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ORDER PRESERVING BIJECTIONS OF $C_+(X)$

Janko Marovt

Abstract. Let X be a compact Hausdorff space which satisfies the first axiom of countability, and $C_+(X)$ the set of all continuous functions from X to $[0,\infty)$. If $\varphi : C_+(X) \to C_+(X)$ is a bijective map which preserves the order in both directions, then there exists a homeomorphism $\omega : X \to X$ and for each $x \in X$ a bijective, increasing map $m_x : [0,\infty) \to [0,\infty)$ such that $\varphi(f)(x) = m_x(f(\omega(x)))$, for all $x \in X$ and $f \in C_+(X)$.

1. Introduction and Statement of the Result

The problem we consider in this paper has been motivated by our result in [5] and by result of L. Molnár in [9] . Molnár and several other authors studied preservers of various operations, relations and quantities on Hilbert space effect algebras (see $[6-9])$.

Let us denote the set of all continuous functions from a compact Hausdorff space X to the unit interval I by $C(X, I)$. From the result of L. Molnár in [9] it follows that the the study of multiplicative bijections on $C(X, I)$ is the crucial step in understanding the sequential automorphisms between the sets of effects in general von Neumann algebras. Our main result in [4] describes the general form of all bijective multiplicative maps of $C(X, I)$ under the technical condition that X satisfies the first axiom of countability. Ercan and Önal proved in [1] that our result does not hold without the assumption of first countability.

It seemed that a necessary step in understanding the structure of preservers of different types on general von Neumann algebra effects is to investigate the transformations of $C(X, I)$. So, in [3] we described bijective maps on $C(X, I)$ which preserve the operation of convex combinations, and in [5] we presented a structural result describing the bijective transformations of $C(X, I)$ which preserve

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order $\leq$ in both directions, i.e., $f \leq g$ if and only if $\varphi(f) \leq \varphi(g)$ for all $f, g \in C(X, I)$. Again, we assumed that X satisfies the first axiom of countability.

Other positive operators on a Hilbert space (beside Hilbert space effects) also play an important role in quantum mechanics (see [2, 10]). For example, the set of all positive operators on $\mathcal{H}$ is important in the definition of so called POV measures. This motivated us to study transformations on $C_+(X)$, where $C_+(X)$ denotes the set of all continuous functions from X to $[0, \infty)$. In this paper we present the form of the bijective maps on $C_+(X)$ which preserve the order $\leq$ in both directions.

Theorem 1.1. *Let X be a compact Hausdorff space which satisfies the first axiom of countability. If $\varphi : C_+(X) \to C_+(X)$ is a bijective map which preserves the order in both directions, then there exists a homeomorphism $\omega : X \to X$ and for each $x \in X$ a bijective, increasing map $m_x : [0, \infty) \to [0, \infty)$ such that*

$$\varphi(f)(x) = m_x(f(\omega(x))), \quad x \in X,$$

for all $f \in C_+(X)$. Conversely, suppose $\omega : X \to X$ is a homeomorphism, let $m_x : [0, \infty) \to [0, \infty)$, $x \in X$, be a bijective, increasing map for every $x \in X$ and $m_x(c) : X \to [0, \infty)$, $x \mapsto m_x(c)$, a continuous map for every $c \in [0, \infty)$. Define

$$\varphi(f)(x) = m_x(f(\omega(x))), \quad x \in X,$$

for all $f \in C_+(X)$. Then $\varphi : C_+(X) \to C_+(X)$ is a bijective map that preserves the order $\leq$ in both directions.

2. Proof of the Theorem

First, we advise the reader to have a good knowledge of [5]. Let us mention that on the one hand some steps in the proof of this theorem are similar as in the proof of Theorem 1.1 in [5]. On the other hand the construction of the homeomorphism ω is more difficult, for example, for difference with the proof in [5], we can not here use the fact that φ maps the identity function into the identity function.

For $a \geq 0$ let a_X be a constant function in $C_+(X)$, i.e., $a_X(x) = a$ for all $x \in X$. If $\varphi : C_+(X) \to C_+(X)$ is a surjective map which preserves the order $\leq$ we obtain

$$\varphi(0_X) = 0_X.$$

Lemma 2.1. *Suppose $\varphi : C_+(X) \to C_+(X)$ is a bijective map where $f \leq g$ if and only if $\varphi(f) \leq \varphi(g)$. Then*

$$fg = 0_X \quad \text{if and only if} \quad \varphi(f)\varphi(g) = 0_X.$$

The proof of Lemma 2.1 is omitted since it is easy.

Throughout the proof we will need the notion of so-called 0-*proper* functions in $C_+(X)$. Let $f \in C_+(X)$. If $f^{-1}(0) \neq X$ and $\text{Int} f^{-1}(0) \neq \emptyset$ then f is called 0-proper and we denote $\text{Int} f^{-1}(0) = Z_f$.

Lemma 2.2. *Let U be an open nonempty subset of X where $\overline{U} \neq X$. Then there exists $f \in C_+(X)$, $f \neq 0_X$, such that $f\left(\overline{U}\right) = \{0\}$. Furthermore, for every such f the function $\varphi(f)$ is 0-proper.*

Lemma 2.2 can be proved in the same way as the similar result in [5].

Lemma 2.3. *The functions $f_1, f_2, ..., f_n$ are 0-proper if and only if $\varphi(f_1)$, $\varphi(f_2), ..., \varphi(f_n)$ are 0-proper. Furthermore, in this case*

$$Z_{f_1} \cap Z_{f_2} \cap ... \cap Z_{f_n} \neq \emptyset \quad \textit{if and only if} \quad Z_{\varphi(f_1)} \cap Z_{\varphi(f_2)} \cap ... \cap Z_{\varphi(f_n)} \neq \emptyset.$$

Proof. Let $f_1, f_2, ..., f_n$ be 0-proper. Each function $\varphi(f_i)$, $i \in \{1,\ 2,\ ...,\ n\}$, is by Lemma 2.2 also 0-proper. Suppose $Z_{f_1} \cap Z_{f_2} \cap ... \cap Z_{f_n} \neq \emptyset$. There exists $a > 0$ such that $a \geq \max f_i$ for every $i \in \{1,\ 2,\ ...,\ n\}$. The finite intersection of open sets is an open set, so by Urysohn's lemma there exist a function $h \in C_+(X)$ and an open set $U \subset Z_{f_1} \cap Z_{f_2} \cap ... \cap Z_{f_n}$, where $h(x) = a$ for every $x \in (Z_{f_1} \cap Z_{f_2} \cap ... \cap Z_{f_n})^c$ and $U = Z_h$. This yields

$$f_i \leq h \qquad \text{for every } i = 1, 2, ..., n$$

and therefore

$$\varphi(f_i) \leq \varphi(h) \qquad \text{for every } i = 1, 2, ..., n. \tag{2.1}$$

Also, it follows by Lemma 2.2 that $\varphi(h)$ is 0-proper. From (2.1) we may conclude that

$$\emptyset \neq Z_{\varphi(h)} \subset Z_{\varphi(f_1)} \cap Z_{\varphi(f_2)} \cap ... \cap Z_{\varphi(f_n)}.$$

This implication is also true in the converse direction since φ^{-1} has the same properties as φ. ■

In the next step we will construct a homeomorphism $\omega : X \to X$.

From now on, let $|X| > 1$. We will use this assumption nearly to the end of the proof. For the point $x_0 \in X$ let A_{x_0}, $\overline{A_{x_0}} \neq X$, be an arbitrary open neighbourhood of $x_0 \in X$. By Urysohn's lemma there exists a 0-proper function f such that $x_0 \in Z_f$, $\overline{Z_f} \subset A_{x_0}$. Let $\mathcal{F}_{A_{x_0}}$ be the set all such 0-proper functions f. Then $x_0 \in \bigcap_{f \in \mathcal{F}_{A_{x_0}}} Z_f$. Let $x_1 \in X$, $x_1 \neq x_0$. Then there exist open sets A_1,

A_2 such that $A_1 \cap A_2 = \emptyset$ and $x_0 \in A_1$, $x_1 \in A_2$. Again by Urysohn's lemma there exists $f \in \mathcal{F}_{A_{x_0}}$ such that $\overline{Z_f} \subset A_1 \cap A_{x_0}$. So, $Z_f \cap A_2 = \emptyset$ and hence $x_1 \notin \bigcap_{f \in \mathcal{F}_{A_{x_0}}} Z_f$. This gives us

$$\bigcap_{f \in \mathcal{F}_{A_{x_0}}} Z_f = \{x_0\}.$$

Let $f \in \mathcal{F}_{A_{x_0}}$. By Lemma 2.2 is then $\varphi(f)$ also 0-proper. We will next show that there exists a point $x_1 \in X$ such that $\bigcap_{f \in \mathcal{F}_{A_{x_0}}} Z_{\varphi(f)} = \{x_1\}$.

Let us first assume that $\bigcap_{f \in \mathcal{F}_{A_{x_0}}} \overline{Z_{\varphi(f)}} = \emptyset$. It is easy to see, since X is a compact space, that

$$\bigcap_{f \in \mathcal{F}_{A_{x_0}}} \overline{Z_{\varphi(f)}} \neq \emptyset.$$

Let us next assume that $\bigcap_{f \in \mathcal{F}_{A_{x_0}}} Z_{\varphi(f)} = \emptyset$. Then there exist

$$x_\lambda \in \bigcap_{f \in \mathcal{F}_{A_{x_0}}} \overline{Z_{\varphi(f)}}$$

and $f_\lambda \in \mathcal{F}_{A_{x_0}}$ such that $x_\lambda \in \overline{Z_{\varphi(f_\lambda)}}$ and $x_\lambda \notin Z_{\varphi(f_\lambda)}$. Since φ preserves the order $\leq$ and is surjective there exists $a > 0$ such that $\varphi(a_X)(x) > 0$ for every $x \in X$. From now on, let a_X be a function with such property. By Urysohn's lemma there exist a 0-proper function h and a nonempty open set U_h such that $x_0 \in U_h \bigcap \overline{Z_h}^c$, $Z_{f_\lambda}^c \subset Z_h$ and $h(x) = a$ for every $x \in \overline{U_h}$. It follows that $\varphi(h)$ is also 0 -proper. Since $hf_\lambda = 0_X$ we obtain by Lemma 2.1

$$\varphi(h)\varphi(f_\lambda) = 0_X.$$

If $\varphi(h)(x) \neq 0$ then $\varphi(f_\lambda)(x) = 0$, $x \in X$. Let us assume that $x_\lambda \notin \overline{Z_{\varphi(h)}}$. Then there exists by the normality of X an open set A_λ such that $x_\lambda \in A_\lambda$ and $\overline{A_\lambda} \bigcap \overline{Z_{\varphi(h)}} = \emptyset$. We may conclude, since $x_\lambda \in \overline{Z_{\varphi(f_\lambda)}} \backslash Z_{\varphi(f_\lambda)}$, that $A_\lambda \cap \overline{Z_{\varphi(f_\lambda)}}^c \neq \emptyset$. So, there exists $x_a \in A_\lambda \bigcap \overline{Z_{\varphi(f_\lambda)}}^c \subset \overline{Z_{\varphi(h)}}^c$. Without loss of generality we may assume that $\varphi(h)(x_a) \neq 0$. Also, since $\varphi(h)$ is continuous there exists an open neighbourhood A_{x_a} of x_a where $\varphi(h)(x) \neq 0$ for all $x \in A_{x_a}$. Again, by the normality of X there exists an open neighbourhood A_1 of x_a where $\overline{A_1} \bigcap \overline{Z_{\varphi(f_\lambda)}} = \emptyset$. Then $\varphi(h)(x) \neq 0$ for all $x \in A_1 \cap A_{x_a}$ and therefore

$$\varphi(f_\lambda)(x) = 0 \quad \text{for all} \quad x \in A_1 \cap A_{x_a}.$$

But $A_1 \cap A_{x_a} \subset \overline{Z_{\varphi(f_\lambda)}}^c$, a contradiction since $A_1 \cap A_{x_a}$ is a nonempty open set and $Z_{\varphi(f_\lambda)} = \text{Int}\varphi(f_\lambda)^{-1}(0)$. So, our assumption was wrong and therefore

$$x_\lambda \in \overline{Z_{\varphi(h)}}.$$

There exists a function $f_\mu \in \mathcal{F}_{A_{x_0}}$ where $\overline{Z_{f_\mu}} \subset U_h$ and $f_\mu(U_h^c) = \{a\}$. For $f \in C_+(X)$ such that $h \leq f$ and $f_\mu \leq f$ it follows that $\varphi(f)(x) \neq 0$ for every $x \in X$. This may be derived from the fact that $\max\{h, f_\mu\} = a_X$ and $\varphi(a_X)(x) > 0$ for every $x \in X$. So, $\max\{\varphi(h), \varphi(f_\mu)\}(x) \neq 0$ for every $x \in X$. We may then conclude that $\overline{Z_{\varphi(f_\mu)}} \subset \overline{Z_{\varphi(h)}}^c$. So, $x_\lambda \notin \overline{Z_{\varphi(f_\mu)}}$. A contradiction, since $f_\mu \in \mathcal{F}_{A_{x_0}}$ and $x_\lambda \in \bigcap_{f\in\mathcal{F}_{A_{x_0}}} \overline{Z_{\varphi(f)}}$.

We have proved that

$$\bigcap_{f\in\mathcal{F}_{A_{x_0}}} Z_{\varphi(f)} \neq \emptyset.$$

Let us now assume that there exist $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $\{x_1, x_2\} \subset \bigcap_{f\in\mathcal{F}_{A_{x_0}}} Z_{\varphi(f)}$. Denote: $b = \max\varphi(a_X)$. Let V' and V'' be disjoint open neighbourhoods of the points x_1 and x_2 respectively. There exists by Urysohn's lemma and the surjectivity of φ a 0-proper function $\varphi(h_1)$ where $\overline{Z_{\varphi(h_1)}} \subset V'$, $x_1 \in Z_{\varphi(h_1)}$ and $\varphi(h_1)(V'^c) = \{b\}$. Similarly, there exists 0-proper a function $\varphi(h_2)$ where $\overline{Z_{\varphi(h_2)}} \subset V''$, $x_2 \in Z_{\varphi(h_2)}$ and $\varphi(h_2)(V''^c) = \{b\}$. It follows that $\max\{\varphi(h_1), \varphi(h_2)\} \geq b_X \geq \varphi(a_X)$ and therefore $\max\{h_1, h_2\} \geq a_X$. So, since $a > 0$, we get $\overline{Z_{h_1}} \bigcap \overline{Z_{h_2}} = \emptyset$. Without loss of generality we may assume that $x_0 \notin \overline{Z_{h_2}}$. By Urysohn's lemma we may find a function $h_3 \in \mathcal{F}_{A_{x_0}}$ such that $Z_{h_3} \bigcap Z_{h_2} = \emptyset$. Then on the one hand we establish that

$$x_2 \in Z_{\varphi(h_3)} \bigcap Z_{\varphi(h_2)}$$

and on the other hand we obtain by Lemma 2.3

$$Z_{\varphi(h_3)} \bigcap Z_{\varphi(h_2)} = \emptyset,$$

which is a contradiction. Therefore

$$\bigcap_{f\in\mathcal{F}_{A_{x_0}}} Z_{\varphi(f)} = \{x_1\}.$$

This intersection is clearly independent of the selection of an open neighbourhood A_{x_0} of the point x_0.

Let now $\psi : X \to X$ be the function which $x_0 \mapsto x_1$. It is easy to prove that ψ is then a homeomorphism. Let us denote the homeomorphism

$$\omega = \psi^{-1}.$$

We will now prove another auxiliary result. We will show that the order $\leq$ is valid also locally.

Lemma 2.4. *Let* $f, g \in C_+(X)$ *such that* $f(\omega(x_1)) < g(\omega(x_1))$, $x_1 \in X$. *Then* $\varphi(f)(x_1) \leq \varphi(g)(x_1)$.

Proof. By the continuity of the functions f and g there exists an open neighbourhood U of $\omega(x_1)$ where $f(x) < g(x)$ for all $x \in U$. Let us assume that $\varphi(g)(x_1) < \varphi(f)(x_1)$. Then by the continuity of $\varphi(f)$ and $\varphi(g)$ there exists an open neighbourhood V of x_1 where $\varphi(g)(x) < \varphi(f)(x)$ for all $x \in V$. By Urysohn's lemma and the surjectivity of φ there exists a 0-proper function $\varphi(h_1)$ where $x_1 \in Z_{\varphi(h_1)} \subset V$. By Lemma 2.2 and since φ^{-1} has the same properties as φ, h_1 is also 0-proper. Note that $\omega(x_1) \in Z_{h_1} \bigcap U$.

The set $\overline{Z_{h_1}}$ is generally not necessarily a subset of U. We may find in any case by Urysohn's lemma a 0-proper function h_1^a such that $h_1^a(U^c) = 1$ and $\omega(x_1) \in Z_{h_1^a}$. Let $h_1^b = \max\{h_1, h_1^a\}$. We notice that $Z_{h_1^b} = Z_{h_1} \bigcap Z_{h_1^a}$, $\omega(x_1) \in Z_{h_1} \bigcap Z_{h_1^a}$ and $\overline{Z_{h_1} \bigcap Z_{h_1^a}} \subset U$. By Lemma 2.2, $\varphi\left(h_2^b\right)$ is 0-proper and $x_1 \in Z_{\varphi(h_1^b)}$. Since $h_1 \leq h_1^b$ we obtain $\varphi(h_1) \leq \varphi\left(h_1^b\right)$ and therefore $Z_{\varphi(h_1^b)} \subset Z_{\varphi(h_1)} \subset V$. So, without loss of generality we may assume that the closure of Z_{h_1} is a subset of U.

Denote: $c_1 = \sup \varphi(f)$, $c_2 = \sup \varphi(g)$ and $d = \max\{c_1, c_2\}$. Again, by Urysohn's lemma and the surjectivity of φ there exist a 0-proper function $\varphi(h_2)$ and an open set V_2 such that $x_1 \in V_2$, $V_2 \subset Z_{\varphi(h_1)}$, $Z^c_{\varphi(h_1)} \subset Z_{\varphi(h_2)}$ and $\varphi(h_2)(x) = d$ for every $x \in V_2$. By the surjectivity of φ there exists the function h_3 such that $\varphi(h_3) = \min\{\varphi(f), \varphi(h_2)\}$. It follows that $\varphi(h_3)$ is 0-proper and $Z_{\varphi(h_3)} = Z_{\varphi(h_2)}$. Also, $\varphi(h_3) \leq \varphi(f)$ and $\varphi(h_3) \nleq \varphi(g)$ and therefore

$$h_3 \leq f \qquad \text{and} \qquad h_3 \nleq g.$$

By Lemma 2.2 it follows that h_3 is a 0-proper function. Also, $\varphi(h_1)\varphi(h_3) = 0_X$ and therefore by Lemma 2.1 $h_1 h_3 = 0_X$. It is easy to see that then $\overline{Z_{h_1}} \bigcup \overline{Z_{h_3}} = X$. So, $h_3(x) \neq 0$ only if $x \in \overline{Z_{h_1}}$. But $\overline{Z_{h_1}} \subset U$ and therefore since $h_3 \leq f$ we obtain

$$h_3 \leq g,$$

a contradiction. So, our assumption was wrong and therefore $\varphi(f)(x_1) \leq \varphi(g)(x_1)$. ■

Let $x \in X$ and let $m_x : [0, \infty) \to [0, \infty)$ be the function defined in the following way:

$$m_x(c) = \varphi(c)(x).$$

By using Lemmas 2.2 and 2.4 and Tietze theorem we prove similarly as in [5] that m_x is a bijective and increasing map. Observe that then m_x, $x \in X$, is also continuous. Using this fact we may prove (see [5] for details) that

$$\varphi(f)(x) = m_x\left(f(\omega(x))\right)$$

for every $f \in C_+(X)$. Let $|X| = 1$ and $\varphi(f)(x) = m(f)$ where $m : [0, \infty) \to [0, \infty)$. Then m is a bijective and increasing function.

The second part of the theorem can be again proved in a similar way as in [5]. ■

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Janko Marovt
EPF-University of Maribor,
Razlagova 14,
2000 Maribor,
Slovenia
E-mail: janko.marovt@uni-mb.si