$$

*with multiplicity $n-1$, and the principal curvature along the coordinate curve $t$ is given by*

$$\mu = \frac{\psi'\varphi'' - \psi''\varphi'}{(\varphi'^2 - \psi'^2)\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}},$$

*where $\varepsilon = \operatorname{sgn}(\varphi'^2 - \psi'^2) = \mp 1$ and, $q = 0$ if $\varepsilon = 1$ and $q = 1$ if $\varepsilon = -1$.*

*Proof.* Taking derivative of (2.2) we have the orthogonal coordinate vector fields on $M_{q,T}$ as

$$\begin{aligned} \frac{\partial f_T}{\partial u_i} &= \varphi(t)\sin u_{n-1}\frac{\partial \Theta}{\partial u_i}, \quad i = 1, \ldots, n-2, \\ \frac{\partial f_T}{\partial u_{n-1}} &= \varphi(t)(\cos u_{n-1}\Theta - \sin u_{n-1}\eta_n), \\ \frac{\partial f_T}{\partial t} &= \varphi'(t)(\sin u_{n-1}\Theta + \cos u_{n-1}\eta_n) + \psi'(t)\eta_{n+1}. \end{aligned} \tag{3.1}$$

The vectors $\partial f_T/\partial u_i$'s are space-like, and however the vector $\partial f_T/\partial t$ is space-like if $\varepsilon = \operatorname{sgn}(\langle \partial f_T/\partial t, \partial f_T/\partial t\rangle) = \operatorname{sgn}(\varphi'^2 - \psi'^2) = 1$ and time-like if $\varepsilon = -1$.

Now we can choose an orthonormal tangent basis on $M_{q,T}$ as

$$e_i = \frac{1}{\|\partial f_T/\partial u_i\|}\frac{\partial}{\partial u_i}, \quad i = 1, \ldots, n-1, \quad e_n = \frac{1}{\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}\frac{\partial}{\partial t}$$

with signatures $\varepsilon_i =< e_i, e_i >= 1,\ i = 1, \ldots, n-1$ and $\varepsilon_n =< e_n, e_n >= \varepsilon$, where $\|\partial f_T/\partial u_i\| = \sqrt{\varepsilon_i \langle \partial f/\partial u_i, \partial f/\partial u_i\rangle}$. We determine a unit normal vector field $N$ on $M_{q,T}$ as

$$N = \frac{1}{\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}[\psi'(t)(\sin u_{n-1}\Theta + \cos u_{n-1}\eta_n) + \varphi'(t)\eta_{n+1}]$$

with $\langle N, N\rangle = -\varepsilon$ . Let $A_N$ denotes the shape operator of $M_{q,T}$ in the direction $N$. By a straightforward calculation we obtain

$$A_N(e_i) = -\frac{\psi'}{\varphi\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}\, e_i, \quad i = 1, \ldots, n-1$$

and

$$A_N(e_n) = \frac{\psi'\varphi'' - \psi''\varphi'}{(\varphi'^2 - \psi'^2)\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}\, e_n.$$

It follows from above that the coordinate curves are lines of curvature, and hence the principal curvatures $\lambda$ and $\mu$ are obtained. ■

Therefore the mean curvature of $M_{q,T}$ is

$$\text{(3.2)} \quad \alpha = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \langle A_N(e_i), e_i\rangle = \frac{1}{n\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}\left(-\frac{(n-1)\psi'}{\varphi} + \frac{\psi'\varphi'' - \psi''\varphi'}{\varphi'^2 - \psi'^2}\right)$$

which is the function of $t$.

Now we will investigate the rotation hypersurfaces of $L^{n+1}$ with time-like axis and constant mean curvature. We consider the rotation hypersurface $M_{q,T}$ defined by (2.2) for the profile curve $\gamma(t) = (\varphi(t), \psi(t)) = (t, g(t)),\ t > 0$, that is,

$$\text{(3.3)} \quad f_T(u_1, \ldots, u_{n-1}, t) = t \sin u_{n-1} \Theta(u_1, \ldots, u_{n-2}) + t \cos u_{n-1}\, \eta_n + g(t)\eta_{n+1},$$

where $g(t)$ is a differentiable function. This rotation hypersurface is space-like if $g'^2 < 1, (\varepsilon = 1,\ q = 0)$ and time-like if $g'^2 > 1, (\varepsilon = -1,\ q = 1)$.

**Theorem 3.2.** *The rotation hypersurface $M_{q,T}$ of $L^{n+1}$ with the index $q$ and time-like axis defined by* (3.3) *has constant mean curvature $\alpha$ if and only if the function $g(t)$ for the profile curve is given by*

$$\text{(3.4)} \quad g(t) = \int^t \frac{a \pm \alpha t^n}{\sqrt{(a \pm \alpha t^n)^2 + \varepsilon t^{2(n-1)}}}\, dt,$$

*where $a$ is an arbitrary constant, and $q = 0$ for $\varepsilon = 1$ and $q = 1$ for $\varepsilon = -1$.*

*Proof.* For the functions $\varphi(t) = t,\ t > 0$ and $\psi(t) = g(t)$, from (3.2) the rotation hypersurface $M_{q,T}$ defined by (3.3) has constant mean curvature if and only if $g = g(t)$ satisfies the differential equation:

$$\text{(3.5)} \quad g'' + \frac{(n-1)(1 - g'^2)g'}{t} + n\alpha\varepsilon[\varepsilon(1 - g'^2)]^{3/2} = 0,$$

for some constant $\alpha$.

Suppose that $M_{q,T}$ has constant mean curvature $\alpha$. Let $\varepsilon = 1$, that is, $g'^2 < 1$. If we substitute $g' = \sin u$, then the differential equation (3.5) becomes

$$\text{(3.6)} \quad u' + \frac{(n-1)}{t}\cos u \sin u + n\mu\alpha \cos^2 u = 0,$$

where $\mu = \operatorname{sgn}(\cos u) = \pm 1$. Now we make another substitution $w = \tan u$, then we obtain

$$u' = \frac{w'}{1+w^2}, \quad \cos u = \frac{1}{\sqrt{1+w^2}}, \quad \sin u = \frac{w}{\sqrt{1+w^2}}.$$

Hence the differential equation (3.6) becomes

$$w'(t) + \frac{(n-1)}{t} w(t) + n\mu\alpha = 0. \tag{3.7}$$

The solution of (3.7) yields $w(t) = \dfrac{a \pm \alpha t^n}{t^{(n-1)}}$ for some constant $a$. Therefore

$$g'(t) = \sin\left(\tan^{-1}\left(\frac{a \pm \alpha t^n}{t^{(n-1)}}\right)\right) = \frac{a \pm \alpha t^n}{\sqrt{(a \pm \alpha t^n)^2 + t^{2(n-1)}}} \tag{3.8}$$

and then

$$g(t) = \int^t \frac{a \pm \alpha t^n}{\sqrt{(a \pm \alpha t^n)^2 + t^{2(n-1)}}} dt. \tag{3.9}$$

Let $\varepsilon = -1$, that is, $g'^2 > 1$. Now if we substitute $g' = \cosh u$, then the differential equation (3.5) turns to

$$u' - \frac{(n-1)}{t} \cosh u \sinh u - n\mu\alpha \sinh^2 u = 0, \tag{3.10}$$

where $\mu = \operatorname{sgn}(\sinh u) = \pm 1$. Let us put $w = \tanh u$. Then we obtain

$$u' = \frac{w'}{1-w^2}, \quad \cosh u = \frac{1}{\sqrt{1-w^2}}, \quad \sinh u = \frac{w}{\sqrt{1-w^2}}.$$

Thus the differential equation (3.10) becomes

$$w'(t) - \frac{(n-1)}{t} w(t) - n\mu\alpha w^2 = 0, \tag{3.11}$$

which has solution $w(t) = \dfrac{t^{(n-1)}}{a \pm \alpha t^n}$ for some constant $a$. Therefore

$$g'(t) = \cosh\left(\tanh^{-1}\left(\frac{t^{(n-1)}}{a \pm \alpha t^n}\right)\right) = \frac{a \pm \alpha t^n}{\sqrt{(a \pm \alpha t^n)^2 - t^{2(n-1)}}} \tag{3.12}$$

and then

$$g(t) = \int^t \frac{a \pm \alpha t^n}{\sqrt{(a \pm \alpha t^n)^2 - t^{2(n-1)}}} dt. \tag{3.13}$$

Conversely, it can be shown that the mean curvature of $M_{q,T}$ is constant if $g(t)$ is given by (3.4). ■

We can have the following corollaries.

**Corollary 3.3.** *Let the mean curvature* $\alpha$ *of* $M_{q,T}$ *be a non-zero constant. If* $a = 0$ *in* (3.4)*, then* $g(t) = \pm\alpha^{-1}\sqrt{\alpha^2 t^2 + \varepsilon} + c$*,* $t > 1/|\alpha|$ *for* $\varepsilon = -1$*, where* $c$ *is an integration constant. Moreover,*

(1) *for* $\varepsilon = 1$ *the space-like rotation hypersurface* $M_{0,T}$ *of* $L^{n+1}$ *with time-like axis defined by* (3.3) *is a part of hyperbolic n-space* $\mathbb{H}^n(c\eta_{n+1}, -1/|\alpha|)$*, hence it is totally umbilical.*

(2) *for* $\varepsilon = -1$ *the Lorentzian rotation hypersurface* $M_{1,T}$ *of* $L^{n+1}$ *with time-like axis defined by* (3.3) *is a part of the de Sitter n-space* $\mathbb{S}^n_1(c\eta_{n+1}, 1/|\alpha|)$*, hence it is totally umbilical.*

*Proof.* If $a = 0$, by integrating (3.4) we get $g(t) = \pm\alpha^{-1}\sqrt{\alpha^2 t^2 + \varepsilon} + c$ for $\varepsilon = \pm 1$, and $t > 1/|\alpha|$ when $\varepsilon = -1$. Using the parameterization (3.3) of $M_{q,T}$ we have

$$\langle f_T - c\eta_{n+1}, f_T - c\eta_{n+1}\rangle = t^2 \sin^2 u_{n-1}\,\langle\Theta,\Theta\rangle + t^2\cos^2 u_{n-1} - \frac{\alpha^2 t^2 + \varepsilon}{\alpha^2} = -\frac{\varepsilon}{\alpha^2}$$

as $\langle\Theta,\Theta\rangle = 1$ from (2.1). Thus the proof follows. ■

**Corollary 3.4.**

(1) *The space-like rotation hypersurface* $M_{0,T}$ *of* $L^{n+1}$ *with time-like axis defined by* (3.3) *is maximal if and only if the function* $g(t)$ *for the profile curve is given by*

$$g(t) = \int^t \frac{a}{\sqrt{a^2 + t^{2(n-1)}}}\,dt. \tag{3.14}$$

(2) *The Lorentzian rotation hypersurface* $M_{1,T}$ *of* $L^{n+1}$ *with time-like axis defined by* (3.3) *has zero mean curvature if and only if the function* $g(t)$ *for the profile curve is given by*

$$g(t) = \int^t \frac{a}{\sqrt{a^2 - t^{2(n-1)}}}\,dt, \quad 0 < t < \sqrt[n-1]{|a|}, \tag{3.15}$$

*where* $a$ *is a non-zero constant.*

For $n > 2$ a non-planer minimal rotation hypersurface of an Euclidean space $E^{n+1}$ is called a generalized catenoid [1, 9]. Similarly we call a rotation hypersurface of a Lorentz-Minkowski space $L^{n+1}$ with zero mean curvature a generalized catenoid. So the maximal rotation hypersurface $M_{0,T}$ of $L^{n+1}$ with time-like axis is a part of the generalized catenoid of first kind. For instance, if $n = 2$, then from (3.14) we get $g(t) = a\sinh^{-1}(\frac{t}{a}) + b$, and then we have from (3.3)

$$f_T(u_1, t) = \left(t\sin u_1, t\cos u_1, a\sinh^{-1}\left(\frac{t}{a}\right) + b\right)$$

which is congruent to the catenoid of first kind given in [6].

Similarly, the Lorentzian rotation hypersurface $M_{1,T}$ of $L^{n+1}$ with time-like axis and zero mean curvature is called a generalized catenoid of the 3rd kind. For instance, if $n = 2$, then from (3.15) we get $g(t) = a\sin^{-1}(\frac{t}{a}) + b$, and then by (3.3) we have

$$f_T(u_1, t) = (t\sin u_1, t\cos u_1, a\sin^{-1}(\frac{t}{a}) + b)$$

which is congruent to a part of the catenoid of the 3rd kind given in [10].

## 4. Rotation Hypersurfaces of First Kind with Space-like Axis

In this section we investigate rotation hypersurfaces of the first kind $M_{q,S_1}$ of $L^{n+1}$ with space-like axis and constant mean curvature. Especially we determine maximal space-like rotation hypersurfaces and Lorentzian rotation hypersurfaces with zero mean curvature of the first kind of $L^{n+1}$ as a generalization of catenoids of the second kind and fourth kind, respectively.

On the hypersurface $M_{q,S_1}$ defined by (2.3), the unit normal field is given by

$$\bar{N} = \frac{1}{\sqrt{\bar{\varepsilon}(\psi'^2 - \varphi'^2)}}[\psi'(t)(\sinh u_{n-1}\Theta + \cosh u_{n-1}\eta_{n+1}) + \varphi'(t)\eta_n],$$

where $\bar{\varepsilon} = \text{sgn}(\psi'^2 - \varphi'^2)$ and $\langle \bar{N}, \bar{N}\rangle = -\bar{\varepsilon}$.

We state the followings without proof because the most of the calculations are the same as in Section 2.

**Proposition 4.1.** *Let $M_{q,S_1}$ be a rotation hypersurface of the first kind of $L^{n+1}$ with the index q and space-like axis parameterized by* (2.3). *Then the directions of parameters are principal directions, and the principal curvatures along the coordinate curves $u_i$, $i = 1, \ldots, n-1$, are all equal and given by*

$$\lambda = -\frac{\psi'}{\varphi\sqrt{\bar{\varepsilon}(\psi'^2 - \varphi'^2)}}$$

*with multiplicity* $n-1$*, and the principal curvature along the coordinate curve* $t$ *is given by*

$$\mu = \frac{\psi''\varphi' - \psi'\varphi''}{(\psi'^2 - \varphi'^2)\sqrt{\bar{\varepsilon}(\psi'^2 - \varphi'^2)}},$$

*where* $\bar{\varepsilon} = \operatorname{sgn}(\psi'^2 - \varphi'^2) = \mp 1$ *and,* $q = 0$ *if* $\bar{\varepsilon} = 1$ *and* $q = 1$ *if* $\bar{\varepsilon} = -1$.

Therefore the mean curvature vector of $M_{q,S_1}$ is

$$\text{(4.1)} \quad \bar{\alpha} = \frac{1}{n}\sum_{i=1}^{n} \bar{\varepsilon}_i \langle A_{\bar{N}}(e_i), e_i\rangle = \frac{1}{n\sqrt{\bar{\varepsilon}(\psi'^2 - \varphi'^2)}}\left(-\frac{(n-1)\psi'}{\varphi} + \frac{\varphi'\psi'' - \psi'\varphi''}{\psi'^2 - \varphi'^2}\right)$$

which is the function of $t$, where $e_1, \ldots, e_n$ are the unit principal directions of the shape operator $A_{\bar{N}}$ with signatures $\bar{\varepsilon}_i = \langle e_i, e_i\rangle$.

We now consider the rotation hypersurface of the first kind $M_{q,S_1}$ of $L^{n+1}$ with space-like axis defined by (2.3) for the profile curve $\gamma(t) = (\varphi(t), \psi(t)) = (t, g(t)),\ t > 0$, that is,

$$\text{(4.2)} \qquad f_{S_1}(u_1, \ldots, u_{n-1}, t) = t \sinh u_{n-1} \Theta(u_1, \ldots, u_{n-2}) + g(t)\eta_n + t\cosh u_{n-1}\eta_{n+1},$$

where $g(t)$ is a differentiable function. This rotation hypersurface is space-like if $g'^2 > 1, (\bar{\varepsilon} = 1)$ and time-like if $g'^2 < 1, (\bar{\varepsilon} = -1)$.

Hence, from (4.1) we can state that the rotation hypersurface of the first kind $M_{q,S_1}$ of $L^{n+1}$ with space-like axis parametrized by (4.2) has constant mean curvature if and only if $g = g(t)$ satisfies the differential equation

$$\text{(4.3)} \qquad g'' - \frac{(n-1)(g'^2 - 1)g'}{t} + n\bar{\alpha}\bar{\varepsilon}[\bar{\varepsilon}(g'^2 - 1)]^{3/2} = 0$$

for some constant $\bar{\alpha}$.

**Theorem 4.2.** *The rotation hypersurface of the first kind* $M_{q,S_1}$ *of* $L^{n+1}$ *with the index* $q$ *and space-like axis defined by* (4.2) *has constant mean curvature* $\bar{\alpha}$ *if and only if the function* $g(t)$ *for the profile curve is given by*

$$\text{(4.4)} \qquad g(t) = \int^t \frac{a \pm \bar{\alpha}t^n}{\sqrt{(a \pm \bar{\alpha}t^n)^2 - \bar{\varepsilon}t^{2(n-1)}}}\,dt,$$

*where* $a$ *is an arbitrary constant, and* $q = 0$ *for* $\bar{\varepsilon} = 1$ *and* $q = 1$ *for* $\bar{\varepsilon} = -1$.

So we can have the following corollaries.

**Corollary 4.3.** *Let the mean curvature* $\bar{\alpha}$ *of* $M_{q,S_1}$ *be a non-zero constant. If* $a = 0$ *in* (4.4)*, then* $g(t) = \pm\bar{\alpha}^{-1}\sqrt{\bar{\alpha}^2 t^2 - \bar{\varepsilon}} + c,\ \ t > 1/|\bar{\alpha}|$ *for* $\bar{\varepsilon} = 1$*, where* $c$ *is an integration constant. Moreover,*

(1) *for* $\bar{\varepsilon} = 1$ *the space-like rotation hypersurface of the first kind* $M_{0,S_1}$ *of* $L^{n+1}$ *with space-like axis defined by* (4.2) *is a part of the hyperbolic n-space* $\mathbb{H}^n(c\eta_n, -1/|\bar{\alpha}|)$*, hence it is totally umbilical.*

(2) *for* $\bar{\varepsilon} = -1$ *the Lorentzian rotation hypersurface of the first kind* $M_{1,S_1}$ *of* $L^{n+1}$ *with space-like axis defined by* (4.2) *is a part of de Sitter n-space* $\mathbb{S}^n_1(c\eta_n, 1/|\bar{\alpha}|)$*, hence it is totally umbilical.*

**Corollary 4.4.**

(1) *The space-like rotation hypersurface of the first kind* $M_{0,S_1}$ *of* $L^{n+1}$ *with space-like axis defined by* (4.2) *is maximal if and only if the function* $g(t)$ *for the profile curve is given by*

$$g(t) = \int^t \frac{a}{\sqrt{a^2 - t^{2(n-1)}}}\,dt, \quad 0 < t < \sqrt[n-1]{|a|}. \tag{4.5}$$

(2) *The Lorentzian rotation hypersurface of the first kind* $M_{1,S_1}$ *of* $L^{n+1}$ *with space-like axis defined by* (4.2) *has zero mean curvature if and only if the function* $g(t)$ *for the profile curve is given by*

$$g(t) = \int^t \frac{a}{\sqrt{a^2 + t^{2(n-1)}}}\,dt, \tag{4.6}$$

*where* $a$ *is a non-zero constant.*

The maximal rotation hypersurface of the first kind $M_{0,S_1}$ of $L^{n+1}$ with space-like axis is called a generalized catenoid of the second kind. For instance, if $n = 2$, then from (4.5) we get $g(t) = a\sin^{-1}(\frac{t}{a}) + b,\quad 0 < t < \sqrt{|a|}$, and then by (4.2) we have

$$f_{S_1}(u_1, t) = (t\sinh u_1, a\sin^{-1}(\frac{t}{a}) + b, t\cosh u_1)$$

which is congruent to a part of the catenoid of the second kind given in [6].

Similarly, the Lorentzian rotation hypersurface of the first kind $M_{1,S_1}$ of $L^{n+1}$ with space-like axis and zero mean curvature is called a generalized catenoid of the 4th kind. For instance, if $n = 2$, then from (4.6) we get $g(t) = a\sinh^{-1}(\frac{t}{a}) + b$, and then by (4.2) we have

$$f_{S_1}(u_1, t) = (t\sinh u_1, a\sinh^{-1}(\frac{t}{a}) + b, t\cosh u_1)$$

which is congruent to the catenoid of the 4th kind given in [10].

## 5. Rotation Hypersurfaces of Second Kind with Space-like Axis

In this section we study rotation hypersurfaces of the second kind $M_{q,S_2}$ of $L^{n+1}$ with space-like axis and constant mean curvature. In the following it is seen that the index $q$ is only one, That is, in this case we only have Lorentzian rotation hypersurface of $L^{n+1}$.

Taking derivative of (2.4) we have the orthogonal coordinate vector fields on $M_{q,S_2}$ as

$$\begin{aligned} \frac{\partial f_{S_2}}{\partial u_i} &= \varphi(t)\cosh u_{n-1}\frac{\partial \Theta}{\partial u_i}, \quad i=1,\ldots,n-2, \\ \frac{\partial f_{S_2}}{\partial u_{n-1}} &= \varphi(t)(\sinh u_{n-1}\Theta + \cosh u_{n-1}\eta_{n+1}), \\ \frac{\partial f_{S_2}}{\partial t} &= \varphi'(t)(\cosh u_{n-1}\Theta + \sinh u_{n-1}\eta_{n+1}) + \psi'(t)\eta_n. \end{aligned} \tag{5.1}$$

The vectors $\partial f_{S_2}/\partial t,\ \ \partial f_{S_2}/\partial u_i\ i=1,\ldots,n-2$ are space-like and the vector $\partial f_{S_2}/\partial u_{n-1}$ is time-like. This means that $M_{q,S_2}$ is Lorentzian, that is, $q=1$. Also, the space-like unit normal field on $M_{1,S_2}$ is given by

$$\tilde{N} = \frac{1}{\sqrt{\varphi'^2+\psi'^2}}[\psi'(t)(\cosh u_{n-1}\Theta + \sinh u_{n-1}\eta_{n+1}) - \varphi'(t)\eta_n].$$

Thus we give the followings without proof because the most of the calculations are the same as in Section 2.

**Proposition 5.1.** *Let $M_{1,S_2}$ be the Lorentzian rotation hypersurface of the second kind of $L^{n+1}$ with space-like axis parameterized by* (2.4)*. Then the directions of parameters are principal directions, and the principal curvatures along the coordinate curves $u_i$, $i=1,\ldots,n-1$, are all equal and given by*

$$\lambda = -\frac{\psi'}{\varphi\sqrt{(\psi'^2+\varphi'^2)}}$$

*with multiplicity $n-1$, and the principal curvature along the coordinate curve $t$ is given by*

$$\mu = \frac{\psi'\varphi'' - \psi''\varphi'}{(\psi'^2+\varphi'^2)\sqrt{(\psi'^2+\varphi'^2)}}.$$

Hence the mean curvature of $M_{1,S_2}$ is

$$\tilde{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\tilde{\varepsilon}_i\langle A_{\tilde{N}}(e_i), e_i\rangle = \frac{1}{n\sqrt{\varphi'^2+\psi'^2}}\left(-\frac{(n-1)\psi'}{\varphi} + \frac{\psi'\varphi''-\varphi'\psi''}{\varphi'^2+\psi'^2}\right), \tag{5.2}$$

where $e_1, \ldots, e_n$ are the unit principal directions of the shape operator $A_{\tilde{N}}$ with signatures $\tilde{\varepsilon}_n = \tilde{\varepsilon}_i = 1,\ i = 1, \ldots, n-2,\ \tilde{\varepsilon}_{n-1} = -1$.

We consider the Lorentzian rotation hypersurface of the second kind $M_{1,S_2}$ of $L^{n+1}$ with space-like axis defined by (2.4) for the profile curve $\gamma(t) = (\varphi(t), \psi(t)) = (t, g(t)),\ t > 0$, that is,

$$f_{S_2}(u_1, \ldots, u_{n-1}, t) = t \cosh u_{n-1} \Theta(u_1, \ldots, u_{n-2}) + g(t)\eta_n + t \sinh u_{n-1} \eta_{n+1}, \tag{5.3}$$

where $g(t)$ is a differentiable function. Hence, from (5.2) we can state that the rotation hypersurface of the second kind $M_{1,S_2}$ of $L^{n+1}$ with space-like axis parametrized by (5.3) has constant mean curvature if and only if the function $g = g(t)$ satisfies the differential equation:

$$g'' + \frac{(n-1)(1+g'^2)g'}{t} + n\tilde{\alpha}(1+g'^2)^{3/2} = 0 \tag{5.4}$$

for some constant $\tilde{\alpha}$.

**Theorem 5.2.** *The Lorentzian rotation hypersurface of the second kind $M_{1,S_2}$ of $L^{n+1}$ with space-like axis defined by* (5.3) *has constant mean curvature $\tilde{\alpha}$ if and only if the function $g(t)$ for the profile curve is given by*

$$g(t) = \int^t \frac{a \pm \tilde{\alpha} t^n}{\sqrt{t^{2(n-1)} - (a \pm \tilde{\alpha} t^n)^2}}\, dt, \tag{5.5}$$

*where $a$ is an arbitrary constant.*

From this theorem we have the following corollaries.

**Corollary 5.3.** *Let the mean curvature $\tilde{\alpha}$ of $M_{1,S_2}$ be a non-zero constant. If $a = 0$ in* (5.5)*, then $g(t) = \mp\tilde{\alpha}^{-1}\sqrt{1 - \tilde{\alpha}^2 t^2} + c,\ \ 0 < t < 1/|\tilde{\alpha}|$. Moreover, the Lorentzian rotation hypersurface of the second kind $M_{1,S_2}$ of $L^{n+1}$ with space-like axis defined by* (5.3) *is a part of the de Sitter n-space $\mathbb{S}^n_1(c\eta_n, 1/|\tilde{\alpha}|)$, hence it is totally umbilical.*

**Corollary 5.4.** *The Lorentzian rotation hypersurface of the second kind $M_{1,S_2}$ of $L^{n+1}$ with space-like axis defined by* (5.3) *has zero mean curvature if and only if the function $g(t)$ for the profile curve is given by*

$$g(t) = \int^t \frac{a}{\sqrt{t^{2(n-1)} - a^2}}\, dt, \quad t > \sqrt[n-1]{|a|}, \tag{5.6}$$

*where $a$ is a non-zero constant.*

The Lorentzian rotation hypersurface of the second kind $M_{1,S_2}$ of $L^{n+1}$ with space-like axis and zero mean curvature is called a generalized catenoid of the 5th kind. For instance, if $n = 2$, then from (5.6) we get $g(t) = a\cosh^{-1}(\frac{t}{a}) + b$, and then by (5.3) we have

$$f_{S_2}(u_1, t) = (t\cosh u_1, a\cosh^{-1}(\frac{t}{a}) + b, t\sinh u_1)$$

which is congruent to the catenoid of the 5th kind given in [10].

## 6. Rotation Hypersurface with Light-like Axis

In this section we study rotation hypersurfaces $M_{q,L}$ of $L^{n+2}$ with light-like axis and constant mean curvature. We determine maximal space-like rotation hypersurface and Lorentzian rotation hypersurface with zero mean curvature of $L^{n+1}$ as a generalization of Enneper surfaces of the second kind and third kind, respectively.

**Proposition 6.1.** *Let $M_{q,L}$ be a rotation hypersurface of $L^{n+1}$ with the index $q$ and light-like axis parameterized by* (2.8)*. Then the directions of parameters are principal directions, and the principal curvatures along the coordinate curves $u_i$, $i = 1, \ldots, n-1$ are all equal and given by*

$$\lambda = -\frac{\varphi'}{2\varphi\sqrt{\hat{\varepsilon}\varphi'\psi'}}$$

*with multiplicity $n-1$, and the principal curvature along the coordinate curve $t$ is given by*

$$\mu = \frac{\varphi'\psi'' - \psi'\varphi''}{4\varphi'\psi'\sqrt{\hat{\varepsilon}\varphi'\psi'}},$$

*where $\hat{\varepsilon} = sgn(\varphi'\psi') = \mp 1$ and, $q = 0$ if $\hat{\varepsilon} = 1$ and $q = 1$ if $\hat{\varepsilon} = -1$.*

*Proof.* Taking derivative of (2.8) we have the orthogonal coordinate vector fields on $M_{q,L}$ as

$$\begin{aligned}
\frac{\partial f_L}{\partial u_i} &= 2\varphi(t)u_{n-1}\frac{\partial \Theta}{\partial u_i}, \quad i = 1, \ldots, n-2, \\
\frac{\partial f_L}{\partial u_{n-1}} &= 2\varphi(t)(\Theta - \sqrt{2}u_{n-1}\hat{\eta}_{n+1}), \qquad (6.1)\\
\frac{\partial f_L}{\partial t} &= \varphi'(t)(2u_{n-1}\Theta + \sqrt{2}\hat{\eta}_n) + \sqrt{2}(\psi'(t) - \varphi'(t)u_{n-1}^2)\hat{\eta}_{n+1}.
\end{aligned}$$

So we have $\left\langle \frac{\partial f_L}{\partial u_i}, \frac{\partial f_L}{\partial u_j} \right\rangle = 4\varphi^2(t)u_{n-1}^2 \left\langle \frac{\partial \Theta}{\partial u_i}, \frac{\partial \Theta}{\partial u_j} \right\rangle = 4\varphi^2(t)u_{n-1}^2 \|\frac{\partial \Theta}{\partial u_i}\|^2\delta_{ij}$, $u_{n-1} \neq 0$, $i, j = 1, \ldots, n-2$ as $\Theta$ is an orthogonal parametrization of the unit sphere

$S^{n-2}(1)$, $\left\langle \frac{\partial f_L}{\partial u_{n-1}}, \frac{\partial f_L}{\partial u_{n-1}} \right\rangle = 4\varphi^2(t)$ and $\left\langle \frac{\partial f_L}{\partial t}, \frac{\partial f_L}{\partial t} \right\rangle = 4\varphi'(t)\psi'(t) \neq 0$ because the profile curve is nonnull.

The vectors $\partial f_L/\partial u_i$'s are space-like, and however the vector $\partial f_L/\partial t$ is space-like if $\hat{\varepsilon} = \text{sgn}(\langle \partial f_T/\partial t, \partial f_T/\partial t \rangle) = \text{sgn}(\varphi'(t)\psi'(t)) = 1$ and time-like if $\hat{\varepsilon} = -1$. Thus we can choose an orthonormal tangent basis on $M_{q,L}$ as

$$e_i = \frac{1}{\|\partial f_L/\partial u_i\|}\frac{\partial}{\partial u_i}, \quad i = 1, \ldots, n-1, \quad e_n = \frac{1}{2\sqrt{\hat{\varepsilon}\varphi'\psi'}}\frac{\partial}{\partial t}$$

with $\hat{\varepsilon}_i = 1,\ i = 1, \ldots, n-1,\ \hat{\varepsilon}_n = \hat{\varepsilon}$. Also we have the unit normal field to $M_{q,L}$ as

$$\hat{N} = \frac{1}{\sqrt{2\hat{\varepsilon}\varphi'\psi'}}[\varphi'(\sqrt{2}u_{n-1}\Theta + \hat{\eta}_n) - (\psi' + \varphi' u_{n-1}^2)\hat{\eta}_{n+1}]$$

with $\langle \hat{N}, \hat{N} \rangle = -\hat{\varepsilon}$. By a straightforward calculation we obtain

$$A_{\hat{N}}(e_i) = -\frac{\varphi'}{2\varphi\sqrt{\hat{\varepsilon}\varphi'\psi'}}\, e_i, \ \ i = 1, \ldots, n-1 \ \text{ and } \ A_{\hat{N}}(e_n) = \frac{\varphi'\psi'' - \psi'\varphi''}{4\varphi'\psi'\sqrt{\hat{\varepsilon}\varphi'\psi'}}\, e_n.$$

It follows from above that the coordinate curves are lines of curvature, and hence the principal curvatures $\lambda$ and $\mu$ are obtained. ■

Therefore, for the mean curvature $\hat{\alpha}$ of $M_{q,L}$ we have

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} \hat{\varepsilon}_i \langle A_{\hat{N}}(e_i), e_i \rangle = \frac{1}{n\sqrt{\hat{\varepsilon}\varphi'\psi'}}\left(-\frac{(n-1)\varphi'}{2\varphi} + \frac{\varphi'\psi'' - \psi'\varphi''}{4\varphi'\psi'}\right) \tag{6.2}$$

which is the function of $t$.

To investigate the rotation hypersurface $M_{q,L}$ of $L^{n+1}$ with light-like axis and constant mean curvature we consider the rotation hypersurface defined by (2.8) for the profile curve $\gamma(t) = (\varphi(t), \psi(t)) = (t, g(t)),\ t > 0$, that is,

$$\begin{aligned} f_L(u_1, \ldots, u_{n-1}, t) &= 2tu_{n-1}\Theta(u_1, \ldots, u_{n-2}) + \sqrt{2}t\hat{\eta}_n \\ &\quad + \sqrt{2}(g(t) - tu_{n-1}^2)\hat{\eta}_{n+1}, \end{aligned} \tag{6.3}$$

where $g(t)$ is a differentiable function. This rotation hypersurface is space-like if $g' > 0, (\hat{\varepsilon} = 1)$ and time-like if $g' < 0, (\hat{\varepsilon} = -1)$ This means that the profile curve is strictly monotonic.

**Theorem 6.2.** *The rotation hypersurface $M_{q,L}$ of $L^{n+1}$ with light-like axis given by* (6.3) *has constant mean curvature $\hat{\alpha}$ if and only if the function $g(t)$ for the profile curve is given by*

$$g(t) = \int^t \hat{\varepsilon}\frac{t^{2(n-1)}}{(a - 2t^n\hat{\alpha})^2}\, dt, \quad \hat{\varepsilon} = \pm 1, \tag{6.4}$$

*where $a$ is an arbitrary constant.*

*Proof.* For the function $\varphi(t) = t,\ t > 0$ and $\psi(t) = g(t)$, from (6.2) the rotation hypersurface $M_{q,L}$ parametrized by (6.3) has constant mean curvature if and only if $g = g(t)$ satisfies the differential equation:

$$g'' - \frac{2(n-1)g'}{t} - 4n\hat{\alpha}g'\sqrt{\hat{\varepsilon}g'} = 0 \tag{6.5}$$

for some constant $\hat{\alpha}$.

Suppose that $M_{q,L}$ has constant mean curvature $\hat{\alpha}$. Let us put $\hat{\varepsilon}g' = w^2$. Then $g'' = 2\hat{\varepsilon}ww'$ and the differential equation (6.5) becomes

$$w' - \frac{(n-1)}{t}w - 2n\mu\hat{\alpha}w^2 = 0, \tag{6.6}$$

where $\mu = \mathrm{sgn}(w) = \pm 1$. The solution of (6.6) gives $w(t) = \mu\dfrac{t^{n-1}}{a - 2\hat{\alpha}t^n}$ for some constant $a$. We then obtain (6.4) by solving $g' = \hat{\varepsilon}w^2$.

Conversely, it can be shown that the mean curvature of $M_{q,L}$ is constant if $g(t)$ is given by (6.4). ∎

Then we have the following corollaries.

**Corollary 6.3.** *Let the mean curvature $\hat{\alpha}$ of $M_{q,L}$ be a non-zero constant. If $a = 0$ in* (6.4)*, then $g(t) = c - \frac{\hat{\varepsilon}}{4t\hat{\alpha}^2}$, $t > 0$, where $c$ is an integration constant. Moreover,*

(1) *for $\hat{\varepsilon} = 1$ the space-like rotation hypersurface $M_{0,L}$ of $L^{n+1}$ with light-like axis parameterized by* (6.3) *is a part of hyperbolic n-space $\mathbb{H}^n(c\hat{\eta}_{n+1}, -1/|\hat{\alpha}|)$, hence it is totally umbilical.*

(2) *for $\varepsilon = -1$ the Lorentzian rotation hypersurface $M_{1,L}$ of $L^{n+1}$ with time-like axis parameterized by* (6.3) *is a part of the de Sitter n-space $\mathbb{S}^n_1(c\hat{\eta}_{n+1}, 1/|\hat{\alpha}|)$, hence it is totally umbilical, where $\hat{\eta}_{n+1}$ is the direction of the light-like rotation axis.*

*Proof.* If $a = 0$, by integrating (6.4) we get $g(t) = c - \frac{\hat{\varepsilon}}{4t\hat{\alpha}^2}$. Using the parameterization (6.3) of $M_{q,L}$ we have

$$\begin{aligned}\left\langle f_L - c\sqrt{2}\hat{\eta}_{n+1}, f_L - c\sqrt{2}\hat{\eta}_{n+1}\right\rangle &= 4t^2u_{n-1}^2\langle\Theta,\Theta\rangle + 4t\left(\frac{\hat{\varepsilon}}{4t\hat{\alpha}^2} - tu_{n-1}^2\right)\langle\hat{\eta}_n, \hat{\eta}_{n+1}\rangle \\ &= -\frac{\hat{\varepsilon}}{\hat{\alpha}^2}\end{aligned}$$

as $\langle\Theta,\Theta\rangle = 1$ from (2.1). Therefore the proof follows. ∎

**Corollary 6.4.**

(1) *The space-like rotation hypersurface $M_{0,L}$ of $L^{n+1}$ with light-like axis given by* (6.3) *is maximal if and only if the function $g(t)$ for the profile curve is given by*

$$g(t) = \frac{1}{a^2}\frac{t^{2n-1}}{2n-1} + b. \tag{6.7}$$

(2) *The Lorentzian rotation hypersurface $\hat{M}_{1,L}$ of $L^{n+1}$ with light-like axis given by* (6.3) *has zero mean curvature if and only if the function $g(t)$ for the profile curve is given by*

$$g(t) = b - \frac{1}{a^2}\frac{t^{2n-1}}{2n-1}, \tag{6.8}$$

*where $a \neq 0$ and $b$ are constants.*

We call the maximal space-like rotation hypersurface $M_{0,L}$ of $L^{n+1}$ with light-like axis the hypersurface of Enneper of the second kind. For instance, for $n = 2,\ a = 1$ and $b = 0$ from (6.3) and (6.7) the maximal space-like rotation surface $M_{0,L}$ with light-like axis is given by

$$\begin{aligned} f_L(u_1,t) =& 2tu_1\eta_1 + \sqrt{2}t\hat{\eta}_2 + \sqrt{2}\left(\frac{t^3}{3} - tu_1^2\right)\hat{\eta}_3 \\ =& \left(2tu_1, \frac{t^3}{3} + t - tu_1^2, \frac{t^3}{3} - t - tu_1^2\right), \end{aligned}$$

which is congruent to the Enneper's surface of the second kind given in [6].

Similarly we call the time-like rotation hypersurface $M_{1,L}$ of $L^{n+1}$ with light-like axis and zero mean curvature the hypersurface of Enneper of the third kind. For instance, for $n = 2,\ a = 1$ and $b = 0$ from (6.3) and (6.8) the time-like rotation surface $M_{1,L}$ with light-like axis and zero mean curvature is given by

$$\begin{aligned} f_L(u_1,t) =& 2tu_1\eta_1 + \sqrt{2}t\hat{\eta}_2 + \sqrt{2}\left(-\frac{t^3}{3} - tu_1^2\right)\hat{\eta}_3 \\ =& \left(2tu_1, -\frac{t^3}{3} + t - tu_1^2, -\frac{t^3}{3} - t - tu_1^2\right), \end{aligned}$$

which congruent to the Enneper's surface of the third kind given in [10].

**Corollary 6.5.** *For $n = 2$ and $a \neq 0$, the rotation surface $M_{q,L}$ with light-like axis given by* (6.3) *has non-zero constant mean curvature if and only if the function*

*$g(t)$ for the profile curve is given by*

$$(6.9)\quad g(t) = \begin{cases} \dfrac{\hat{\varepsilon}}{8\hat{\alpha}^2}\left(\dfrac{t}{\rho^2-t^2}-\dfrac{1}{\rho}\tanh^{-1}\left(\dfrac{t}{\rho}\right)\right)+b,\ 0<\dfrac{t}{\rho}<1, & \dfrac{a}{2\hat{\alpha}}=\rho^2>0 \\ \dfrac{\hat{\varepsilon}}{8\hat{\alpha}^2}\left(\dfrac{-t}{t^2+\rho^2}+\dfrac{1}{\rho}\tan^{-1}\left(\dfrac{t}{\rho}\right)\right)+b,\ t>0, & \dfrac{a}{2\hat{\alpha}}=-\rho^2<0, \end{cases}$$

*where $a$ is a non-zero constant, $q = 0$ when $\hat{\varepsilon} = 1$ and $q = 1$ when $\hat{\varepsilon} = -1$.*

*Proof.* The proof is followed from the evaluation of the integral in (6.4) for $n = 2$. ■

The results given in Corollary 6.5 for the space-like surface ($\hat{\varepsilon} = 1$) was also obtained in [4].

Now we state a classification theorem for rotation hypersurfaces of $L^{n+1}$ with constant mean curvature.

**Theorem 6.6.** *Let $M$ be a rotation hypersurface of a Lorentz-Minkowski space $L^{n+1}$. If $M$ has constant mean curvature, then it is locally congruent to a part of one of the following rotation hypersurfaces:*

(1) *a space-like hyperplane or a time-like hyperplane of $L^{n+1}$;*

(2) *a Lorentz cylinder $M_{1,T} = \mathbb{S}^{n-1}(0,\varphi_0)\times L^1$ or a hyperbolic cylinder $M_{0,S_1} = \mathbb{H}^{n-1}(0,-\varphi_0)\times\mathbb{R}$ or a pseudo-spherical cylinder $M_{1,S_2} = \mathbb{S}_1^{n-1}(0,\varphi_0)\times\mathbb{R}$, where $\varphi_0$ is a positive real number;*

(3) *the rotation hypersurface $M_{q,T}$ of $L^{n+1}$ with time-like axis defined by* (3.3) *for the profile curve $g(t)$ given by* (3.4)*;*

(4) *the rotation hypersurface of the first kind $M_{q,S_1}$ of $L^{n+1}$ with space-like axis defined by* (4.2) *for the profile curve $g(t)$ given by* (4.4)*;*

(5) *the rotation hypersurface of the second kind $M_{1,S_2}$ of $L^{n+1}$ with space-like axis defined by* (5.3) *for the profile curve $g(t)$ given by* (5.5)*;*

(6) *the rotation hypersurface $M_{q,L}$ of $L^{n+1}$ with light-like axis defined by* (6.3) *for the profile curve $g(t)$ given by* (6.4).

Note that the cases (3), (4), (5), and (6) in Theorem 6.6 include a hyperbolic n-space $\mathbb{H}^n$ or a de Sitter n-space $\mathbb{S}_1^n$ with time-like, space-like or light-like axis by following Corollaries 3.3, 4.3, 5.3, and 6.3.

When $n = 2$, Theorem 6.6 gives all time-like and space-like rotation surfaces of Lorentz-Minkowski 3-space with constant mean curvature which includes locally the results on the space-like rotation surfaces with constant mean curvature given in [4].

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