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EXISTENCE OF THREE POSITIVE SOLUTIONS FOR M-POINT BOUNDARY-VALUE PROBLEM WITH ONE-DIMENSIONAL P-LAPLACIAN

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Abstract. In this paper, we consider the multipoint boundary value problem for the one-dimensional p-Laplacian

$$(\phi_p(u'(t)))' + q(t)f(t, u(t), u'(t)) = 0, \qquad t \in (0,1),$$

subject to the boundary value conditions:

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ \xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $a_i, b_i \in [0,1),\ 0 \le \sum_{i=1}^{m-2} a_i < 1,\ 0 \le \sum_{i=1}^{m-2} b_i < 1$. Using a fixed point theorem due to Avery and Peterson, we study the existence of at least three positive solutions to the above boundary value problem. The interesting point is the nonlinear term f is involved with the first-order derivative explicitly.

1. Introduction

In this paper, we study the existence of multiple positive solutions to the boundary value problem (BVP for short) for the one-dimensional p-Laplacian

$$(1.1) \qquad \big(\phi_p(u'(t))\big)' + q(t)f\big(t, u(t), u'(t)\big) = 0, \qquad t \in (0,1),$$

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$$u(0)=\sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u(1)=\sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

where $\phi_p(s)=|s|^{p-2}s,\ p>1,\ \xi_i\in(0,1)$ with $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$ and $a_i,\ b_i,\ q,\ f$ satisfy

(H_1) $a_i, b_i \in [0,1)$ satisfy $0\le \sum\limits_{i=1}^{m-2} a_i<1, 0\le \sum\limits_{i=1}^{m-2} b_i<1$;
(H_2) $f\in C([0,1]\times[0,+\infty)\times R,(0,+\infty))$;
(H_3) $q(t)\in L^1[0,1]$ is nonnegative on $(0,1)$ and $q(t)\not\equiv 0$ on any subinterval of $(0,1)$.

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1]. They investigated a mathematical model leading to the solution of a Sturm-Liouville equation, with nonlocal boundary conditions of the second kind expressed in terms of given linear combinations of flows. Since then there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see [2-9].

For problem (1.1), when the nonlinear term f does not depend on the first-order derivative, many authors studied the equation

$$\left(\phi_p(u'(t))\right)'+q(t)f\left(t,u(t)\right)=0, \qquad t\in(0,1), \tag{1.3}$$

with different boundary conditions. For example, in [5], Ma and Ge studied the existence of positive solutions for the multipoint BVP

$$\left(\phi_p(u'(t))\right)'+q(t)f\left(t,u(t)\right)=0, \qquad t\in(0,1),$$
$$u'(0)=\sum_{i=1}^{m-2}\alpha_i u'(\xi_i), \qquad u(1)=\sum_{i=1}^{m-2}\beta_i u(\xi_i).$$

The main tool is the monotone iterative technique.

In [6,7], Wang and Ge considered multipoint BVPs for the one-dimensional p-Laplcian successively

$$(\phi_p(u'))'+f(t,u)=0, \qquad t\in(0,1),$$
$$\phi_p(u'(0))=\sum_{i=1}^{n-2}a_i\phi_p(u'(\xi_i)), \qquad u(1)=\sum_{i=1}^{n-2}a_i u(\xi_i)$$

and

$$(\phi_p(u'))'+f(t,u)=0, \qquad t\in(0,1),$$
$$u'(0)=\sum_{i=1}^{n-2}\alpha_i u'(\xi_i), \qquad u(1)=\sum_{i=1}^{n-2}\beta_i u(\xi_i).$$

They provided sufficient conditions for the existence of multiple positive solutions to the above BVPs by applying the fixed point theorem in a cone.

However, all the above works about positive solutions were done under the assumption that the first order derivative is not involved explicitly in the nonlinear term. Recently, Bai, Gui and Ge in [8] investigated the following two-point BVPs

$$\big(\phi_p(u'(t))\big)' + q(t)f\left(t, u(t), u'(t)\right) = 0, \qquad t \in (0,1),$$

$$\alpha\phi_p(u(0)) - \beta\phi_p(u'(0)) = 0, \qquad \gamma\phi_p(u(1)) + \delta\phi_p(u'(1)) = 0$$

or

$$u(0) - g_1(u'(0)) = 0, \qquad u(1) + g_2(u'(1)) = 0.$$

Where $g_1(t)$ and $g_2(t)$ are both continuous functions defined on $(-\infty, +\infty), \alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0$.

Wang and Ge in [9] considered the following two-point BVPs

$$\big(\phi_p(u'(t))\big)' + q(t)f\left(t, u(t), u'(t)\right) = 0, \qquad t \in (0,1),$$

$$u(0) = 0, \qquad u(1) = 0,$$

or

$$u(0) = 0, \qquad u'(1) = 0.$$

The authors obtained sufficient conditions that guarantee the existence of at least three positive solutions by using fixed point theorems due to Avery-Peterson.

Motivated by works in [8,9], our purpose of this paper is to show the existence of at least three positive solutions to the multipoint BVP $(1.1), (1.2)$.

2. Preliminary

For the convenience of readers, we provide some background material from the theory of cones in Banach spaces. We also state in this section the Avery-Peterson's fixed point theorem.

Definition 2.1. Let E be a real Banach space over R. A nonempty closed set $P \subset E$ is said to be a cone provide that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0, b \geq 0$, and

(ii) $u, \ -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map γ is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\gamma : P \to [0, \infty)$ is continuous and

$$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on a cone P, α be a nonnegative continuous concave functional on a cone P, and ψ be a nonnegative continuous functional on a cone P. Then for positive real numbers a, b, c and d, we define the following convex sets:

$$\begin{aligned} &P(\gamma, d) = \{u \in P \mid \gamma(u) < d\}, \\ &P(\gamma, \alpha, b, d) = \{u \in P \mid b \leq \alpha(u), \gamma(u) \leq d\}, \\ &P(\gamma, \theta, \alpha, b, c, d) = \{u \in P \mid b \leq \alpha(u), \theta(u) \leq c, \gamma(u) \leq d\} \end{aligned}$$

and a closed set

$$R(\gamma, \psi, a, d) = \{u \in P \mid a \leq \psi(u), \gamma(u) \leq d\}.$$

To prove our results, we need the following fixed point theorem due to Avery and Peterson in [10].

Theorem 2.1. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda u) \leq \lambda \psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers $\overline{M}$ and d,*

$$\alpha(u) \leq \psi(u) \ \ \text{and} \ \ \|u\| \leq \overline{M}\gamma(u) \tag{2.1}$$

for all $u \in \overline{P(\gamma, d)}$. Suppose

$$T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$$

is completely continuous and there exist positive numbers a, b and c with $a < b$ such that

$(S1)$ $\{u \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(u) > b\} \neq \phi$ *and* $\alpha(Tu) > b$ *for* $u \in P(\gamma, \theta, \alpha, b, c, d)$*;*

$(S2)$ $\alpha(Tu) > b$ *for* $u \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tu) > c$*;*

$(S3)$ $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tu) < a$ *for* $u \in R(\gamma, \psi, a, d)$ *with* $\psi(u) = a$*.*

Then T *has at least three fixed points* $u_1, u_2, u_3 \in \overline{P(\gamma, d)}$*, such that*

$$\begin{aligned}
&\gamma(u_i) \le d \quad for \quad i = 1, 2, 3,\\
&b < \alpha(u_1),\\
&a < \psi(u_2), \quad with \quad \alpha(u_2) < b,\\
&\psi(u_3) < a.
\end{aligned}$$

3. Related Lemmas

Let the Banach space $E = C^1[0, 1]$ be endowed with the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0, 1]$ and the maximum norm

$$\|u\| = \max\left\{\max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)|\right\}.$$

Define the cone $P \subset E$ by

$$P = \left\{u \in E \mid u(t) \ge 0,\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u \text{ is concave on } [0, 1]\right\}.$$

It follows from (H_3) that there exists a natural number $k > \max\left\{\frac{1}{\xi_1}, \frac{2}{1 - \xi_{m-2}}\right\}$ such that $0 < \int_{\frac{1}{k}}^{1-\frac{1}{k}} q(t)dt < \infty$.

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\begin{aligned}
&\gamma(u) = \max_{0\le t\le 1} |u'(t)|, \psi(u) = \theta(u) = \max_{0\le t\le 1} |u(t)|,\\
&\alpha(u) = \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}} |u(t)| \quad for\ u \in P.
\end{aligned}$$

Lemma 3.1. *For* $u \in P$*, then*

$$\max_{0\le t\le 1} |u(t)| \le \overline{M} \max_{0\le t\le 1} |u'(t)|,$$

where $\overline{M} = 1 + \frac{\sum\limits_{i=1}^{m-2} a_i\xi_i}{1 - \sum\limits_{i=1}^{m-2} a_i}$.

Proof. By the concavity of u, one arrives at

$$u(t) - u(0) \leq u'(0) \leq \max_{0\leq t\leq 1} |u'(t)|, \quad t \in [0,1].$$

On the other hand,

$$\begin{aligned}\left(1 - \sum_{i=1}^{m-2} a_i\right) u(0) &= u(0) - \sum_{i=1}^{m-2} a_i u(0) \\ &= \sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i u(0) \\ &= \sum_{i=1}^{m-2} a_i [u(\xi_i) - u(0)] \\ &= \sum_{i=1}^{m-2} a_i \xi_i u'(\eta_i),\end{aligned}$$

where $\eta_i \in (0, \xi_i)$. so

$$|u(0)| = \left|\frac{\sum\limits_{i=1}^{m-2} a_i\xi_i u'(\eta_i)}{1 - \sum\limits_{i=1}^{m-2} a_i}\right| \leq \frac{\sum\limits_{i=1}^{m-2} a_i\xi_i |u'(\eta_i)|}{1 - \sum\limits_{i=1}^{m-2} a_i} \leq \frac{\sum\limits_{i=1}^{m-2} a_i\xi_i}{1 - \sum\limits_{i=1}^{m-2} a_i} \max_{0\leq t\leq 1} |u'(t)|.$$

Thus one has

$$\max_{0\leq t\leq 1} |u(t)| \leq \left(1 + \frac{\sum\limits_{i=1}^{m-2} a_i\xi_i}{1 - \sum\limits_{i=1}^{m-2} a_i}\right) \max_{0\leq t\leq 1} |u'(t)| = \overline{M} \max_{0\leq t\leq 1} |u'(t)|.$$

Lemma 3.2. *For* $u \in P$, *then*

$$\min_{0\leq t\leq 1} |u(t)| \geq \frac{1}{k} \max_{0\leq t\leq 1} |u(t)|.$$

Proof. It is well known that if u is nonnegative and concave on $[0,1]$ then

$$u(t) \geq \min\{t, 1-t\} \max_{0\leq t\leq 1} |u(t)|.$$

Thus we have

$$\min_{0\leq t\leq 1} |u(t)| \geq \frac{1}{k} \max_{0\leq t\leq 1} |u(t)|.$$

With Lemma 3.1 and Lemma 3.2, for all $u \in P$, we obtain

$$\frac{1}{k}\theta(u) \leq \alpha(u) \leq \theta(u) = \psi(u), \quad \|u\| = max\{\theta(u), \gamma(u)\} \leq \overline{M}\gamma(u). \tag{3.1}$$

Therefore the condition (2.1) of Theorem 2.1 is satisfied.

Lemma 3.3. *Assume that $(H_1) - (H_3)$ hold. Then, for any $x \in C^{1+}[1,0] = \{x \in C^1[0,1] : x(t) \geq 0\}$,*

$$\big(\phi_p(u'(t))\big)' + q(t)f(t, x(t), x'(t)) = 0, \qquad t \in (0,1), \tag{3.2}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{3.3}$$

has a unique solution $u(t)$ as

$$\begin{aligned} u(t) = &\frac{1}{1-\sum\limits_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\left(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right) ds \\ &+ \int_0^t \phi_p^{-1}\left(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right) ds \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} u(t) = &-\frac{1}{1-\sum\limits_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^{1} \phi_p^{-1}\left(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right) ds \\ &- \int_t^1 \phi_p^{-1}\left(A_x - \int_0^s q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right) ds, \end{aligned} \tag{3.5}$$

where A_x satisfies

$$
\begin{aligned}
&\frac{1-\sum\limits_{i=1}^{m-2} b_i}{1-\sum\limits_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds\\
&+\int_0^1 \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds\\
&-\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds=0.
\end{aligned}
\tag{3.6}
$$

Proof. For any $x \in C^{1+}[1,0]$, suppose u is a solution of BVP (3.2), (3.3). By integration of (3.1), it follows that

$$
\begin{aligned}
u'(t) &= \phi_p^{-1}\left(A_x-\int_0^t q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right),\\
u(t) &= u(0)+\int_0^t \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds
\end{aligned}
$$

or

$$
u(t) = u(1)-\int_t^1 \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds.
$$

Using the boundary condition (3.3), we can easily have

$$
\begin{aligned}
u(t) =& \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds\\
&+\int_0^t \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds
\end{aligned}
$$

or

$$
\begin{aligned}
u(t) =& -\frac{1}{1-\sum\limits_{i=1}^{m-2} b_i}\sum_{i=1}^{m-2} b_i \int_{\xi_i}^{1} \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds\\
&-\int_t^1 \phi_p^{-1}\left(A_x-\int_0^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds,
\end{aligned}
$$

where A_x satisfies (3.6).

On the other hand, it is easy to verify that if u is a solution of (3.4) or (3.5), then u is one of (3.2), (3.3).

Lemma 3.4. *Assume that* $(H_1) - (H_3)$ *hold. For any* $x \in C^{1+}[0,1]$, *there exists a unique* $A_x \in (-\infty, +\infty)$ *satisfying* (3.6). *Moreover, there is a unique* $\sigma_x \in (0,1)$ *such that* $A_x = \int_0^{\sigma_x} q(\tau) f(\tau, x(\tau)) d\tau$.

Proof. For any $x \in C^{1+}[0,1]$, define

$$\begin{aligned} H_x(c) = & \frac{1 - \sum\limits_{i=1}^{m-2} b_i}{1 - \sum\limits_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1} \left(c - \int_0^s q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right) ds \\ & + \int_0^1 \phi_p^{-1} \left(c - \int_0^s q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right) ds \\ & - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_p^{-1} \left(c - \int_0^s q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right) ds. \end{aligned}$$

It is easy to see that

$$\begin{aligned} H_x(c) = & \frac{1 - \sum\limits_{i=1}^{m-2} b_i}{1 - \sum\limits_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1} \left(c - \int_0^s q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right) ds \\ & + (1 - \sum_{i=1}^{m-2} b_i) \int_0^1 \phi_p^{-1} \left(c - \int_0^s q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right) ds \\ & + \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 \phi_p^{-1} \left(c - \int_0^s q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right) ds. \end{aligned}$$

So $H_x : R \to R$ is continuous and strictly increasing. One has

$$H_x(0) < 0, \qquad H_x \left(\int_0^1 q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right) > 0.$$

Therefore there exists a unique $A_x \in \left(0,\ \int_0^1 q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right) \subset (-\infty, +\infty)$ satisfying (3.6). Furthermore let

$$F(t) = \int_0^t q(\tau) f(\tau, x(\tau), x'(\tau)) d\tau,$$

then $F(t)$ is continuous and strictly increasing on $[0,1]$ and $F(0)=0,\ F(1)=\int_0^1 q(\tau)f(\tau,x(\tau),x'(\tau))d\tau$. So,

$$0=F(0)<A_x<F(1)=\int_0^1 q(\tau)f(\tau,x(\tau),x'(\tau))d\tau.$$

Thus the intermediate value theorem guarantees that there exists a unique $\sigma_x\in(0,1)$ such that $A_x=\int_0^{\sigma_x} q(\tau)f(\tau,x(\tau),x'(\tau))d\tau$.

Lemma 3.5. *Assume that* $(H_1)-(H_3)$ *hold. For any* $x\in C^{1+}[0,1]$, *then the unique solution* $u(t)$ *of* $(3.2),(3.3)$ *has the following properties:*

(i) *the graph of* $u(t)$ *is concave*;

(ii) $u(t)\geq 0$;

(iii) *there exists a unique* $t_0\in(0,1)$ *such that* $u(t_0)=\max\limits_{0\leq t\leq 1}u(t)$, *moreover* $u'(t_0)=0$;

(iv) $\sigma_x=t_0$.

Proof. Suppose that $u(t)$ is the solution of (3.2), (3.3). Then

(i) From the fact that $(\phi_p(u'(t)))'(t)=-q(t)f(t,x(t),x'(t))\leq 0$, we know that $\phi_p(u'(t))$ is nonincreasing. It follows that $u'(t)$ is also nonincreasing. Thus we know that the graph of $u(t)$ is concave down on $(0,1)$.

(ii) Without loss of generality, we assume that $u(0)=\min\{u(0),u(1)\}$. By the concavity of u, we know that $u(\xi_i)\geq u(0)$ $(i=0,1,2,\dots,n)$. So we get

$$u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i)\geq\sum_{i=1}^{m-2}a_iu(0),$$

by $1-\sum\limits_{i=1}^{m-2}a_i>0$, it is obvious that $u(0)\geq 0$. Hence $u(1)\geq u(0)\geq 0$. So from the concavity of u, we know that $u(t)\geq 0$, $t\in[0,1]$.

(iii) From $u(0)-\sum\limits_{i=1}^{m-2}a_iu(\xi_i)=0\leq(1-\sum\limits_{i=1}^{m-2}a_i)u(0)$, one has

$$\sum_{i=1}^{m-2}a_iu(0)\leq\sum_{i=1}^{m-2}a_iu(\xi_i).$$

It follows that there exists ξ_{i_1} such that $u(\xi_{i_1}) \geq u(0)$. On the other hand, from $u(1) - \sum_{i=1}^{m-2} b_i u(\xi_i) = 0 \leq (1 - \sum_{i=1}^{m-2} b_i)u(1)$, one arrives at

$$\sum_{i=1}^{m-2} b_i u(1) \leq \sum_{i=1}^{m-2} b_i u(\xi_i).$$

This implies that there exists ξ_{i_2} such that $u(\xi_{i_2}) \geq u(1)$. So there exists $t_0 \in (0,1)$ such that $u(t_0) = \max_{0\leq t\leq 1} u(t)$. Furthermore, $u'(t_0) = 0$.

If there exist $t_1, t_2 \in (0,1),\ t_1 < t_2$ such that $u'(t_1) = 0 = u'(t_2)$, then

$$0 = \phi_p(u'(t_1)) - \phi_p(u'(t_2)) = -\left(\int_{t_1}^{t_2} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right) < 0,$$

which is a contradiction.

(iv) By Lemma 3.3 and Lemma 3.4, it is easy to check $u'(t) = \phi_p^{-1}\Big(\int_t^{\sigma_x} q(\tau) f(\tau, x(\tau), x'(t))d\tau\Big)$. Hence we obtain that $u'(\sigma_x) = u'(t_0) = 0$, this implies $\sigma_x = t_0$.

Lemma 3.6. *For any $u \in P$, define the operator*

$$(Tu)(t) = \begin{cases} \dfrac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\left(\int_s^{\sigma_u} q(\tau)f(\tau, u(\tau), u'(\tau))d\tau\right) ds \\ \quad + \int_0^t \phi_p^{-1}\left(\int_s^{\sigma_u} q(\tau)f(\tau, u(\tau), u'(\tau))d\tau\right) ds, \quad 0 \leq t \leq \sigma_u, \\ \dfrac{1}{1-\sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_{\sigma_u}^{s} q(\tau)f(\tau, u(\tau), u'(\tau))d\tau\right) ds \\ \quad + \int_t^1 \phi_p^{-1}\left(\int_{\sigma_u}^{s} q(\tau)f(\tau, u(\tau), u'(\tau))d\tau\right) ds, \quad \sigma_u \leq t \leq 1. \end{cases} \tag{3.7}$$

Then $T: P \longrightarrow P$ is completely continuous.

Proof. According to the definition of T and Lemma 3.5, it is easy to know that $TP \subset P$. By similar arguments in [11,12], $T : P \to P$ is completely continuous.

4. Existence of Triple Positive Solutions to BVP (1.1), (1.2)

We are now ready to apply the Avery-Peterson's fixed point theorem to the operator T to give sufficient conditions for the existence of at least three positive solutions to BVP (1.1), (1.2).

Now for convenience we introduce following notations. Let

$$t_i^* = \frac{\xi_i + \xi_{i+1}}{2}, \qquad (i = 0, 1, \ldots, m-2, \text{ denote } \xi_0 = 1/k,\ \xi_{m-1} = 1 - 1/k \text{ here}),$$

$$L = \phi_p^{-1}\left(\int_0^1 q(\tau)d\tau\right),$$

$$M_i = \min\left\{\int_{\xi_i}^{t_i^*} \phi_p^{-1}\left(\int_s^{t_i^*} q(\tau)d\tau\right) ds,\ \int_{t_i^*}^{\xi_{i+1}} \phi_p^{-1}\left(\int_{t_i^*}^{s} q(\tau)d\tau\right) ds\right\},$$

$$M = \min_{0\le i\le m-2}\{M_i\},$$

$$N_i = \max\left\{\frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i} \phi_p^{-1}\left(\int_s^1 q(\tau)d\tau\right) ds + \int_0^{t_i^*} \phi_p^{-1}\left(\int_s^{t_i^*} q(\tau)d\tau\right) ds,\right.$$
$$\left.\frac{\sum_{i=1}^{m-2} b_i}{1-\sum_{i=1}^{m-2} b_i}\int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right) ds + \int_{t_i^*}^1 \phi_p^{-1}\left(\int_{t_i^*}^s q(\tau)d\tau\right) ds\right\},$$

$$N = \max_{0\le i\le m-2}\{N_i\},$$

$$K = \frac{k}{2}\left(1 + \frac{1-\sum_{i=1}^{m-2} a_i}{\sum_{i=1}^{m-2} a_i\xi_i}\right),$$

where $0 < 1/k = \xi_0 < \xi_1 < \ldots < \xi_{m-2} < \xi_{m-1} = 1 - 1/k < 1$.

Remark 4.1. Instead of $t_i^* = \dfrac{\xi_i + \xi_{i+1}}{2}$, we can take any $t_i^* \in (\xi_i, \xi_{i+1})$.

Theorem 2.2. *Assume* $(H_1)-(H_3)$ *hold. Let* $0 < a < b \le 2(\overline{M}-1)\frac{d}{k}$, *and suppose that* f *satisfies the following conditions:*
$(A1)$ $f(t,u,v) \le \phi_p(d/L),\ for\ (t,u,v) \in [0,1]\times[0,\overline{M}d]\times[-d,d];$
$(A2)$ $f(t,u,v) > \phi_p(kb/M),\ for\ (t,u,v) \in [1/k,1-1/k]\times[b,Kb]\times[-d,d];$
$(A3)$ $f(t,u,v) < \phi_p(a/N),\ for\ (t,u,v) \in [0,1]\times[0,a]\times[-d,d].$

Then BVP $(1.1),\ (1.2)$ *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\begin{aligned}
&\max_{0\le t\le 1}|u_i'(t)| \le d \quad for \quad i=1,2,3,\\
&b < \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}}|u_1(t)|, \quad \max_{0\le t\le 1}|u_1(t)| \le \overline{M}d,\\
&a < \max_{0\le t\le 1}|u_2(t)| < Kb, \quad with \quad \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}}|u_2(t)| < b,\\
&\max_{0\le t\le 1}|u_3(t)| < a.
\end{aligned}$$

Proof. BVP (1.1), (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$. Thus we set out to verify that the operator T satisfies the Avery-Peterson's fixed point theorem which will prove the existence of three fixed points of T which satisfy the conclusion of the theorem. Now the proof is divided into some steps.

(1) We will show that (A_1) implies that

$$T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}. \tag{4.1}$$

In fact, for $u \in \overline{P(\gamma,d)}$, there is $\gamma(u) = \max_{0\le t\le 1}|u'(t)| \le d$. With Lemma 3.1, there is $\max_{0\le t\le 1}|u(t)| \le \overline{M}d$, then condition (A_1) implies $f\,(t,u(t),u'(t)) \le \phi_p(d/L)$. On the other hand, for $u \in P$, there is $Tu \in P$, then Tu is concave on $[0,1]$, and $\max_{0\le t\le 1}|(Tu)'(t)| = \max\{(Tu)'(0),\ -(Tu)'(1)\}$, so

$$\begin{aligned}
\gamma(Tu) &= \max_{0\le t\le 1}|(Tu)'(t)| = \max\{(Tu)'(0), -(Tu)'(1)\}\\
&= \max\left\{\phi_p^{-1}\left(\int_0^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right),\right.\\
&\qquad \left.\phi_p^{-1}\left(\int_{\sigma_u}^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)\right\}\\
&\le \phi_p^{-1}\left(\int_0^1 q(\tau)d\tau\ \phi_p(d/L)\right)\\
&= \frac{d}{L}\phi_p^{-1}\left(\int_0^1 q(\tau)d\tau\right) = \frac{d}{L}L = d.
\end{aligned}$$

Thus (4.1) holds.

(2) We show that the condition (S_1) in Theorem 2.1 holds.

We take

$$u_0(t) = \frac{kb\left(1-\sum\limits_{i=1}^{m-2} a_i\right)}{2\sum\limits_{i=1}^{m-2} a_i\xi_i} t + \frac{kb}{2} = \frac{kb}{2(\overline{M}-1)} t + \frac{kb}{2},$$

it is easy to see that $u_0(t) \in P(\gamma,\theta,\alpha,b,Kb,d)$ and $\alpha(u_0) = \min\limits_{\frac{1}{k}\le t\le 1-\frac{1}{k}} |u_0(t)|$ $= u_0(\frac{1}{k}) > \frac{kb}{2} > b$, So $\{u \in P(\gamma,\theta,\alpha,b,Kb,d) \mid \alpha(u) > b\} \neq \phi$. Thus for $u \in P(\gamma,\theta,\alpha,b,Kb,d)$, there is $b \le u(t) \le Kb$, $|u'(t)| \le d$, for $1/k \le t \le 1-1/k$. Hence by condition (A_2) of this theorem, one has $f(t,u(t),u'(t)) > \phi_p(kb/M)$, for $t \in [1/k, 1-1/k]$. Noticing $(Tu)(0) \ge 0$ and $(Tu)(1) \ge 0$ from Lemma 3.5 and combining the conditions of α and P, one arrives at

$$\begin{aligned}
\alpha(Tu) &= \min_{1/k\le t\le 1-1/k} |(Tu)(t)| \\
&\ge \frac{1}{k}\max_{0\le t\le 1} |(Tu)(t)| = \frac{1}{k}(Tu)(\sigma_u) \\
&= \frac{1}{k}\left(\frac{1}{1-\sum\limits_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1}\left(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds\right. \\
&\quad \left.+ \int_0^{\sigma_u} \phi_p^{-1}\left(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds\right) \\
&= \frac{1}{k}\left(\frac{1}{1-\sum\limits_{i=1}^{m-2} b_i}\sum_{i=1}^{m-2} b_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_{\sigma_u}^{s} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds\right. \\
&\quad \left.+ \int_{\sigma_u}^{1} \phi_p^{-1}\left(\int_{\sigma_u}^{s} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds\right) \\
&\ge \frac{1}{k}\min\left\{\int_0^{\sigma_u} \phi_p^{-1}\left(\int_s^{\sigma_u} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds,\right. \\
&\quad \left.\int_{\sigma_u}^{1} \phi_p^{-1}\left(\int_{\sigma_u}^{s} q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds\right\}
\end{aligned}$$

$$\geq \frac{1}{k}\min\left\{\int_{\xi_i}^{t_i^*}\phi_p^{-1}\left(\int_s^{t_i^*}q(\tau)f(\tau,u(\tau),u'(t))d\tau\right)ds,\right.$$
$$\left.\int_{t_i^*}^{\xi_{i+1}}\phi_p^{-1}\left(\int_{t_i^*}^{s}q(\tau)f(\tau,u(\tau),u'(t))d\tau\right)ds\right\}$$
$$> \frac{1}{k}\min\left\{\int_{\xi_i}^{t_i^*}\phi_p^{-1}\left(\int_s^{t_i^*}q(\tau)d\tau\ \phi_p(kb/M)\right)ds,\right.$$
$$\left.\int_{t_i^*}^{\xi_{i+1}}\phi_p^{-1}\left(\int_{t_i^*}^{s}q(\tau)d\tau\ \phi_p(kb/M)\right)ds\right\}$$
$$= \frac{1}{k}\frac{kb}{M}\min\left\{\int_{\xi_i}^{t_i^*}\phi_p^{-1}\left(\int_s^{t_i^*}q(\tau)d\tau\right)ds,\ \int_{t_i^*}^{\xi_{i+1}}\phi_p^{-1}\left(\int_{t_i^*}^{s}q(\tau)d\tau\right)ds\right\}$$
$$= \frac{1}{k}\frac{kb}{M}M_i \geq b, \quad (i=0,1,\cdots m-2).$$

Therefore we have

$$\alpha(Tu) > b, \text{ for all } u\in P(\gamma,\theta,\alpha,b,Kb,d).$$

Consequently condition (S_1) in Theorem 2.1 is satisfied.

(3) We now prove (S_2) in Theorem 2.1 holds.

With (3.1), we have

$$\alpha(Tu)\geq \frac{1}{k}\theta(Tu) > \frac{1}{k}kb = b,$$

for $u\in P(\gamma,\alpha,b,d)$ with $\theta(Tu)>kb$. Hence condition (S_2) in Theorem 2.1 is satisfied.

(4) Finally, we prove (S_3) in Theorem 2.1 is also satisfied.

Since $\psi(0)=0<a$, so $0\notin R(\gamma,\psi,a,d)$. Suppose that $u\in R(\gamma,\psi,a,d)$ with $\psi(u)=a$. Then by the condition (A_3) of this theorem

$$\begin{aligned}\psi(Tu) &= \max_{0\leq t\leq 1}|Tu(t)| = Tu(\sigma_u)\\ &= \frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\left(\int_s^{\sigma_u}q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds\\ &\quad+\int_0^{\sigma_u}\phi_p^{-1}\left(\int_s^{\sigma_u}q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\right)ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-\sum\limits_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_{\sigma_u}^{s} q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau\right) ds \\
&\quad + \int_{\sigma_u}^{1} \phi_p^{-1}\left(\int_{\sigma_u}^{s} q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau\right) ds \\
&\leq \max\left\{ \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_{0}^{\xi_i} \phi_p^{-1}\left(\int_{s}^{1} q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau\right) ds \right. \\
&\quad + \int_{0}^{t_i^*} \phi_p^{-1}\left(\int_{s}^{t_i^*} q(\tau) f(\tau, u(\tau), u'(t)) d\tau\right) ds, \\
&\quad \frac{1}{1-\sum\limits_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau\right) ds \\
&\quad \left. + \int_{t_i^*}^{1} \phi_p^{-1}\left(\int_{t_i^*}^{s} q(\tau) f(\tau, u(\tau), u'(t)) d\tau\right) ds \right\}. \\
&< \max\left\{ \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_{0}^{\xi_i} \phi_p^{-1}\left(\int_{s}^{1} q(\tau) d\tau \ \phi_p(a/N)\right) ds \right. \\
&\quad + \int_{0}^{t_i^*} \phi_p^{-1}\left(\int_{s}^{t_i^*} q(\tau) d\tau \ \phi_p(a/N)\right) ds, \\
&\quad \frac{}{1-\sum\limits_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_{0}^{s} q(\tau) d\tau \ \phi_p(a/N)\right) ds \\
&\quad \left. + \int_{t_i^*}^{1} \phi_p^{-1}\left(\int_{t_i^*}^{s} q(\tau) d\tau \ \phi_p(a/N)\right) ds \right\}. \\
&= \frac{a}{N} \max\left\{ \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_{0}^{\xi_i} \phi_p^{-1}\left(\int_{s}^{1} q(\tau) d\tau\right) ds \right.
\end{aligned}
$$

$$+\int_0^{t_i^*}\phi_p^{-1}\left(\int_s^{t_i^*}q(\tau)d\tau\right)ds,$$

$$\left.\frac{1}{1-\sum\limits_{i=1}^{m-2}b_i}\sum_{i=1}^{m-2}b_i\int_{\xi_i}^1\phi_p^{-1}\left(\int_0^s q(\tau)d\tau\right)ds+\int_{t_i^*}^1\phi_p^{-1}\left(\int_{t_i^*}^s q(\tau)d\tau\right)ds\right\}$$

$$=\frac{a}{N}N_i\le a,\quad (i=0,1,\ldots,m-2).$$

Thus condition (S_3) in Theorem 2.1 holds.

Therefore an application of Theorem 2.1 implies the BVP (1.1), (1.2) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{0\le t\le 1}|u_i'(t)|\le d\ \ for\ \ i=1,2,3,$$

$$b<\min_{\frac{1}{k}\le t\le 1-\frac{1}{k}}|u_1(t)|,\quad \max_{0\le t\le 1}|u_1(t)|\le \overline{M}d,$$

$$a<\max_{0\le t\le 1}|u_2(t)|<Kb,\quad with\quad \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}}|u_2(t)|<b,$$

$$\max_{0\le t\le 1}|u_3(t)|<a.$$

5. Example

Let $p=3,\ q(t)=1$ in (1.1) and $m=4,\ \xi_1=1/3,\ \xi_2=2/3,\ a_i=b_i=1/3\ (i=1,2)$ in (1.2). Now we consider following BVP

(5.1) $$(|u'(t)|u'(t))'+f\left(t,u(t),u'(t)\right)=0,\qquad t\in(0,1),$$

(5.2) $$u(0)=\frac{1}{3}u(\frac{1}{3})+\frac{1}{3}u(\frac{2}{3}),\qquad u(1)=\frac{1}{3}u(\frac{1}{3})+\frac{1}{3}u(\frac{2}{3}),$$

where

$$f(t,u,v)=\begin{cases}\dfrac{t}{2}+1.7\times10^5\cdot u^{35}+\dfrac{1}{10}\left(\left(\dfrac{v}{4\times10^{21}}\right)^2+\dfrac{1}{10}\right), & u\le 12;\\[2ex] \dfrac{t}{2}+1.7\times10^5\cdot 12^{35}+\dfrac{1}{10}\left(\left(\dfrac{v}{4\times10^{21}}\right)^2+\dfrac{1}{10}\right), & u>12.\end{cases}$$

Choose $a=2/3,\ b=1,\ k=12,\ d=4\times10^{22}$, we note $L=1,\ \overline{M}=2,\ M=\sqrt{2}/48,\ N=\dfrac{2}{3}\left(2-\left(\dfrac{1}{3}\right)^{\frac{3}{2}}-\left(\dfrac{2}{3}\right)^{\frac{3}{2}}+\left(\dfrac{19}{24}\right)^{\frac{3}{2}}\right)\doteq 1.312,\ K=12.$

Consequently $f(t,u,v)$ satisfies

$$
\begin{aligned}
&(1)\ \ f(t,u,v) < 1.1\times 10^{43} < \phi_3(d/L) = 1.6\times 10^{43},\\
&\qquad for\ \ (t,u,v)\in[0,1]\times[0,8\times 10^{21}]\times[-4\times 10^{21},4\times 10^{21}];\\
&(2)\ \ f(t,u,v) > 170000 > \phi_3(kb/M) = 165888,\\
&\qquad for\ \ (t,u,v)\in[1/12,11/12]\times[1,12]\times[-4\times 10^{21},4\times 10^{21}];\\
&(3)\ \ f(t,u,v) < 0.727 < \phi_3(a/N) \doteq 0.765,\\
&\qquad for\ \ (t,u,v)\in[0,1]\times[0,2/3]\times[-4\times 10^{21},4\times 10^{21}].
\end{aligned}
$$

Then all conditions of Theorem 4.1 hold. Thus, with Theorem 4.1, BVP (5.1), 5.2) has at least three positive solutions u_1, u_2, u_3 such that

$$
\begin{aligned}
&\max_{0\le t\le 1}|u_i'(t)| \le 4\times 10^{21}\quad for\quad i=1,2,3,\\
&1 < \min_{\frac{1}{12}\le t\le\frac{11}{12}}|u_1(t)|,\quad \max_{0\le t\le 1}|u_1(t)| \le 8\times 10^{21},\\
&\frac{2}{3} < \max_{0\le t\le 1}|u_2(t)| < 12,\quad with\quad \min_{\frac{1}{12}\le t\le\frac{11}{12}}|u_2(t)| < 1,\\
&\max_{0\le t\le 1}|u_3(t)| < \frac{2}{3}.
\end{aligned}
$$

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