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CERTAIN CLASS OF CONTACT CR-SUBMANIFOLDS OF AN ODD-DIMENSIONAL UNIT SPHERE

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Abstract. In this paper we investigate $(n+1)(n \geq 5)$-dimensional contact CR-submanifolds M of $(n-1)$ contact CR-dimension in a $(2m+1)$-dimensional unit sphere S^{2m+1} which satisfy the condition $h(FX,Y) - h(X,FY) = g(FX,Y)\zeta$ for a normal vector field ζ to M, where h and F denote the second fundamental form and a skew-symmetric endomorphism (defined by (2.3)) acting on tangent space of M, respectively.

1. INTRODUCTION

Let S^{2m+1} be a $(2m+1)$-unit sphere in the complex $(m+1)$-space $\mathbb{C}^{m+1}$, i.e.,

$$S^{2m+1} := \{(z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1} | \sum_{j=1}^{m+1} |z_j|^2 = 1\}.$$

For any point $z \in S^{2m+1}$ we put $\xi = Jz$, where J denotes the complex structure of $\mathbb{C}^{m+1}$. Denoting by π the orthogonal projection : $T_z\mathbb{C}^{m+1} \to T_zS^{2m+1}$ and putting $\phi = \pi \circ J$, we can see that the set (ϕ, ξ, η, g) defines a Sasakian structure on S^{2m+1}, where g is the standard metric on S^{2m+1} induced from that of $\mathbb{C}^{m+1}$ and η is a 1-form dual to ξ. Hence S^{2m+1} can be considered as a Sasakian manifold of constant curvature 1 (cf. [1, 2, 10]).

Let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field ξ of S^{2m+1} and denote by $\mathcal{D}_x$ the ϕ-invariant subspace $T_xM \cap \phi T_xM$ of the tangent space T_xM of M at $x \in M$. Then ξ cannot be contained in $\mathcal{D}_x$ at any point

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$x \in M$ (cf. [5]). Thus the assumption $\dim\mathcal{D}_x^{\perp}$ being constant and equal to 2 at each point $x \in M$ yields that M can be dealt with a contact CR-submanifold in the sense of Yano-Kon (cf. [1,10]), where $\mathcal{D}_x^{\perp}$ denotes the complementary orthogonal subspace to $\mathcal{D}_x$ in T_xM. In fact, if there exists a non-zero vector U which is orthogonal to ξ and contained in $\mathcal{D}_x^{\perp}$, then $N := \phi U$ must be normal to M. In particular we can easily see that real hypersurfaces tangent to ξ of S^{2m+1} are typical examples of such submanifolds.

On the other hand, in [7] Nakagawa and Yokote have studied real hypersurfaces M of S^{2m+1} which satisfy the condition

$$AF + FA = \rho F$$

for a function ρ and determined such submanifolds under the additional condition that the scalar curvature is constant, where F denotes a skew-symmetric endomorphism induced from ϕ acting on the tangent bundle TM and A the shape operator of M(see also [10, Theorem 6.2, p.196]).

In this paper we study contact CR-submanifolds M of maximal contact CR-dimension in S^{2m+1}, namely, those with $\dim\mathcal{D}_x = n-1$ at each point $x \in M$ and investigate such submanifolds under the condition

$$h(FX, Y) - h(X, FY) = g(FX, Y)\zeta$$

for a normal vector field ζ to M, where F is a skew-symmetric endomorphism given by (2.3) acting on TM and h the second fundamental form on M.

Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be connected, differentiable and of class C^∞.

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2. Fundamental Properties of Contact CR-Submanifolds

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure (ϕ, ξ, η, g). By definition it follows that

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{2.1}$$

for any tangent vector fields X, Y to $\overline{M}$(cf. [1, 10]).

Let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field ξ of $\overline{M}$. If the ϕ-invariant subspace $\mathcal{D}_x$ has constant dimension for any $x \in M$, then M is called a *contact CR-submanifold* and the constant is called *contact CR-dimension* of M (cf. [1, 5, 6, 8]).

From now on we assume that M is a contact CR-submanifold of $(n-1)$ contact CR-dimension in $\overline{M}$, where $n-1$ must be even. Then, as already mentioned in the previous section, the structure vector ξ is always contained in $\mathcal{D}_x^{\perp}$ and $\phi\mathcal{D}_x^{\perp} \subset T_xM^{\perp}$ at any point $x \in M$. Further, by definition we can see that $\dim\mathcal{D}_x^{\perp} = 2$ at any point $x \in M$, and thus there exists a unit vector field U contained in $\mathcal{D}^{\perp}$ which is orthogonal to ξ. Since $\phi\mathcal{D}^{\perp} \subset TM^{\perp}$, ϕU is a unit normal vector field to M, which will be denoted by N, that is,

$$N := \phi U. \tag{2.2}$$

Moreover, it is clear that $\phi TM \subset TM \oplus \mathrm{Span}\{N\}$. Hence we have, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\ldots,p}$ $(N_1 := N,\ p := 2m-n)$ of normal vectors to M, the following decomposition in tangential and normal components:

$$\phi X = FX + u(X)N, \tag{2.3}$$

$$\phi N_\alpha = \sum_{\beta=2}^{p} P_{\alpha\beta}N_\beta, \quad \alpha = 2, \ldots, p. \tag{2.4}$$

It is easily shown that F is a skew-symmetric endomorphism acting on T_xM and $P_{\alpha\beta} = -P_{\beta\alpha}$. Since the structure vector field ξ is tangent to M, (2.1), (2.2) and (2.3) imply

$$F\xi = 0,\ FU = 0,\ u(X) = g(U, X),\ u(\xi) = g(U, \xi) = \eta(U) = 0, \tag{2.5}$$

Next, applying ϕ to (2.3) and using (2.1), (2.2), (2.3) and (2.5), we also have

$$F^2X = -X + u(X)U + \eta(X)\xi, \quad u(FX) = 0. \tag{2.6}$$

On the other hand, it is clear from (2.1) and (2.5) that

$$\phi N = -U, \tag{2.7}$$

which combined with (2.4) yields the existence of a local orthonormal basis $\{N, N_a, N_{a^*}\}_{a=1,\cdots,q}$ of normal vectors to M such that

$$N_{a^*} := \phi N_a, \quad a = 1, \cdots, q := (p-1)/2. \tag{2.8}$$

We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $\overline{M}$ and M, respectively, and by $\nabla^{\perp}$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^{\perp}$ of M. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$

$$\overline{\nabla}_X N = -AX + \nabla^{\perp}_X N = -AX + \sum_{a=1}^{q}\{s_a(X)N_a + s_{a^*}(X)N_{a^*}\}, \tag{2.10}$$

$$\overline{\nabla}_X N_a = -A_a X - s_a(X)N + \sum_{b=1}^{q}\{s_{ab}(X)N_b + s_{ab^*}(X)N_{b^*}\}, \tag{2.11}$$

$$\overline{\nabla}_X N_{a^*} = -A_{a^*} X - s_{a^*}(X)N + \sum_{b=1}^{q}\{s_{a^*b}(X)N_b + s_{a^*b^*}(X)N_{b^*}\} \tag{2.12}$$

for any tangent vector fields X, Y to M, where $s's$ are coefficients of the normal connection $\nabla^{\perp}$. Here and in the sequel h denotes the second fundamental form and A, A_a, A_{a^*} the shape operators corresponding to the normals N, N_a, N_{a^*}, respectively. They are related by

$$h(X,Y) = g(AX,Y)N + \sum_{a=1}^{q}\{g(A_aX,Y)N_a + g(A_{a^*}X,Y)N_{a^*}\}. \tag{2.13}$$

From now on we specialize to the case of an ambient Sasakian manifold $\overline{M}$, that is,

$$\overline{\nabla}_X \xi = \phi X, \tag{2.14}$$

$$(\overline{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{2.15}$$

Since the structure vector ξ is tangent to M, it follows from (2.1), (2.3), (2.7), (2.8), (2.11), (2.12) and (2.15) that

$$A_a X = -FA_{a^*}X + s_{a^*}(X)U, \quad A_{a^*}X = FA_aX - s_a(X)U, \tag{2.16}$$

$$s_a(X) = -u(A_{a^*}X), \quad s_{a^*}(X) = u(A_aX). \tag{2.17}$$

Moreover, since F is skew-symmetric, (2.16) implies

$$g((FA_a + A_aF)X,Y) = s_a(X)u(Y) - s_a(Y)u(X), \tag{2.18}$$

$$g((FA_{a^*} + A_{a^*}F)X,Y) = s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X). \tag{2.19}$$

Differentiating (2.3) and (2.7) covariantly and comparing the tangential and normal parts, we have

$$(\nabla_Y F)X = -g(Y,X)\xi + \eta(X)Y - g(AY,X)U + u(X)AY, \tag{2.20}$$

$$\nabla_X U = FAX, \quad (\nabla_Y u)X = g(FAY,X), \tag{2.21}$$

where we have used (2.3), (2.7), (2.8), (2.9), (2.10), (2.13) and (2.15).

On the other hand, since ξ is tangent to M, (2.14) combined with (2.9) and (2.13) yields

$$\nabla_X \xi = FX, \quad (\nabla_X \eta)Y = g(FX,Y) \tag{2.22}$$

$$\eta(AX) = g(A\xi, X) = u(X), \quad \text{i.e.,} \quad A\xi = U, \tag{2.23}$$

$$A_a \xi = 0, \quad A_{a^*}\xi = 0, \quad a = 2, \ldots, q. \tag{2.24}$$

If the ambient manifold $\overline{M}$ is a $(2m+1)$-dimensional unit sphere S^{2m+1}, then its curvature tensor $\overline{R}$ satisfies

$$\overline{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y$$

for tangent vector fields X, Y, Z to $\overline{M}$. In this case, from (2.3) and (2.4), we can see that the equations of Codazzi and Ricci imply

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = \sum_{a=1}^{q}\{&s_a(X)A_aY - s_a(Y)A_aX \\ &+ s_{a^*}(X)A_{a^*}Y - s_{a^*}(Y)A_{a^*}X\},\end{aligned} \tag{2.25}$$

$$\begin{aligned}(\nabla_X A_a)Y - (\nabla_Y A_a)X = s_a(Y)AX - s_a(X)AY + \sum_{b=1}^{q}\{&s_{ab}(X)A_bY \\ - s_{ab}(Y)A_bX + s_{ab^*}(X)A_{b^*}Y - s_{ab^*}(Y)A_{b^*}X\},&\end{aligned} \tag{2.26}$$

$$\begin{aligned}(\nabla_X A_{a^*})Y - (\nabla_Y A_{a^*})X = s_{a^*}(Y)AX - s_{a^*}(X)AY + \sum_{b=1}^{q}\{&s_{a^*b}(X)A_bY \\ - s_{a^*b}(Y)A_bX + s_{a^*b^*}(X)A_{b^*}Y - s_{a^*b^*}(Y)A_{b^*}X\},&\end{aligned} \tag{2.27}$$

$$g(R^{\perp}(X,Y)N, N_\alpha) + g([A_\alpha, A]X, Y) = 0, \quad \alpha = 2, \cdots, p \tag{2.28}$$

for any vector fields X, Y tangent to M, where R and $R^{\perp}$ denote the Riemannian curvature tensor and the normal curvature tensor of M, respectively(cf. [1, 2, 10]).

3. Some Lemmas

Let M be an $(n+1)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in S^{2m+1} which is considered as a Sasakian manifold of constant curvature 1 and let us use the same notations as stated in the previous section.

We assume that the equality

$$h(FX, Y) - h(X, FY) = g(FX, Y)\zeta \tag{3.1}$$

holds on M for a normal vector field ζ to M. We also use the orthonormal basis (2.8) of normal vectors to M and set

$$\zeta = \rho N + \sum_{a=1}^{q}(\rho_a N_a + \rho_{a^*} N_{a^*}).$$

Then by means of (2.13) the condition (3.1) is equivalent to

$$(AF + FA)X = \rho FX, \tag{3.2}$$

$$(A_a F + FA_a)X = \rho_a FX, \quad (A_{a^*}F + FA_{a^*})X = \rho_{a*}FX \tag{3.3}$$

for all $a = 1, \ldots, q$. Moreover, the last two equations combined with (2.18) and (2.19) yield

$$s_a(X)u(Y) - s_a(Y)u(X) = \rho_a g(FX, Y), \tag{3.3}$$

$$s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X) = \rho_{a*}g(FX, Y), \tag{3.4}$$

from which, putting $Y = U$ and $Y = \xi$ into (3.4), respectively, and using (2.5), we obtain

$$s_a(X) = s_a(U)u(X), \quad s_{a^*}(X) = s_{a^*}(U)u(X), \tag{3.5}$$

$$s_a(\xi) = 0, \quad s_{a^*}(\xi) = 0, \quad a = 1, \cdots, q. \tag{3.6}$$

Substituting (3.5) into (3.4), we have

$$\rho_a = 0, \quad \rho_{a^*} = 0, \quad a = 1, \cdots, q \tag{3.7}$$

and consequently

$$FA_a + A_aF = 0, \quad FA_{a^*} + A_{a^*}F = 0, \quad a = 1, \cdots, q \tag{3.8}$$

with the aid of (3.3)

As a direct consequence of (3.2) and (3.8), it follows from (2.5), (2.6), (2.17), (2.23) and (2.24) that

$$AU = \lambda U + \xi, \quad \lambda := u(AU) \tag{3.9}$$

and, for $a = 1, \cdots, q$,

$$A_aU = u(A_aU)U = s_{a^*}(U)U, \quad A_{a^*}U = u(A_{a^*}U)U = -s_a(U)U. \tag{3.10}$$

Inserting FX into (3.2) instead of X and using (2.6), (2.23) and (3.9), we have

$$-AX+\{(\lambda-\rho)u(X)+\eta(X)\}U+\{u(X)-\rho\eta(X)\}\xi+FAFX=-\rho X. \tag{3.11}$$

On the other hand, $F\mathcal{D}_x = \mathcal{D}_x$ at each point $x \in M$, and thus there exists a local orthonormal basis $\{E_\kappa\}_{\kappa=1,\cdots,n+1} := \{E_i, E_{i^*}, U, \xi\}_{i=1,\cdots,l}$ of tangent vectors to M such that

$$E_{i^*} = FE_i, \quad i = 1, \cdots, l := (n-1)/2 \tag{3.12}$$

Taking the trace of the both side of (3.11) by using this orthonormal basis, we have

$$trA = \lambda + \rho(n-1)/2. \tag{3.13}$$

because of $\sum_{i=1}^{l}\{g(FAFE_i, E_i) + g(FAFE_{i^*}, E_{i^*})\} = -trA + \lambda$.

Differentiating (3.9) covariantly and using (2.21), (2.22) and the symmetry of A, we can easily show that

$$g((\nabla_X A)Y, U) + g(FAX, AY) = (X\lambda)u(Y) + \lambda g(FAX, Y) + g(FX, Y),$$

from which, taking the skew-symmetric part and substituting (2.25) into the equation thus obtained,

$$-(2+\lambda\rho)g(FX, Y) + 2g(FAX, AY) = (X\lambda)u(Y) - (Y\lambda)u(X) \tag{3.14}$$

with the help of (2.17), (3.2) and (3.5). Putting $Y = U$ into (3.14) and using (2.5) and (3.9), we have

$$(X\lambda) = (U\lambda)u(X), \tag{3.15}$$

which together with (3.2) and (3.14) implies

$$-(2+\lambda\rho)FX + 2(\rho FAX - FA^2X) = 0.$$

Applying F to this equation and using (2.5), (2.6), (2.23) and (3.9), we can easily obtain

$$\begin{aligned} &2A^2X - 2\rho AX + (2+\lambda\rho)X \\ &+\{(\lambda\rho - 2\lambda^2 - 4)u(X) + 2(\rho-\lambda)\eta(X)\}U \\ &+\{2(\rho-\lambda)u(X) - (\lambda\rho+4)\eta(X)\}\xi = 0, \end{aligned} \tag{3.16}$$

and thus, it is clear that, at a point $x \in M$ with $\rho(x) = 0$,

$$A^2X + X - \{(\lambda^2+2)u(X) + \lambda\eta(X)\}U - \{\lambda u(X) + 2\eta(X)\}\xi = 0.$$

Therefore, the eigenvalue ν corresponding to an eigenvector of A, orthogonal to U and ξ, satisfies $\nu^2 + 1 = 0$ which is a contradiction because A is a real symmetric tensor.

Thus we have

Remark. The function ρ given by (3.2) takes a value zero nowhere.

Now we prepare some lemmas for later use.

Lemma 3.1. *Let M be an $(n+1)(n \geq 3)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in S^{2m+1}. If the equality (3.1) holds on M for a normal vector field ζ to M, then λ determined by (3.9) is constant. Moreover,*

$$A(\mu_1 U + \xi) = \mu_1(\mu_1 U + \xi), \quad A(\mu_2 U + \xi) = \mu_2(\mu_2 U + \xi), \tag{3.17}$$

where $\mu_i (i = 1, 2)$ denote the solutions of the quadratic equation

$$\mu^2 - \lambda\mu - 1 = 0. \tag{3.18}$$

Proof. Tentatively we denote by $\beta := U\lambda$ in (3.15) and differentiate the equation thus obtained covariantly. Then, from (2.5), (2.21) and (3.2), we have

$$(Y\beta)u(X) - (X\beta)u(Y) + \beta\rho g(FY, X) = 0,$$

from which, putting $Y = U$ and using (2.5), it follows that $(X\beta) = (U\beta)u(X)$ and so

$$\beta\rho g(FY, X) = 0.$$

As already shown in the above remark, ρ is nowhere vanishing and consequently $\beta = 0$, which together with (3.15) implies that λ is constant. The last assertion (3.17) can be easily obtained from (2.23), (3.9) and (3.18). ∎

Differentiating (3.2) covariantly and using (2.20), (2.23), (3.2) and (3.9), we have

$$\begin{aligned}&(\nabla_X A)FY + F(\nabla_X A)Y + u(Y)A^2X + \{(\lambda-\rho)u(Y) + 2\eta(Y)\}AX\\ &+\{u(Y) - \rho\eta(Y)\}X - \{g(X,Y) + (\lambda-\rho)g(AX,Y) + g(AX,AY)\}U\\ &-\{2g(AX,Y) - \rho g(X,Y)\}\xi = (X\rho)FY,\end{aligned}$$

from which, using (2.5) and the orthonormal basis given by (3.12),

$$\begin{aligned}&\sum_{\kappa=1}^{n+1} g((\nabla_{E_\kappa}A)FY, E_\kappa) - \sum_{i=1}^{l} g((\nabla_{E_i}A)FE_i - (\nabla_{FE_i}A)E_i, Y)\\ &+\mathrm{tr}A^2u(Y) + \mathrm{tr}A\{(\lambda-\rho)u(Y) + 2\eta(Y)\} + n\{u(Y) - \rho\eta(Y)\}\\ &-(\lambda-\rho)u(AY) - u(A^2Y) - 2\eta(AY) = (FY)\rho.\end{aligned} \tag{3.19}$$

On the other hand, using (2.5), (2.17), (2.25) and (3.5), we have

$$\sum_{\kappa=1}^{n+1} g((\nabla_{E_\kappa}A)FY, E_\kappa) = \sum_{\kappa=1}^{n+1} g((\nabla_{FY}A)E_\kappa, E_\kappa)$$

and

$$\sum_{i=1}^{l} g((\nabla_{E_i}A)FE_i - (\nabla_{FE_i}A)E_i, Y) = 0.$$

Moreover, taking the trace of (3.16) with respect to the orthonormal bais (3.12) and using (2.23), (3.9) and (3.13), we can find

$$\mathrm{tr}A^2 = (n-1)\rho(\rho-\lambda)/2 + \lambda^2 - n + 3.$$

Substituting these equations into (3.19) and taking account of (2.23), (3.9), (3.13), (3.16) and Lemma 3.1, we can see that

$$(n-3)(FY)\rho = 0,$$

which together with (2.6) implies

$$(n-3)\{Y\rho - u(Y)U\rho - \eta(Y)\xi\rho\} = 0. \tag{3.20}$$

Thus we have

Lemma 3.2. *Let M be an $(n+1)(n \geq 5)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in S^{2m+1}. If the equality (3.1) holds on M for a normal vector field ζ to M, then the function ρ determined by (3.2) is a non-zero constant.*

Proof. Differentiating (3.20) covariantly and using (2.21), (2.22) and (3.2), we can easily obtain

$$(X\alpha)u(Y) - (Y\alpha)u(X) + (X\beta)\eta(Y) - (Y\beta)\eta(X) + (\alpha\rho + 2\beta)g(FX, Y) = 0,$$

where we have put $\alpha := U\rho$ and $\beta := \xi\rho$. Putting $X = U$ and $Y = \xi$ into this equation, respectively, and using (2.5), we obtain

$$X\alpha = (U\alpha)u(X) + (U\beta)\eta(X), \quad X\beta = (\xi\alpha)u(X) + (\xi\beta)\eta(X)$$

and consequently

$$\begin{aligned}&(U\beta)\{\eta(X)u(Y) - \eta(Y)u(X)\} + (\xi\alpha)\{u(X)\eta(Y) - u(Y)\eta(X)\}\\ &+(\alpha\rho + 2\beta)g(FX, Y) = 0.\end{aligned}$$

Putting $X = \xi$ and $Y = U$ into the last equation and using (2.1) and (2.5), we have $U\beta = \xi\alpha$ and so

$$\alpha\rho + 2\beta = 0. \tag{3.21}$$

On the other hand, differentiating (2.23) covariantly and using (2.21), (2.22) and (3.2), we have

$$g((\nabla_X A)\xi, Y) = g(2FAX - \rho FX, Y),$$

which together with (2.24), (2.25) and (3.6) implies

$$g((\nabla_\xi A)X, Y) = g(2FAX - \rho FX, Y).$$

Taking the trace of the last equation with respect to the basis (3.12) and using (2.5) and (3.2), we obtain

$$\begin{aligned}\sum_{\kappa=1}^{n+1} g((\nabla_\xi A)E_\kappa, E_\kappa) &= 2\sum_{i=1}^{l}\{g(FAE_i, E_i) + g(FAFE_i, FE_i)\}\\ &= 2\sum_{i=1}^{l}\{-g(AE_i, FE_i) + g(AFE_i, E_i)\} = 0,\end{aligned}$$

and thus $\xi(\mathrm{tr}A) = 0$, which combined with Lemma 3.1 yields $\beta = \xi\rho = 0$. Therefore we can see from (3.21) that $\alpha\rho = 0$ and consequently $\alpha = 0$ because

ρ takes a value zero nowhere. Hence (3.20) with $\alpha = \beta = 0$ implies that ρ is constant. ∎

Finally, differentiating the first equation of (3.10) covariantly and using (2.21), we have

$$g((\nabla_X A_a)Y, U) + g(A_a FAX, Y) = X(s_{a^*}(U))u(Y) + s_{a^*}(U)g(FAX, Y),$$

from which, taking the skew-symmetric part with respect to X and Y and using (2.26), (3.2), (3.5), (3.9) and (3.10), the last equation turns out to be

$$\begin{aligned} &s_a(U)u(Y)\eta(X) - s_a(U)u(X)\eta(Y) + \textstyle\sum_{b=1}^{q}\{s_{ab}(X)s_{b^*}(U)u(Y) \\ &-s_{ab}(Y)s_{b^*}(U)u(X) - s_{ab^*}(X)s_b(U)u(Y) + s_{ab^*}(Y)s_b(U)u(X)\} \\ &+g(A_aFAX, Y) - g(A_aFAY, X) \\ &= X(s_{a^*}(U))u(Y) - Y(s_{a^*}(U))u(X) + \rho s_{a^*}(U)g(FX, Y). \end{aligned} \tag{3.22}$$

Taking $Y = U$ in (3.22) and using (2.5), (3.9) and (3.10), it follows that

$$\begin{aligned} X(s_{a^*}(U)) = U(s_{a^*}(U))u(X) + s_a(U)\eta(X) + \sum_{b=1}^{q}[s_{ab}(X)s_{b^*}(U) \\ -s_{ab^*}(X)s_b(U) - u(X)\{s_{ab}(U)s_{b^*}(U) - s_{ab^*}(U)s_b(U)\}]. \end{aligned}$$

Inserting the last equation back into (3.22) and using (3.2) and (3.8), we have

$$-g(FA_aAX, Y) - g(FAA_aX, Y) + \rho g(FA_aX, Y) = \rho s_{a^*}(U)g(FX, Y).$$

Replacing Y by FY in the last equation and using (2.6), we can easily obtain

$$\begin{aligned} g((A_aA + AA_a)X, Y) = 2\lambda s_{a^*}(U)u(X)u(Y) + s_{a^*}(U)\{\eta(X)u(Y) \\ +u(X)\eta(Y)\} + \rho\{g(A_aX, Y) - s_{a^*}(U)g(X, Y) + s_{a^*}(U)\eta(X)\eta(Y)\}, \end{aligned} \tag{3.23}$$

where we have used

$$u(A_aAX) = s_{a^*}(U)\{\lambda u(X) + \eta(X)\}, \quad u(AA_aX) = \lambda s_{a^*}(U)u(X),$$
$$\eta(AA_aX) = s_{a^*}(U)u(X)$$

which are direct consequences of (2.23), (2.24), (3.5), (3.9) and (3.10).

On the other hand, taking account of (2.10) – (2.12), we can easily see that

$$\begin{aligned} g(R^{\perp}(X, Y)N, N_c) = (\nabla_X s_c)Y - (\nabla_Y s_c)X + \sum_{a=1}^{q}\{s_a(Y)s_{ac}(X) \\ -s_a(X)s_{ac}(Y) + s_{a^*}(Y)s_{a^*c}(X) - s_{a^*}(X)s_{a^*c}(Y)\}, \end{aligned}$$

which combined with (2.28) yields

$$
\begin{aligned}
&g(A_cAX,Y)-g(AA_cX,Y)+(\nabla_X s_c)Y-(\nabla_Y s_c)X\\
&+\sum_{a=1}^{q}\{s_a(Y)s_{ac}(X)-s_a(X)s_{ac}(Y)\\
&+s_{a^*}(Y)s_{a^*c}(X)-s_{a^*}(X)s_{a^*c}(Y)\}=0.
\end{aligned}
\tag{3.24}
$$

As a direct consequence of (3.5), we have

$$
(\nabla_X s_a)Y-(\nabla_Y s_a)X = X(s_a(U))u(Y)-Y(s_a(U))u(X)+\rho s_a(U)g(FX,Y),
$$

and consequently (3.24) reduces to

$$
\begin{aligned}
&g(A_aAX,Y)-g(AA_aX,Y)+X(s_a(U))u(Y)-Y(s_a(U))u(X)\\
&+\rho g(FX,Y)s_a(U)+\sum_{b=1}^{q}\{s_b(Y)s_{ba}(X)-s_b(X)s_{ba}(Y)\\
&+s_{b^*}(Y)s_{b^*a}(X)-s_{b^*}(X)s_{b^*a}(Y)\}=0.
\end{aligned}
\tag{3.25}
$$

Taking $Y=U$ in (3.25) and using (2.5), (2.24), (3.5), (3.9) and (3.10), we have

$$
\begin{aligned}
X(s_a(U))u(Y) &= U(s_a(U))u(X)u(Y)-s_{a^*}(U)\eta(X)u(Y)\\
&\quad-\sum_{b=1}^{q}\{s_b(Y)s_{ba}(X)+s_{b^*}(Y)s_{b^*a}(X)\\
&\quad-s_b(U)s_{ba}(U)u(X)u(Y)-s_{b^*}(U)s_{b^*a}(U)u(X)u(Y)\},
\end{aligned}
$$

which together with (3.25) implies

$$
\begin{aligned}
&g(A_aAX,Y)-g(AA_aX,Y)\\
&=s_{a^*}(U)\{\eta(X)u(Y)-\eta(Y)u(X)\}-\rho s_a(U)g(FX,Y).
\end{aligned}
\tag{3.26}
$$

Adding (3.23) and (3.26), we obtain

$$
\begin{aligned}
&2g(A_aAX,Y)=2\lambda s_{a^*}(U)u(X)u(Y)+2s_{a^*}(U)\eta(X)u(Y)\\
&+\rho\{g(A_aX,Y)-s_{a^*}(U)g(X,Y)+s_{a^*}(U)\eta(X)\eta(Y)-s_a(U)g(FX,Y)\}
\end{aligned}
\tag{3.27}
$$

Now, let X be an eigenvector of A, orthogonal to U and ξ, with the corresponding eigenvalue ν. Then, it follows from (3.27) that

$$
(2\nu-\rho)A_aX=-\rho\{s_{a^*}(U)X+s_a(U)FX\},
\tag{3.28}
$$

and also

$$
(2\nu-\rho)A_aFX=-\rho\{s_a(U)X-s_{a^*}(U)FX\}
$$

because of $AFX = (\rho - \nu)FX$. Similarly, from the second equation of (3.10) we can obtain

$$(2\nu - \rho)A_{a^*}X = \rho\{s_a(U)X + s_{a^*}(U)FX\},$$

$$(2\nu - \rho)A_{a^*}FX = \rho\{s_{a^*}(U)X - s_a(U)FX\}.$$

Hence, if the distinguished normal vector field N is parallel with respect to the normal connection , i.e., $\nabla^{\perp}N = 0$, then it is clear from (2.10) that $s_a = s_{a^*} = 0$, and therefore the above equations imply that $A_a = 0$ and $A_{a^*} = 0$ with the help of (2.24) and (3.10), provided $\rho \neq 2\nu$.

Thus we have

Lemma 3.3. *Let M be as in Lemma 3.1 and let the distinguished normal vector field N is parallel with respect to the normal connection. If the equality (3.1) holds on M for a normal vector field ζ to M and $\rho \neq 2\nu$, then*

$$A_a = 0, \quad A_{a^*} = 0, \quad a = 1, \cdots, q.$$

4. Main Results

We first prepare the following lemma:

Lemma 4.1. *Let M be an $(n+1)(n \geq 5)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in S^{2m+1}. If the equality (3.1) holds on M for a normal vector field ζ to M, then the shape operator A has 2 constant eigenvalues $\{\lambda \pm \sqrt{\lambda^2+4}\}/2$ of multiplicities 1 and n, or 4 constant eigenvalues*

$$\{\lambda \pm \sqrt{\lambda^2+4}\}/2, \quad \{\rho \pm \sqrt{\rho^2 - 2(2+\lambda\rho)}\}/2$$

of multiplicities 1, 1, $(n-1)/2$ and $(n-1)/2$, respectively. Moreover, if A has exactly 2 eigenvalues $\{\lambda \pm \sqrt{\lambda^2+4}\}/2$, then the eigenvalue ν corresponding to an eigenvector of A, orthogonal to U and ξ, satisfies $2\nu = \rho = \lambda \pm \sqrt{\lambda^2+4}$ and vice-versa.

Proof. If we denote by ν the eigenvalue corresponding to an eigenvector of A, orthogonal to U and ξ, then it is clear from (3.16) that ν satisfies

$$2\nu^2 - 2\rho\nu + \lambda\rho + 2 = 0 \tag{4.1}$$

and consequently the shape operator A has at most 4 constant eigenvalues

$$\{\lambda \pm \sqrt{\lambda^2+4}\}/2, \quad \{\rho \pm \sqrt{\rho^2 - 2(2+\lambda\rho)}\}/2$$

whose multiplicities are 1, 1, $(n-1)/2$ and $(n-1)/2$, respectively, with the help of (3.13). Moreover, if A has exactly 2 eigenvalues $\{\lambda \pm \sqrt{\lambda^2+4}\}/2$, then $2\nu = \lambda \pm \sqrt{\lambda^2+4}$, which together with (3.18) and (4.1) implies

$$\lambda\{\lambda \pm \sqrt{\lambda^2+4}\} + 4 - \rho\{\lambda \pm \sqrt{\lambda^2+4}\} + \lambda\rho = 0$$

and hence $\rho = \lambda \pm \sqrt{\lambda^2+4} = 2\nu$. ■

In the sense of Lemma 4.1, we first consider the case of $\rho = \lambda \pm \sqrt{\lambda^2+4} = 2\nu$. In this case, the shape operator A has exactly 2 constant eigenvalues

$$\mu_1 := \{\lambda + \sqrt{\lambda^2+4}\}/2, \quad \mu_2 := \{\lambda - \sqrt{\lambda^2+4}\}/2.$$

of multiplicities, say 1 and n, respectively. Moreover, since ρ is a non-zero constant, (3.5) and (3.28) with $\rho = 2\nu$ imply $s_a = s_{a^*} = 0$, which and (2.10) yield $\nabla^\perp N = 0$. It is also clear from (2.25) that A is of Codazzi type because of $s_a = s_{a^*} = 0$.

Now, we denote by

$$T_k := \{X \in TM \mid AX = \mu_k X\}, \quad k = 1, 2.$$

Since A is of Codazzi type and $\mu_1 \neq \mu_2$, we can easily see that the distributions $T_k (k = 1, 2)$ are both involutive and that the integral submanifolds M_k of T_k are totally geodesic and parallel along T_j, $j \neq k$(cf. [3]). Hence M is locally a Riemannian product $M_1 \times M_2$, where $\dim M_1 = 1$ and $M_2 = n$.

In order to investigate the integral submanifolds M_k more precisely, we consider the Gauss and Weingarten formulae for $S^{2m+1} \subset \mathbb{R}^{2m+2}$ which are given by

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y + g(X, Y)\widetilde{N}, \tag{4.2}$$

$$\widetilde{\nabla}_X \widetilde{N} = -X, \tag{4.3}$$

where $\widetilde{\nabla}$ denotes the Euclidean connection of $\mathbb{R}^{2m+2}$ and $\widetilde{N}$ the inward unit normal to S^{2m+1}. Then it follows from $\nabla^\perp N = 0$ and (4.2) that

$$\widetilde{\nabla}_X N = -AX \tag{4.4}$$

for any vector field X tangent to M.

On the other hand, by means of (3.17) M_1 is a curve on S^{2m+1} with unit tangent vector

$$Z = \frac{1}{\sqrt{\mu_1^2+1}}(\mu_1 U + \xi).$$

Further, using (2.5), (2.21), (2.22), (2.23), (2.24) and (3.9), it follows from (2.9) that $\overline{\nabla}_Z Z = \mu_1 N$. Moreover, it also clear from (2.10) that $\overline{\nabla}_Z N = -\mu_1 Z$. Hence we easily deduce that M_1 belongs to a circle S^1 on S^{2m+1}.

Next, we consider the integral submanifold M_2. Let P be the position vector of M_2 in $\mathbb{R}^{2m+2}$ and put

$$Q = P + (1+{\mu_2}^2)^{-1}(\mu_2 N + \widetilde{N}).$$

Then, for $X \in T_2$, we have $\widetilde{\nabla}_X Q = 0$ because of $AX = \mu_2 X$, (4.3) and (4.4), and so Q is a fixed point for M_2. Moreover, it is clear that

$$\|Q - P\|^2 = (1+{\mu_2}^2)^{-1}$$

which means that P belongs to a sphere S_2 with radius $(1+{\mu_2}^2)^{-1/2}$ and center Q.

We consider M_2 as a submanifold of S^{2m+1}. Since M_2 is totally geodesic in M, it is clear that ${A_Y}^{(2)} = 0$ where ${A_Y}^{(2)}$ is the shape operator of M_2 in S^{2m+1} with respect to the tangent vector Y to M_1. This means that the first normal space (cf. [4]) of M_2 is contained in Span$\{N, N_2, \ldots, N_p\}$.

We now prove

Lemma 4.2. Span $\{N, N_2, \ldots, N_p\}$ is invariant under parallel translation with respect to the normal connection $D^{(2)}$ of M_2 in S^{2m+1}.

Proof. Since S^{2m+1} is of constant curvature 1 and $\nabla^\perp_X N = 0$, (2.28) implies

$$g([A, A_{N'}]X, Y) = g(R^\perp(X,Y)N, N') = 0$$

for any normal vector N' to M. Hence $AA_{N'} = A_{N'}A$ and so, for $X \in T_2$ we have $A_{N'}X \in T_2$, i.e.,

$$A_{N'}T_2 \subset T_2. \tag{4.6}$$

On the other hand, for any vector field X tangent to M_2, we have

$$\overline{\nabla}_X N_\alpha = -A_\alpha X + \nabla^\perp_X N_\alpha.$$

But $\nabla^\perp_X N_\alpha \in$ Span$\{N, N_2, \ldots, N_p\}$ and $A_\alpha X \in T_2$ as a consequence of (4.5). Hence

$$D^{(2)}_X N_\alpha = \nabla^\perp_X N_\alpha \in \text{Span}\{N, N_2, \ldots, N_p\},$$

which completes the proof. ∎

As a consequence of Lemma 4.2 we can apply Erbacher's reduction theorem ([4, p. 339]) and this yields that M_2 belongs to a totally geodesic submanifold $S(1)$ of dimension (dim$M_2 + p$) in S^{2m+1}. Therefore M_2 belongs to the intersection of this sphere $S(1)$ and the sphere $S_2((1+{\mu_2}^2)^{-1/2}, Q)$ obtained above. Note that Q

belongs to the Euclidean space of dimension (dim$M_2 + p + 1$) through the origin and containing $S(1)$. Since dim$M_2 + p$ is even, we may conclude

Theorem 4.3. *Let M be an $(n+1)(n \geq 5)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in S^{2m+1}. If the equality* (3.1) *holds on M for a normal vector field ζ to M and $\rho = \lambda \pm \sqrt{\lambda^2 + 4}$, then M is locally a product $S^1 \times M_2$, where M_2 belongs to some sphere of odd-dimension.*

Finally, we consider the case of $\rho \neq \lambda \pm \sqrt{\lambda^2 + 4}$ under the assumption that the distinguished normal vector field N be parallel with respect to the normal connection. In this case, by means of Lemma 3.3 and Erbacher's reduction theorem ([4, p. 339]), we have

Theorem 4.4. *Let M be as in Theorem* 4.3 *and let the distinguished normal vector field N be parallel with respect to the normal connection. If the equality* (3.1) *holds on M for a normal vector field ζ to M and $\rho \neq \lambda \pm \sqrt{\lambda^2 + 4}$, then there exists an $(n+2)$-dimensional unit sphere S^{n+2} which is totally geodesic in S^{2m+1} and $M \subset S^{n+2}$.*

In Lemma 4.4, since the tangent space T_xS^{n+2} of the totally geodesic submanifold S^{n+2} at $x \in M$ is $T_xM \oplus \text{Span}\{N\}$, S^{n+2} is an invariant submanifold of S^{2m+1} because of (2.2) and (2.3). Therefore M can be regarded as a real hypersurface of S^{n+2} which is a totally geodesic invariant submanifold of S^{2m+1}. Hence, under the assumptions stated in Lemma 4.4, Lemma 4.1 implies that M is a real hypersurface of an odd-dimensional unit sphere S^{n+2} whose shape operator A has exactly 4 constant eigenvalues of multiplicities $1, 1, (n-1)/2, (n-1)/2$, respectively. Thus a theorem of Takagi [9](see also [10, Example 1.1, p. 159] and [7, Theorem 4.1, p. 239]) implies

Theorem 4.5. *Let M be an $(n+1)(n \geq 5)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in S^{2m+1} and let the distinguished normal vector field N be parallel with respect to the normal connection. If the equality* (3.1) *holds on M for a normal vector field ζ to M and $\rho \neq \lambda \pm \sqrt{\lambda^2 + 4}$, then M is locally a hypersurface $M'(n+1, t)$ of S^{n+2} defined by*

$$M'(n+1,t) := \{(z_1, \ldots, z_k) \in \mathbb{C}^k || \sum_{j=1}^{k} z_j^2|^2 = t,\ \sum_{j=1}^{k} |z_j|^2 = 1\}$$

where $k := (n+3)/2$.

Combining Theorem 4.3 and Theorem 4.5, we have

Theorem 4.6. *Let M be an $(n+1)(n \geq 5)$-dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in S^{2m+1} and let the distinguished normal vector field N be parallel with respect to the normal connection. If the equality* (3.1) *holds on M for a normal vector field ζ to M, then M is locally one of the following:*

(1) *a product $S^1 \times M_2$, where M_2 belongs to some sphere of odd-dimension.*

(2) *a hypersurface $M'(n+1, t)$ of S^{n+2} given in Theorem* 4.5.

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