TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 2, pp. 611-628, April 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

THE HYPER ORDER OF SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS AND SUBNORMAL SOLUTIONS OF PERIODIC EQUATIONS

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Abstract. In this paper, we obtain a precise estimation of the hyper order of solutions for a class of second order linear differential equations, and investigate the condition of the existence of nontrivial subnormal solution for a class of second order periodic equations. These results generalize the results of Gundersen, Steinbart [5] and Wittich [10]. Our methods of proofs are different from the methods applied in [5, 10].

1. Introduction and Results

Consider the second order homogeneous linear periodic differential equation

$$f'' + P_0(e^z)f' + Q_0(e^z)f = 0 \tag{1.1}$$

where $P_0(\zeta)$ and $Q_0(\zeta)$ are polynomials in $\zeta = e^z$ ($z \in \mathbf{C}$) and are not both constant. It is well known that every solution f of (1.1) is an entire function.

In this paper we will use standard notations from the value distribution theory of meromorphic functions (see [8, 9, 11]). We suppose that $f(z)$ is a meromorphic function in whole complex plane $\mathbf{C}$. The Nevanlinna characteristic of $f(z)$, denoted by $T(r, f)$, i.e.

$$T(r, f) = m(r, f) + N(r, f)$$

where

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r$$

Received January 20, 2007, accepted June 26, 2008.
Communicated by Yingfei Yi.
2000 *Mathematics Subject Classification*: 30D35, 34M10.
Key words and phrases: Periodic differential equation, Subnormal solution, Hyper order.
Project was supported by the Brain Pool Program of Korean Federation of Science and Technology Societies (No. 072-1-3-0164) and by the National Natural Science Foundation of China (No. 10871076) and by KOSEF, 2004, Project No. F01-2004-000-10262-0.

is the pole-counting contribution, where $n(r,f)$ is number of poles of f, including multiplicities, for $|z|\le r$. The proximity function $m(r,f)$ is given by

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta}|d\theta$$

where $\log^+ x=\max\{0,\log x\}$ for all $x>0$. In addition, we denote the order of growth of $f(z)$ by $\sigma(f)$, and also use the notation $\sigma_2(f)$ to denote the hyper-order of $f(z)$, is defined as

$$\sigma_2(f)=\overline{\lim_{r\to\infty}}\frac{\log\log T(r,f)}{\log r}.$$

Suppose $f\not\equiv 0$ is a solution of equation (1.1). If f satisfies the condition

$$\overline{\lim_{r\to\infty}}\frac{\log T(r,f)}{r}=0, \tag{1.2}$$

then we say that f is a nontrivial subnormal solution of (1.1) (see [5, 10]).

Wittich [10] investigated the subnormal solution of (1.1), and obtained the form of all subnormal solutions in the following theorem.

Theorem A. If $f(\not\equiv 0)$ is a subnormal solution of (1.1), then f must have the form

$$f(z)=e^{cz}(h_0+h_1e^z+\cdots+h_me^{mz}) \tag{1.3}$$

where $m\ge 0$ is an integer and $c,h_0,\cdots,h_m$ are constants with $h_0\ne 0$ and $h_m\ne 0$.

Gundersen and Steinbart [5] refined Theorem A and got the following theorem.

Theorem B. Under the assumption of Theorem A, the following statements hold.

(i) If $\deg P_0>\deg Q_0$ and $Q_0\not\equiv 0$, then any subnormal solution $f\not\equiv 0$ of (1.1) must have the form

$$f(z)=\sum_{k=0}^{m}h_ke^{-kz} \tag{1.4}$$

where $m\ge 1$ is an integer and $h_0,h_1,\cdots,h_m$ are constants with $h_0\ne 0$ and $h_m\ne 0$.

(ii) If $Q_0\equiv 0$ and $\deg P_0\ge 1$, then any subnormal solution of (1.1) must be a constant.

(iii) If $\deg P_0<\deg Q_0$, then the only subnormal solution of (1.1) is $f\equiv 0$.

For the subnormal solution, we are only interested in the non-trivial subnormal solution. Theorem B show that if $\deg P_0 < \deg Q_0$, then (1.1) must have no a non-trivial subnormal solution. But if $\deg P_0 > \deg Q_0$, then Theorem B shows that any non-trivial subnormal solution must have the form (1.4).

Thus three natural questions are:

(i) What condition will guarantee that (1.1) does not have a non-trivial subnormal solution under the condition $\deg P_0 > \deg Q_0$?

(ii) What condition will P_0 and Q_0 satisfy if (1.1) has a non-trivial subnormal solution under the condition $\deg P_0 > \deg Q_0$?

(iii) What can be said about the growth of all other solutions of (1.1) except subnormal solutions?

In this paper, we investigate more general equations than (1.1), and get the following theorems 1 and 2. For more general equations (1.7) and (1.8), we prove that all solutions satisfy $\sigma_2(f) = 1$. Applying these results to periodic differential equation (1.1), we obtain the following theorem 3 and corollaries 1 and 2. These results answer the above three questions. Theorem 3 shows that any one of three additional hypotheses (i)-(iii) in Theorem 3 guarantees that (1.1) has no non-trivial subnormal solution. Corollary 1 shows that if (1.1) with $n > s$, has a non-trivial subnormal solution, then the constant terms $c_0,\ d_0$ of $P_0,\ Q_0$ must satisfy (1.10). Corollary 2 gives a precise estimation of growth of all other solutions of (1.1) except subnormal solutions, and shows that, for any two linearly independent solutions of (1.1), at least one of them satisfies $\sigma_2(f) = 1$.

These results generalize the results of [5, 10]. Our methods of proofs are also different from the methods applied in [5, 10].

Now we let polynomials

$$(1.5)\quad a_j(z) = a_{jd_j} z^{d_j} + a_{j(d_j-1)} z^{d_j-1} + \cdots + a_{j1} z + a_{j0},\ (j = 0, 1, \cdots, n);$$

$$(1.6)\quad b_k(z) = b_{km_k} z^{m_k} + b_{k(m_k-1)} z^{m_k-1} + \cdots + b_{k1} z + b_{k0},\ (k = 0, 1, \cdots, s),$$

where $d_j \geq 0,\ m_k \geq 0\ (j = 1, \cdots, n,\ k = 1, \cdots, s)$ are integers, $a_{jd_j}, \cdots, a_{j0}$; $b_{km_k}, \cdots, b_{k0}$ are constants, $a_{jd_j} \neq 0,\ b_{km_k} \neq 0$.

Theorem 1. *Let* $a_n(z), \cdots, a_1(z), b_s(z), \cdots, b_1(z)$ *be polynomials and satisfy* (1.5) *and* (1.6)*, and* $a_n(z)b_s(z) \not\equiv 0$*. Suppose that*

$$P(e^z) = a_n(z)e^{nz} + \cdots + a_1(z)e^z, \quad Q(e^z) = b_s(z)e^{sz} + \cdots + b_1(z)e^z.$$

If $n \neq s$*, then every solution* $f(\not\equiv 0)$ *of equation*

$$(1.7)\qquad f'' + P(e^z)f' + Q(e^z)f = 0$$

satisfies $\sigma_2(f)=1$.

Theorem 2. *Let* $a_n(z),\cdots,a_1(z),a_0(z),b_s(z),\cdots,b_1(z),b_0(z)$ *be polynomials and satisfy* (1.5) *and* (1.6)*, and* $a_n(z)b_s(z)\not\equiv 0$. *Suppose that*

$$P^*(e^z)=a_n(z)e^{nz}+\cdots+a_1(z)e^z+a_0(z),\quad Q^*(e^z)=b_s(z)e^{sz}+\cdots+b_1(z)e^z+b_0(z).$$

If $n<s$*, then every solution* $f(\not\equiv 0)$ *of equation*

$$f''+P^*(e^z)f'+Q^*(e^z)f=0 \tag{1.8}$$

satisfies $\sigma_2(f)=1$.

Theorem 3. *Let*

$$P_0(e^z)=c_ne^{nz}+\cdots+c_1e^z+c_0,\ Q_0(e^z)=d_se^{sz}+\cdots+d_1e^z+d_0,\ c_nd_s\neq 0, \tag{1.9}$$

where $c_j,\ d_k\ (j=0,1,\cdots,n;\ k=0,1,\cdots,s)$ *are constants. Suppose that* P_0 *and* Q_0 *satisfy any one of the following three additional hypotheses:*

(i) $s>n$;

(ii) $n>s$ *and* $c_0=d_0=0$;

(iii) $n>s$ *and equation* $x^2-c_0x+d_0=0$ *has no positive integer solution.*

Then (1.1) *has no non-trivial subnormal solution, and every non-trivial solution* f *satisfies* $\sigma_2(f)=1$.

Corollary 1. *Suppose that* $P_0(e^z)$ *and* $Q_0(e^z)$ *satisfy (1.9) and* $n>s$, $c_0=c_{01}+c_{02}i$ *and* $d_0=d_{01}+d_{02}i$*, where* $c_{01},\ c_{02},\ d_{01},\ d_{02}$ *are real constants. If (1.1) has a non–trivial subnormal solution, then* c_0 *and* d_0 *satisfy*

$$\begin{cases} d_{02}=c_{02}m \\ m^2-c_{01}m+d_{01}=0 \end{cases} \tag{1.10}$$

where m *is a positive integer and* $m\geq n-s$.

Wittich [10] showed that of any two linearly independent solutions of (1.1) at most one of them can be subnormal. By the result of Wittich, we can get the following corollary 2.

Corollary 2. *Suppose that* $P_0(e^z)$ *and* $Q_0(e^z)$ *satisfy (1.9). If* f_1 *and* f_2 *are two linearly independent solutions of (1.1), then at least one of* f_1 *and* f_2*, say* f_2*, satisfies* $\sigma_2(f_2)=1$.

Example 1. The equation

$$f'' + \Big(- \frac{1}{2}e^{3z} + \frac{1}{2}e^{z} + 2\Big) f' + e^{z} f = 0$$

has a subnormal solution $f_0 = e^{-2z} - 1$, and all subnormal solutions must have the form $f = cf_0$ where c is any constant. Here $c_0 = 2$ and $d_0 = 0$, the equation

$$x^2 - c_0 x + d_0 = 0$$

has a positive integer solution $x = 2 = m = n - s$.

Example 2. The equation

$$f'' + \big(- e^{2z} + 2\big) f' + \big(e^{z} + 1\big) f = 0$$

has a subnormal solution $f_0 = e^{-z} - 1$, and all subnormal solutions must have the form $f = cf_0$ where c is any constant. Here $c_0 = 2$ and $d_0 = 1$, equation

$$x^2 - 2x + 1 = 0$$

has a positive integer solution $x = 1 = m = n - s$.

2. Lemmas for Proofs of Theorems

Lemma 1. [7]. *Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta} (n = 1, 2, \ldots)$, where $r_n \to \infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\left| \frac{f^{(j)}(z_n)}{f^{(k)}(z_n)} \right| \le |z_n|^{k-j}(1 + o(1)) \ (j = 0, \ldots, k - 1). \tag{2.1}$$

Lemma 2. [6]. *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$. Let $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \ge 0$, for $i = 1, \ldots, q$. And let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi)\backslash E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \ge R_0$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.2}$$

Remark 1. Obviously, in Lemma 2, if $\psi \in [0, 2\pi)\backslash E$ is replaced by $\psi \in [-\frac{\pi}{2}, \frac{3\pi}{2})\backslash E$, then (2.2) still holds.

Lemma 3. [2]. *Let $f(z)$ be an entire function with $\sigma(f) = \sigma < \infty$. Suppose that there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for any ray* $\arg z = \theta_0 \in [0, 2\pi)\backslash E, |f(re^{i\theta_0})| \le Mr^k (M = M(\theta_0) > 0$ *is a constant and* $k(>0)$ *is a constant independent of* θ_0*, then $f(z)$ is a polynomial with* $\deg f \le k$.

Lemma 4. [1]. *Let A, B be entire functions of finite order. If $f(z)$ is a solution of the equation*

$$f'' + Af' + Bf = 0, \tag{2.3}$$

then $\sigma_2(f) \le \max\{\sigma(A), \sigma(B)\}$.

Lemma 5. [3]. *Let $g(z)$ be an entire function of infinte order with the hyper-order $\sigma_2(g) = \sigma$ and let $\nu(r)$ be the central index of g. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log\log \nu(r)}{\log r} = \sigma. \tag{2.4}$$

Using a similar proof as in the proof of Remark 1 of [1], we can obtain the following lemma 6.

Lemma 6. *Let $f(z)$ be an entire function of infinite order with $\sigma_2(f) = \alpha$ $(0 \le \alpha < \infty)$ and a set $E \subset [1, \infty)$ have finite logarithmic measure. Then there exists $\{z_k = r_k e^{i\theta_k}\}$ such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $r_k \notin E$, $r_k \to \infty$, and such that*

(*i*) *if $\sigma_2(f) = \alpha$ $(0 < \alpha < \infty)$, then for any given ε_1 $(0 < \varepsilon_1 < \alpha)$,*

$$\exp\{r_k^{\alpha-\varepsilon_1}\} < \nu(r_k) < \exp\{r_k^{\alpha+\varepsilon_1}\}; \tag{2.5}$$

(*ii*) *if $\sigma(f) = \infty$ and $\sigma_2(f) = 0$, then for any given ε_2 $(0 < \varepsilon_2 < \frac{1}{2})$ and for any large $M(>0)$, we have as r_k sufficiently large*

$$r_k^M < \nu(r_k) < \exp\{r_k^{\varepsilon_2}\}. \tag{2.6}$$

Proof. By Lemma 5, $\sigma(f) = \infty$ and $\sigma_2(f) = \alpha$, we have

$$\overline{\lim_{r\to\infty}} \frac{\log \nu(r)}{\log r} = \infty, \quad \overline{\lim_{r\to\infty}} \frac{\log\log \nu(r)}{\log r} = \sigma_2(f) = \alpha < \infty.$$

There is a sequence $\{r'_k\}(r'_k \to \infty)$ satisfying

$$\lim_{r'_k\to\infty} \frac{\log \nu(r'_k)}{\log r'_k} = \infty, \quad \lim_{r'_k\to\infty} \frac{\log\log \nu(r'_k)}{\log r'_k} = \alpha.$$

Set the logarithmic measure of E, $lmE = \delta < \infty$. Then there is a point $r_k \in [r'_k, (\delta+1)r'_k]\backslash E$. Since

$$\frac{\log\log \nu(r_k)}{\log r_k} \geq \frac{\log\log \nu(r'_k)}{\log[(\delta+1)r'_k]} = \frac{\log\log \nu(r'_k)}{\log r'_k[1 + \frac{\log(\delta+1)}{\log r'_k}]},$$

we have

$$\lim_{r_k\to\infty} \frac{\log \nu(r_k)}{\log r_k} = \infty, \quad \lim_{r_k\to\infty} \frac{\log\log \nu(r_k)}{\log r_k} = \alpha. \tag{2.7}$$

Now we take $z_k = r_k e^{i\theta_k}$ and $\theta_k \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, such that $|f(z_k)| = M(r_k, f)$. Thus there exists a subset of $\{\theta_k\}$. For convenience, we still suppose that the subset is $\{\theta_k\}$. Then it satisfies $\lim_{k\to\infty} \theta_k = \theta_0 \in [-\frac{\pi}{2}, \frac{3\pi}{2})$. Thus we easily obtain (2.5) and (2.6) by (2.7), i.e. (i) and (ii) hold.

Lemma 7. [6]. *Let f be a transcendental meromorphic function and let $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and i, j $(0 \leq i < j \leq 2)$, such that for all z satisfying $|z| = r \notin [0, 1] \bigcup E$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B \left(\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f)\right)^{j-i}. \tag{2.8}$$

Remark 2. From the proof of Lemma 7 (i.e. Theorem 3 in [6]), we can see that the exceptional set E satisfies that if a_n and b_m $(n, m = 1, 2, \cdots)$ denote all zeros and poles of f respectively and if $O(a_n)$ and $O(b_m)$ denote sufficiently small neighborhoods of a_n and b_m respectively, then

$$E = \left\{|z| : z \in \left(\bigcup_{n=1}^{+\infty} O(a_n)\right) \bigcup \left(\bigcup_{m=1}^{+\infty} O(b_m)\right)\right\}.$$

Hence, if $f(z)$ is a transcendental entire function, and z is a point that satisfies $|f(z)|$ is sufficiently large, then (2.8) holds.

Lemma 8. [4] *Let equation*

$$w^{(k)} + a_{k-1} w^{(k-1)} + \cdots + a_0 w = 0$$

be satisfied in the complex plane by linearly independent meromorphic functions $f_1, \cdots, f_k$. *Then the coefficients* a_j $(j = 0, \cdots, k-1)$ *are meromorphic in the plane satisfying the properties*

$$m(r, a_j) = O\{\log[\max(T(r, f_s) : s = 1, \cdots, k)]\}.$$

Lemma 9. [9, p. 5]. *Let* $g : (0, +\infty) \to R$ *and* $h : (0, +\infty) \to R$ *be monotone increasing functions such that* $g(r) \le h(r)$ *outside of an exceptional set* E *of finite logarithmic measure. Then, for any* $\alpha > 1$*, there exists* $r_0 > 0$ *such that* $g(r) \le h(\alpha r)$ *holds for all* $r > r_0$.

3. Proof of Theorem 2

Suppose that $f (\not\equiv 0)$ is a solution of (1.8), then f is an entire function. Since $\sigma(P^*) = \sigma(Q^*) = 1$, by Lemma 4, we see that

$$\sigma_2(f) \le \max\{\sigma(P^*),\ \sigma(Q^*)\} = 1. \tag{3.1}$$

By Lemma 7, we see that there exist a subset $E \subset (1, \infty)$ having logarithmic measure $lmE < \infty$, and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \bigcup E$, we have

$$\left|\frac{f''(z)}{f(z)}\right| \le B[T(2r, f)]^3, \quad \left|\frac{f'(z)}{f(z)}\right| \le B[T(2r, f)]^2. \tag{3.2}$$

Taking $z = r$, by (1.5) and (1.6), we obtain that for sufficiently large r

$$|P^*(z)| = |a_n(r)e^{nr} + \cdots + a_1(r)e^r + a_0(r)| \le 2|a_{nd_n}|r^{d_n}e^{nr}(1 + o(1)); \tag{3.3}$$

$$|Q^*(z)| = |b_s(r)e^{sr} + \cdots + b_1(r)e^r + b_0(r)| \ge \frac{1}{2}|b_{sm_s}|r^{m_s}e^{sr}(1 + o(1)). \tag{3.4}$$

Now taking $z = r \notin [0, 1] \bigcup E$, by (1.8) and (3.2)-(3.4), we deduce that

$$\begin{aligned} & \frac{1}{2}|b_{sm_s}|r^{m_s}e^{sr}(1 + o(1)) \\ \le\ & |-Q^*(z)| = \left|\frac{f''(z)}{f(z)} + P^*(z)\frac{f'(z)}{f(z)}\right| \\ \le\ & \left|\frac{f''(z)}{f(z)}\right| + |P^*(z)|\left|\frac{f'(z)}{f(z)}\right| \\ \le\ & B[T(2r, f)]^3 + 2|a_{nd_n}|r^{d_n}e^{nr}B[T(2r, f)]^2(1 + o(1)) \\ \le\ & 4B[T(2r, f)]^3|a_{nd_n}|r^{d_n}e^{nr}(1 + o(1)). \end{aligned} \tag{3.5}$$

By (3.5), we get when $z = r \notin [0,1] \bigcup E$;

$$\frac{1}{2}|b_{sm_s}|r^{m_s}e^{(s-n)r}(1+o(1)) \le 4B[T(2r,f)]^3|a_{nd_n}|r^{d_n}(1+o(1)). \tag{3.6}$$

Since $s - n > 0$, by (3.6) and Lemma 9, we get

$$\sigma_2(f) \ge \varlimsup_{\substack{r\to\infty \\ r\notin[0,1]\bigcup E}} \frac{\log\log T(r,f)}{\log r} = 1. \tag{3.7}$$

By (3.1) and (3.7), we have $\sigma_2(f) = 1$. Thus Theorem 2 is proved.

4. Proof of Theorem 1

If $s > n$, by Theorem 2, we have $\sigma_2(f) = 1$.

Now we suppose $n > s$. Suppose that $f(\not\equiv 0)$ is a solution of (1.7), then f is an entire function.

First step. We prove that $\sigma(f) = \infty$. If f is a polynomial, then we take $z = r$, by observing the growth of the left side of (1.1), we can obtain a contradiction. Now we assume f is transcendental with $\sigma(f) = \sigma < \infty$.

By Lemma 2 and Remark 1, we know that for any given $\varepsilon > 0$, there exists a set $E \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ having linear measure zero, such that if $\psi \in [-\frac{\pi}{2}, \frac{3\pi}{2})\backslash E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| = r > R_0$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \le r^{\sigma-1+\varepsilon}, \quad \left|\frac{f''(z)}{f(z)}\right| \le r^{2(\sigma-1+\varepsilon)}, \quad \left|\frac{f'(z)}{f(z)}\right| \le r^{\sigma-1+\varepsilon}. \tag{4.1}$$

Now we take a ray $\arg z = \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\backslash E$. Then we have $\cos\theta > 0$. We assert that $|f'(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f'(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 1, there exists a sequence $\{z_q = r_q e^{i\theta}\}$ such that as $r_q \to \infty$, $f'(z_q) \to \infty$ and

$$\left|\frac{f(z_q)}{f'(z_q)}\right| \le r_q(1+o(1)). \tag{4.2}$$

By (1.7), we get that

$$\begin{aligned} &-(a_n(z_q)e^{nz_q} + \cdots + a_1(z_q)e^{z_q}) \\ &= \frac{f''(z_q)}{f'(z_q)} + (b_s(z_q)e^{sz_q} + \cdots + b_1(z_q)e^{z_q})\frac{f(z_q)}{f'(z_q)}. \end{aligned} \tag{4.3}$$

Since when $r_q \to \infty$,

$$
\begin{aligned}
(4.4)\qquad & |a_n(z_q)e^{nz_q} + \cdots + a_1(z_q)e^{z_q}| \\
& \geq |a_n(z_q)e^{nz_q}| - \left[\left|a_{n-1}(z_q)e^{(n-1)z_q}\right| + \cdots + |a_1(z_q)e^{z_q}|\right] \\
& = |a_{nd_n}|r^{d_n}(1+o(1))e^{nr_q\cos\theta} \\
& \quad - \left[|a_{n-1,d_n-1}|r_q^{d_n-1}e^{(n-1)r_q\cos\theta}(1+o(1))\right. \\
& \quad \left. + \cdots + |a_{1d_1}|r^{d_1}(1+o(1))e^{r_q\cos\theta}\right] \\
& = |a_{nd_n}|r^{d_n}(1+o(1))e^{nr_q\cos\theta}(1+o(1))
\end{aligned}
$$

and

$$
(4.5)\qquad |b_s(z_q)e^{sz_q} + \cdots + b_1(z_q)e^{z_q}| \leq |b_{sm_s}|r_q^{m_s}e^{sr_q\cos\theta}(1+o(1)),
$$

by substituting (4.1), (4.2), (4.4) and (4.5) into (4.3), we obtain that

$$
(4.6)\quad |a_{nd_n}|r^{d_n}e^{nr_q\cos\theta}(1+o(1)) \leq |b_{sm_s}|r_q^{m_s}e^{sr_q\cos\theta}r_q(1+o(1)) + r_q^{\sigma-1+\varepsilon}.
$$

By $n > s$, we know that when $r_q \to \infty$, (4.6) is a contradiction. Hence when $\arg z = \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\backslash E$, we have $|f'(re^{i\theta})| \leq M$, so, on the ray $\arg z = \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\backslash E$,

$$
(4.7)\qquad |f(re^{i\theta})| \leq Mr.
$$

Now we take a ray $\arg z = \theta \in (\frac{\pi}{2}, \frac{3\pi}{2})\backslash E$, then $\cos\theta < 0$. We assert that $|f''(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f''(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 1, there exists a sequence $\{z_q = r_q e^{i\theta}\}$ such that as $r_q \to \infty$, $f''(z_q) \to \infty$ and

$$
(4.8)\qquad \left|\frac{f'(z_q)}{f''(z_q)}\right| \leq r_q(1+o(1)), \quad \left|\frac{f(z_q)}{f''(z_q)}\right| \leq r_q^2(1+o(1)).
$$

By (1.7), we get that

$$
\begin{aligned}
(4.9)\qquad -1 &= \frac{f'(z_q)}{f''(z_q)}\left(a_n(z_q)e^{nz_q} + \cdots + a_1(z_q)e^{z_q}\right) \\
& \quad + \frac{f(z_q)}{f''(z_q)}\left(b_s(z_q)e^{sz_q} + \cdots + b_1(z_q)e^{z_q}\right).
\end{aligned}
$$

Since when $r_q \to \infty$,

$$
\begin{aligned}
(4.10)\qquad & |a_n(z_q)e^{nz_q} + \cdots + a_1(z_q)e^{z_q}| \\
& \leq |a_{nd_n}|r_q^{d_n}e^{nr_q\cos\theta}(1+o(1)) + \cdots + |a_{1d_1}|r_q^{d_1}e^{r_q\cos\theta}(1+o(1))
\end{aligned}
$$

and

$$\tag{4.11} \begin{aligned} &|b_s(z_q)e^{sz_q} + \cdots + b_1(z_q)e^{z_q}| \\ &\le |b_{sm_s}|r_q^{m_s}e^{sr_q\cos\theta}(1+o(1)) + \cdots + |b_{1m_1}|r_q^{m_1}e^{r_q\cos\theta}(1+o(1)). \end{aligned}$$

Substituting (4.8), (4.10) and (4.11) into (4.9), we obtain that

$$\tag{4.12} \begin{aligned} 1 \le\ & r_q(1+o(1))\Big[|a_{nd_n}|r_q^{d_n}e^{nr_q\cos\theta}(1+o(1)) \\ & + \cdots + |a_{1d_1}|r_q^{d_1}e^{r_q\cos\theta}(1+o(1))\Big] \\ & + r_q^2(1+o(1))\Big[|b_{sm_s}|r_q^{m_s}e^{sr_q\cos\theta}(1+o(1)) \\ & + \cdots + |b_{1m_1}|r_q^{m_1}e^{r_q\cos\theta}(1+o(1))\Big]. \end{aligned}$$

Since $\cos\theta < 0$, when $r_q \to \infty$, by (4.12), we get $1 \le 0$. This is a contradiction. Hence $|f''(re^{i\theta})| \le M$ on the ray $\arg z = \theta \in (\frac{\pi}{2}, \frac{3\pi}{2})\backslash E$. So, on the ray $\arg z = \theta \in (\frac{\pi}{2}, \frac{3\pi}{2})\backslash E$, we have

$$\tag{4.13} |f(re^{i\theta})| \le Mr^2.$$

Since the linear measure of $E\bigcup\{-\frac{\pi}{2}, \frac{\pi}{2}\}$ is zero, by Lemma 3, (4.7) and (4.13), we know that $f(z)$ is a polynomial. This contradicts our assumption that $f(z)$ is transcendental. Therefore $\sigma(f) = \infty$.

Second step. We prove that $\sigma_2(f) = 1$. By Lemma 4 and $\sigma(P) = \sigma(Q) = 1$, we see that

$$\tag{4.14} \sigma_2(f) \le \max\{\sigma(P),\ \sigma(Q)\} = 1.$$

Set $\sigma_2(f) = \alpha$, then $\alpha \le 1$. We assert that $\alpha = 1$. Now we assume that $\alpha < 1$ and prove that $\sigma_2(f) = \alpha < 1$ is failing.

By the Wiman-Valiron theory (see [9, p.51]), there is a set $E_1 \subset (1, \infty)$ having logarithmic measure $lmE_1 < \infty$, such that we can choose a z satisfying $|z| = r \notin [0,1]\bigcup E_1$ and $|f(z)| = M(r, f)$, then we get

$$\tag{4.15} \frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r)}{z}\right)^j (1+o(1)), \quad j = 1, 2,$$

where $\nu(r)$ is the central index of $f(z)$.

By Lemma 6, we see that there exists a sequence $\{z_k = r_k e^{i\theta_k}\}$ such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $\lim \theta_k = \theta_0 \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $r_k \notin [0,1]\bigcup E_1$, r_k

$\to \infty$, and if $\alpha > 0$, then for any given ε_1 $(0 < \varepsilon_1 < \min\{\alpha,\ 1-\alpha\})$ and for sufficiently large r_k, we get by (2.5) that

$$\exp\{r_k^{\alpha-\varepsilon_1}\} < \nu(r_k) < \exp\{r_k^{\alpha+\varepsilon_1}\}. \tag{4.16}$$

If $\alpha = 0$, then by $\sigma(f) = \infty$ and (2.6), we see that for any given ε_2 $(0 < \varepsilon_2 < \frac{1}{2})$ and for any sufficiently large $M > 0$, we have for sufficiently large r_k,

$$r_k^M < \nu(r_k) < \exp\{r_k^{\varepsilon_2}\}. \tag{4.17}$$

Since θ_0 may belong to $(-\frac{\pi}{2},\ \frac{\pi}{2})$, or $(\frac{\pi}{2},\ \frac{3\pi}{2})$, or $\{-\frac{\pi}{2},\ \frac{\pi}{2}\}$, we divide this proof into three cases to prove.

Case 1. Suppose $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. By $\cos\theta_0 > 0$ and $\theta_k \to \theta_0$, we see that for sufficiently large k, we have $\cos\theta_k > 0$. By (1.7) and (4.15), we get for sufficiently large r_k,

$$-\frac{f'(z_k)}{f(z_k)}\,(a_n(z_k)e^{nz_k} + \cdots + a_1(z_k)e^{z_k}) = \frac{f''(z_k)}{f(z_k)} + (b_s(z_k)e^{sz_k} + \cdots + b_1(z_k)e^{z_k})$$

and

$$\begin{aligned} &\frac{\nu(r_k)}{r_k}|a_n(z_k)|e^{nr_k\cos\theta_k}(1+o(1)) \\ \le\ &\frac{\nu^2(r_k)}{r_k^2}(1+o(1)) + |b_s(z_k)|e^{sr_k\cos\theta_k}(1+o(1)). \end{aligned} \tag{4.18}$$

If $\alpha > 0$, then by (4.16) and (4.18), we get for sufficiently large r_k,

$$\begin{aligned} &\exp\{r_k^{\alpha-\varepsilon_1}\}r_k^{-1}|a_n(z_k)|e^{nr_k\cos\theta_k}(1+o(1)) \\ \le\ &\exp\{2r_k^{\alpha+\varepsilon_1}\}|b_s(z_k)|e^{sr_k\cos\theta_k}(1+o(1)). \end{aligned} \tag{4.19}$$

Since $n > s,\ \alpha+\varepsilon_1 < 1$ and $\cos\theta_k > 0$, we see (4.19) is a contradiction. If $\alpha = 0$ then by (4.17) and (4.18), we get for sufficiently large r_k,

$$\begin{aligned} &r_k^{M-1}|a_n(z_k)|e^{nr_k\cos\theta_k}(1+o(1)) \\ \le\ &\exp\{2r_k^{\varepsilon_2}\}r_k^{-2}|b_s(z_k)|e^{sr_k\cos\theta_k}(1+o(1)). \end{aligned} \tag{4.20}$$

Since $n > s,\ \varepsilon_2 < \frac{1}{2}$ and $\cos\theta_k > 0$, we see (4.20) is also a contradiction.

Case 2. Suppose $\theta_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$. By $\cos\theta_0 < 0$ and $\theta_k \to \theta_0$, we see that for sufficiently large k, we have $\cos\theta_k < 0$. By (1.7), (4.15) and $\cos\theta_k < 0$, we get for sufficiently large r_k,

$$e^{-nz_k}\frac{f''(z_k)}{f(z_k)} = \frac{f'(z_k)}{f(z_k)}\left(a_n(z_k)+\cdots+a_1(z_k)e^{-(n-1)z_k}\right) + \left(b_s(z_k)e^{-(n-s)z_k}+\cdots+b_1(z_k)e^{-(n-1)z_k}\right)$$

and

$$\begin{aligned}&\frac{\nu^2(r_k)}{r_k^2}e^{-nr_k\cos\theta_k}(1+o(1))\\ &\le (|a_n(z_k)|+\cdots+|a_1(z_k)|)e^{-(n-1)r_k\cos\theta_k}\frac{\nu(r_k)}{r_k}(1+o(1))\\ &\quad+(|b_s(z_k)|+\cdots+|b_1(z_k)|)e^{-(n-1)r_k\cos\theta_k}(1+o(1)).\end{aligned}\tag{4.21}$$

By (4.16) and (4.21) (if $\alpha > 0$), or (4.17) and (4.21) (if $\alpha = 0$), we get for sufficiently large r_k, respectively,

$$\begin{aligned}&\exp\{2r_k^{\alpha-\varepsilon_1}\}r_k^{-2}e^{-nr_k\cos\theta_k}(1+o(1))\\ &\le (|a_n(z_k)|+\cdots+|a_1(z_k)|)e^{-(n-1)r_k\cos\theta_k}\exp\{r_k^{\alpha+\varepsilon_1}\}\\ &\quad+(|b_s(z_k)|+\cdots+|b_1(z_k)|)e^{-(n-1)r_k\cos\theta_k}(1+o(1))\quad (if\ \alpha>0)\end{aligned}\tag{4.22}$$

or

$$\begin{aligned}&e^{-nr_k\cos\theta_k}r_k^{2(M-1)}(1+o(1))\\ &\le (|a_n(z_k)|+\cdots+|a_1(z_k)|)e^{-(n-1)r_k\cos\theta_k}\exp\{r_k^{\varepsilon_2}\}\\ &\quad+(|b_s(z_k)|+\cdots+|b_1(z_k)|)e^{-(n-1)r_k\cos\theta_k}(1+o(1))\ (if\ \alpha=0).\end{aligned}\tag{4.23}$$

Since $a_1,\cdots,a_n,b_1,\cdots,b_s$ are polynomials, $\alpha+\varepsilon_1<1$ (or $\varepsilon_2<\frac{1}{2}$) and $-nr_k\cos\theta_k > -(n-1)r_k\cos\theta_k > 0$, we see that both (4.22) and (4.23) are absurd.

Case 3. Suppose that $\theta_0 = \frac{\pi}{2}$ or $\theta_0 = -\frac{\pi}{2}$. Since the proof for $\theta_0 = -\frac{\pi}{2}$ is the same as the proof for $\theta_0 = \frac{\pi}{2}$, we only prove the case that $\theta_0 = \frac{\pi}{2}$. Since $\theta_k \to \theta_0$, for any given ε_3 $(0<\varepsilon_3<\frac{1}{10})$, we see that there is an integer $K(>0)$, as $k>K$, $\theta_k \in [\frac{\pi}{2}-\varepsilon_3,\ \frac{\pi}{2}+\varepsilon_3]$, and

$$z_k = r_ke^{i\theta_k} \in \overline{\Omega} = \left\{z : \frac{\pi}{2}-\varepsilon_3 \le \arg z \le \frac{\pi}{2}+\varepsilon_3\right\}.\tag{4.24}$$

By Lemma 7, we see that there exist a subset $E_2 \subset (1,\infty)$ having logarithmic measure $lmE_2 < \infty$, and a constant $B>0$ such that for all z satisfying $|z| = r \notin [0,1]\bigcup E_2$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \le B[T(2r,f')]^2.\tag{4.25}$$

Now we consider the property of $f(re^{i\theta})$ on a ray $\arg = \theta \in \overline{\Omega}\backslash\{\frac{\pi}{2}\}$. If $\theta \in [\frac{\pi}{2} - \varepsilon_3,\ \frac{\pi}{2})$, then $\cos\theta > 0$. We assert that $|f'(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f'(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 1, there exists a sequence $\{u_j = R_j e^{i\theta}\}$ such that as $R_j \to \infty, \quad f'(u_j) \to \infty$ and

$$\left|\frac{f(u_j)}{f'(u_j)}\right| \leq R_j(1 + o(1)). \tag{4.26}$$

Since Remark 2 and $f'(u_j) \to \infty$, we know that $|u_j| = R_j \notin E_2$. By (4.25), we have for sufficiently large j,

$$\left|\frac{f''(u_j)}{f'(u_j)}\right| \leq B[T(2R_j, f')]^2. \tag{4.27}$$

Since $a_1, \cdots, a_n$ are polynomials, by (1.7), (4.26) and (4.27), we deduce that for any given ε_4 $(0 < \varepsilon_4 < 1 - \alpha)$

$$\begin{aligned}
&\frac{1}{2}|a_n(u_j)|e^{nR_j\cos\theta}(1 + o(1)) \\
&\leq |a_n(u_j)e^{nu_j} + \cdots + a_1(u_j)e^{u_j}| \\
&\leq \left|\frac{f''(u_j)}{f'(u_j)}\right| + |b_s(u_j)e^{su_j} + \cdots + b_1(u_j)e^{u_j}|\left|\frac{f(u_j)}{f'(u_j)}\right| \\
&\leq B[T(2R_j, f')]^2 + [|b_s(u_j)| + \cdots + |b_1(u_j)|]\, e^{sR_j\cos\theta}R_j(1 + o(1)) \\
&\leq e^{2R_j^{\alpha+\varepsilon_4}} + [|b_s(u_j)| + \cdots + |b_1(u_j)|]\, e^{sR_j\cos\theta}R_j(1 + o(1)).
\end{aligned} \tag{4.28}$$

Since $n > s$, $\alpha + \varepsilon_4 < 1$ and $b_1, \cdots, b_s$ are polynomials, we know that when $R_j \to \infty$, (4.28) is a contradiction. Hence $|f'(re^{i\theta})|$ is bounded on the ray $\arg z = \theta \in [\frac{\pi}{2} - \varepsilon_3,\ \frac{\pi}{2})$. Set $|f'(re^{i\theta})| \leq M$, then on the ray $\arg z = \theta \in [\frac{\pi}{2} - \varepsilon_3,\ \frac{\pi}{2})$

$$|f(re^{i\theta})| \leq Mr. \tag{4.29}$$

If $\theta \in (\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon_3]$, then $\cos\theta < 0$. We assert that $|f''(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f''(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 1, there exists a sequence $\{u_j^* = R_j^* e^{i\theta}\}$ such that as $R_j^* \to \infty, \quad f''(u_j^*) \to \infty$ and

$$\left|\frac{f'(u_j^*)}{f''(u_j^*)}\right| \leq R_j^*(1 + o(1)), \quad \left|\frac{f(u_j^*)}{f''(u_j^*)}\right| \leq (R_j^*)^2(1 + o(1)). \tag{4.30}$$

Since $\cos\theta < 0$ and $a_1, \cdots, a_n, b_1, \cdots, b_s$ are polynomials, by (1.7) and (4.30), we deduce that as $R_j^* \to \infty$

$$
\begin{aligned}
1 \le\ & \left|a_n(u_j^*)e^{nu_j^*} + \cdots + a_1(u_j^*)e^{u_j^*}\right| \left|\frac{f'(u_j^*)}{f''(u_j^*)}\right| \\
& + \left|b_s(u_j^*)e^{su_j^*} + \cdots + b_1(u_j^*)e^{u_j^*}\right| \left|\frac{f(u_j^*)}{f''(u_j^*)}\right| \\
\le\ & \left\{|a_n(u_j^*)|e^{nR_j^*\cos\theta} + \cdots + |a_1(u_j^*)|e^{R_j^*\cos\theta}\right\} R_j^*(1+o(1)) \\
& + \left\{|b_s(u_j^*)|e^{sR_j^*\cos\theta} + \cdots + |b_1(u_j^*)|e^{R_j^*\cos\theta}\right\} (R_j^*)^2(1+o(1)) \\
\to\ & 0.
\end{aligned} \tag{4.31}
$$

Thus (4.31) is a contradiction. Hence $|f''(re^{i\theta})|$ is bounded on the ray $\arg z = \theta \in (\frac{\pi}{2},\ \frac{\pi}{2} + \varepsilon_3]$. Set $|f''(re^{i\theta})| \le M$, then on the ray $\arg z = \theta \in (\frac{\pi}{2},\ \frac{\pi}{2} + \varepsilon_3]$

$$|f(re^{i\theta})| \le Mr^2. \tag{4.32}$$

By (4.29) and (4.32), we see that $|f(re^{i\theta})|$ satisfies on the ray $\arg z = \theta \in \overline{\Omega}\backslash\{\frac{\pi}{2}\}$

$$|f(re^{i\theta})| \le Mr^2. \tag{4.33}$$

But since $f(z)$ is of infinite order and every point z_k of $\{z_k = r_k e^{i\theta_k}\}$ satisfies $|f(z_k)| = M(r_k, f)$, we see that for any large N (> 2), as k sufficiently large,

$$|f(z_k)| = |f(r_k e^{i\theta_k})| \ge \exp\{r_k^N\}. \tag{4.34}$$

Since $z_k \in \overline{\Omega}$, by (4.33) and (4.34), we see that for sufficiently large k

$$\theta_k = \frac{\pi}{2}.$$

Thus $\cos\theta_k = 0$ and

$$|a_n(z_k)e^{nz_k} + \cdots + a_1(z_k)e^{z_k}| \le r^C,\ \ |b_s(z_k)e^{sz_k} + \cdots + b_1(z_k)e^{z_k}| \le r^C, \tag{4.35}$$

where C (> 0) is some constant. By (1.7) and (4.15), we obtain that

$$
\begin{aligned}
-\left(\frac{\nu(r_k)}{z_k}\right)^2 (1+o(1)) =\ & (a_n(z_k)e^{nz_k} + \cdots + a_1(z_k)e^{z_k})\,\frac{\nu(r_k)}{z_k}(1+o(1)) \\
& + (b_s(z_k)e^{sz_k} + \cdots + b_1(z_k)e^{z_k}).
\end{aligned} \tag{4.36}
$$

By (4.35) and (4.36), we obtain that

$$\nu(r_k) \le 3r_k^C. \tag{4.37}$$

By (4.16) (or (4.17)), we see that (4.37) is a contradiction. Theorem 1 is thus proved.

5. Proof of Theorem 3

By Theorems 1 and 2, we know that if $P_0,\ Q_0$ satisfy (i) or (ii), then all solutions of (1.1) satisfy $\sigma_2(f)=1$. Thus by Theorems A and B, we know that if $P_0,\ Q_0$ satisfy (i) or (ii), then (1.1) has no nontrivial subnormal solution.

Now suppose P_0 and Q_0 satisfy (iii) and f_0 is a non-trivial subnormal solution of (1.1), by Theorem B, we see that as $n>s\geq 1,\ f_0$ must have the form

$$f_0(z)=h_0+h_1e^{-z}+\cdots+h_me^{-mz}, \tag{5.1}$$

where $m\geq 1$ is an integer and $h_0,\cdots,h_m$ are constants with $h_0\neq 0$ and $h_m\neq 0$.

Substituting (5.1) into (1.1), we get

$$\begin{aligned} & h_1e^{-z}+2^2h_2e^{-2z}+\cdots+m^2h_me^{-mz}\\ & -\big[c_nh_1e^{(n-1)z}+c_{n-1}h_1e^{(n-2)z}+\cdots+c_0h_1e^{-z}\\ & +2c_nh_2e^{(n-2)z}+2c_{n-1}h_2e^{(n-3)z}+\cdots+2c_0h_2e^{-2z}\\ & +\cdots\\ & +mc_nh_me^{(n-m)z}+mc_{n-1}h_me^{(n-m-1)z}+\cdots+mc_0h_me^{-mz}\big]\\ & +\big[d_sh_0e^{sz}+d_{s-1}h_0e^{(s-1)z}+\cdots+d_0h_0\\ & +d_sh_1e^{(s-1)z}+d_{s-1}h_1e^{(s-2)z}+\cdots+d_0h_1e^{-z}\\ & +\cdots\\ & +d_sh_me^{(s-m)z}+d_{s-1}h_me^{(s-m-1)z}+\cdots+d_0h_me^{-mz}\big]\\ =\ & 0 \end{aligned} \tag{5.2}$$

Taking $z=-r$, by observing the growth of the left side of (5.2) and noting that $h_m\neq 0$, we see that the sum of coefficients of all terms containing e^{-mz} must equal to zero, i.e.

$$m^2-c_0m+d_0=0. \tag{5.3}$$

Since m is a positive integer, (5.3) contradicts our assumption that equation $x^2-c_0x+d_0=0$ has no a positive integer solution. Hence (1.1) has no non-trivial subnormal solution.

Now suppose f is a non-trivial solution of (1.1), then we see that f is not subnormal by the above results. Thus we have

$$\overline{\lim_{r\to\infty}}\frac{\log T(r,f)}{r}=M,(0<M\leq\infty). \tag{5.4}$$

By (5.4), we have $\sigma_2(f)\geq 1$. Combining Lemma 4, we get $\sigma_2(f)=1$.

6. Proofs of Corollaries 1 and 2

Proof of Corollary 1. Suppose that f is a non-trivial subnormal solution of (1.1), then by $n > s$ and Theorem 3, we see that there exists a positive integer m, such that

$$m^2 - (c_{01} + c_{02}i)m + (d_{01} + d_{02}i) = 0. \tag{6.1}$$

Thus we can easily obtain (1.10) from (6.1). If $m < n-s$, i.e. $n-m > s$, by taking $z = r$ and observing the growth of the left side of (5.2) and noting $c_n \neq 0$, since $(n-1)r > (n-2)r > \cdots > (n-m)r > sr$, we obtain $h_1 = h_2 = \cdots = h_m = 0$. This contradicts $h_m \neq 0$. So, $m \geq n - s$.

Proof of Corollary 2. Suppose that f_1 and f_2 are two linearly independent solutions of (1.1), and f_1 is a non-trivial subnormal solution. By Theorem A, we see that f_1 is of the form (1.3).

By Lemma 8, we see that

$$m(r, Q_0) \leq M\big\{ \log[\max(T(r, f_1),\ T(r, f_2))]\big\}. \tag{6.2}$$

Since $m(r, Q_0) = \frac{s}{\pi}r$ and $T(r, f_1) = O(r)$, by (6.2), we see that $\sigma_2(f_2) \geq 1$. Combining with Lemma 4, we obtain that $\sigma_2(f_2) = 1$.

Acknowledgments

The authors are grateful to the referee for a number of helpful suggestions to improve the paper.

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